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Secondary domination in graphs

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Abstract

Given a dominating set $S \subseteq V$ in a graph $G = (V, E)$, place one guard at each vertex in S. Should there be a problem at a vertex $v \in V - S$, we can send a guard at a vertex $u \in S$ adjacent to v to handle the problem. If for some reason this guard needs assistance, a second guard can be sent from S to v, but the question is: how long will it take for a second guard to arrive? This is the issue of what we call *secondary domination*. We will focus primarily on dominating sets in which a second guard can arrive in at most two time steps. A $(1,2)$ *-dominating set* in a graph $G = (V, E)$ is a set S having the property that for every vertex $v \in V - S$ there is at least one vertex in S at distance 1 from v and a second vertex in S at distance at most 2 from v.

We present a variety of results about secondary domination, relating this to several other well-studied types of domination.

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1. Introduction

Consider a dominating set $S \subseteq V$ in a connected graph $G = (V, E)$. That is, S is a set such that every vertex $v \in V - S$ is adjacent to at least one vertex $u \in S$. Place one guard at each vertex in S. If there is a problem at a vertex $v \in V - S$, we could send a guard at an adjacent vertex $u \in S$ to v in one step. If for some reason this guard needed assistance, a second guard could be sent from S to v (assuming that $|S| \geq 2$), but the question is: how long will it take for a second guard to arrive at v? The focus on the length of time it takes a second guard to arrive at any vertex $v \in V$ is called *secondary domination*. The primary focus of this paper is on dominating sets for which secondary domination can be guaranteed in at most two time steps, but we first give a general definition. Let k be a positive integer. A subset S of vertices is called a $(1, k)$-*dominating set* in G if for every vertex $v \in V - S$ there are two distinct vertices $u, w \in S$ such that u is adjacent to v, and w is within distance k of v (i.e. $d_G(v, w) \leq k$). In what follows we will call a $(1, k)$-dominating set simply a $(1, k)$-*set* since the definition guarantees it is a dominating set.

Let us define the $(1, k)$-*domination number*, denoted $\gamma_{1,k}(G)$, to equal the minimum cardinality of a $(1, k)$-set in G. Such a set of vertices in G is called a $\gamma_{1,k}(G)$-set or simply a $\gamma_{1,k}$-set if the graph G is understood from the context. These sets are interesting for several reasons.

First, recall the definition of a 2-*dominating set*; that is, a set S having the property that every vertex $v \in V - S$ is dominated by at least two vertices in S. The minimum cardinality of a 2-dominating set is denoted $\gamma_2(G)$. In this new terminology, a 2-dominating set is the same as a $(1, 1)$-set. Therefore, $\gamma_{1,1}(G) = \gamma_2(G)$, and thus, since any 2-dominating set is a $(1, 2)$-set,

$$\gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G).$$

Another related notion is that of a *total dominating set*. This is a set S having the property that for every vertex $v \in V$, there is a vertex $u \in S$, not equal to v, that is adjacent to v. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality of a total dominating set in G. Notice that every total dominating set is also a $(1, 2)$-set. Therefore, for every graph G,

$$\gamma_{1,2}(G) \leq \gamma_t(G).$$

A set S is called a P_3-*dominating set* if for every vertex $v \in V - S$, there are two vertices u and w in S such that the subgraph induced by $\{u, v, w\}$ contains a path P_3. The P_3-*domination number* is denoted $\gamma_{P_3}(G)$. It is easy to see that every P_3-dominating set is also a $(1, 2)$-set. Therefore,

$$\gamma_{1,2}(G) \leq \gamma_{P_3}(G).$$

P_3-dominating sets were introduced by Haynes, Hedetniemi, Henning and Slater in [2].

Two other concepts are immediately related to $(1,2)$-sets. A dominating set S is called a *paired dominating set* if the subgraph $G[S]$ induced by the set S has a perfect matching. The *paired domination number* is the smallest cardinality of a paired dominating set and is denoted $\gamma_{\mathrm{pr}}(G)$. Paired dominating sets were introduced by Haynes and Slater [3, 4]. Notice that a paired dominating set is a total dominating set, and therefore is also a $(1,2)$-set.

Putting these observations together, we get the following inequality chains.

Theorem 1. *For any graph G for which $\gamma(G) > 1$,*

1. $\gamma(G) \leq \gamma_{1,2}(G) \leq \gamma_{P_3}(G) \leq \gamma_t(G) \leq \gamma_{\mathrm{pr}}(G)$, *and*
2. $\gamma(G) \leq \gamma_{1,2}(G) \leq \gamma_{P_3}(G) \leq \gamma_2(G)$.

It is worth pointing out that $\gamma(G) < \gamma_{1,2}(G) < \gamma_{P_3}(G)$ is possible. Consider for example the path P_6 with vertices labeled in order $1,2,3,4,5,6$. The set $\{2,5\}$ shows that $\gamma(P_6) = 2$; the set $\{1,4,6\}$ shows that $\gamma_{1,2}(P_6) = 3$; and it is easy to see that $\gamma_{P_3}(P_6) = 4$.

A dominating set $S \subseteq V$ is an *independent dominating set* if no two vertices in S are adjacent, that is, S is an *independent set*. The minimum cardinality of an independent dominating set is denoted $i(G)$ and is called the *independent domination number*. Similarly, the maximum cardinality of an independent dominating set is denoted $\beta_0(G)$ and is called the *independence number*. Any independent set in G having cardinality $\beta_0(G)$ will be called a *$\beta_0(G)$-set*. The familiar relationship, $\gamma(G) \leq i(G) \leq \beta_0(G)$, follows directly from the definitions.

2. Independent $(1,2)$-sets

In the same way we can define an *independent $(1,2)$-set* to be a $(1,2)$-set S that is also an independent set. Note that the definition of a $(1,2)$-set S implies that S has at least two vertices, so that an independent $(1,2)$-set must have at least two non-adjacent vertices. Consequently, complete graphs do not have independent $(1,2)$-sets. Although it might be tempting to conjecture that all connected graphs having at least two non-adjacent vertices have an independent $(1,2)$-set, the graph consisting of two disjoint triangles connected by a single edge shows that this is not the case. This graph has independence number two, but no maximal independent set is a $(1,2)$-set.

This, therefore, raises the interesting question, what is the complexity of the following decision problem?

INDEPENDENT $(1,2)$-SET
INSTANCE: A graph $G = (V,E)$
QUESTION: Does G have an independent $(1,2)$-set?

In a subsequent paper [5] we will demonstrate the NP-completeness of this and other decision problems related to secondary domination, and we will construct polynomial algorithms for computing several secondary domination numbers for trees.

We can prove that several classes of graphs have independent $(1,2)$-sets.

Proposition 2. *Every connected graph G having at least two non-adjacent vertices and no triangles has at least one independent $(1,2)$-set.*

Proof. Let S be any maximum independent set in a connected graph G having at least two non-adjacent vertices and no triangles. Then $|S| \geq 2$. We can assume that if G has any leaves, then they belong to S. For if a leaf u is not in S, then its only neighbor, say vertex v, must be in S since S is a maximum independent set. Therefore, in this case we could exchange $v \in S$ for u and still have a maximum independent set.

Let x be an arbitrary vertex in $V - S$. Since S is a dominating set, x is adjacent to at least one vertex $y \in S$. By the above we know that x is not a leaf of G, and thus x is adjacent to another vertex, say w, in G. If $w \in V - S$, then the fact that S is a dominating set implies w is adjacent to at least one vertex $z \in S$. Since G has no triangles, we know that $y \neq z$. Otherwise, $w \in S$. In either case, it follows that x is within distance two of a second vertex (other than y) in S, and consequently S is an independent $(1,2)$-set in G. □

In fact, the proof of Proposition 2 shows that such a graph always has an independent set of (maximum) cardinality $\beta_0(G)$ that is also a $(1,2)$-set.

Corollary 3. *Every triangle-free, connected graph G of order at least three has an independent $(1,2)$-set of cardinality $\beta_0(G)$. In particular, every tree T of order at least three has an independent $(1,2)$-set of cardinality $\beta_0(T)$.*

Theorem 4. *Every graph G of diameter 2 has an independent $(1,2)$-set of cardinality $\beta_0(G)$.*

Proof. Let $S \subseteq V$ be any maximum independent set in a graph $G = (V,E)$ having diameter 2. Since G has diameter 2 we know that $|S| \geq 2$. Since S is a maximum independent set it is a dominating set. Therefore every vertex in $V - S$ is within distance 1 of at least one vertex in S. But since the diameter of G is 2, every vertex in $V - S$ must be within distance 2 of every other vertex in S. Therefore, S is an independent $(1,2)$-set of cardinality $\beta_0(G)$. □

The *Cartesian product* of two graphs $G = (V,E)$ and $H = (W,F)$ is the graph $G \square H$ whose vertex set is the Cartesian product $V \times W$ and two pairs (u,v) and (w,x) of vertices are adjacent in $G \square H$ if and only if either $u = w$ and v is adjacent to x in H, or u is adjacent to w in G and $v = x$.

We now show that any graph that is a nontrivial Cartesian product (one in which both factors have at least two vertices) has an independent $(1,2)$-set.

Theorem 5. *Let G and H be connected graphs of order at least two. Then $G\Box H$ has an independent $(1,2)$ -set.*

Proof. We recursively form a partition $I_1, I_2, \ldots, I_t$ of $V(G)$ into independent sets chosen as follows. Let I_1 be a maximal independent set of G. Choose I_2 to be any maximal independent set in $G - I_1$. If $I_1, I_2, \ldots, I_j$ have been chosen, let I_{j+1} be any maximal independent set in $G - (I_1 \cup I_2 \cup \cdots \cup I_j)$. Continue this until the sequence of independent sets $I_1, I_2, \ldots, I_t$ partitions $V(G)$. Using an identical procedure in H form a partition $J_1, J_2, \ldots, J_r$ of $V(H)$ into independent sets. Without loss of generality we assume that $t \le r$. For notational ease let $M_s = \cup_{i=1}^{s} I_i$ and let $N_s = \cup_{i=1}^{s} J_i$ for each s, $1 \le s \le t$. Note that $M_t = V(G)$ but it is possible that $V(H) - N_t \neq \emptyset$.

Let $S = \cup_{i=1}^{t}(I_i \times J_i)$. By the choice of the sets $I_1, I_2, \ldots, I_t$ and $J_1, J_2, \ldots, J_r$ and the definition of $G\Box H$, it is clear that S is an independent set of $G\Box H$. We will show that S is, in fact, a $(1,2)$-set. See Figure 1 where we have illustrated the structure of S for a situation where $t = 8 < r$ and $N_t \neq V(H)$.

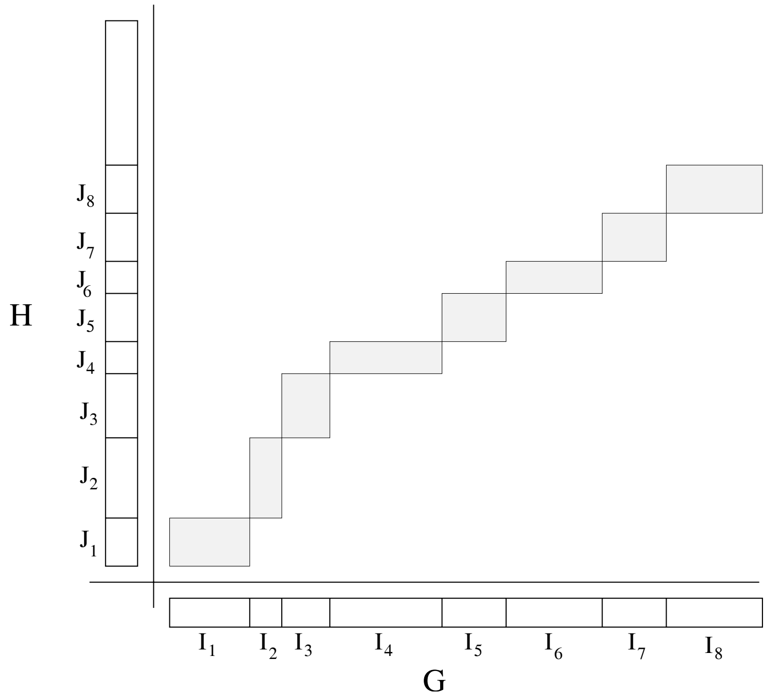

Figure 1: Illustration of the independent $(1,2)$-set in $G\Box H$

Let $(a,b) \in V(G\Box H) - S$. Since $M_t = V(G)$ it follows that $a \in I_k$ for some k, $1 \le k \le t$. We first consider the case when $k > 1$. There are three subcases to consider depending on where b is located.

If $b \in V(H) - N_k$, then b has distinct neighbors $b' \in J_k$ and $b'' \in J_1$. Now (a,b)

is adjacent to $(a, b') \in I_k \times J_k \subseteq S$. Since I_1 is a maximal independent set in G, there exists a vertex $a'' \in I_1$ that is adjacent to a. Consequently, $(a, b), (a, b''), (a'', b'')$ is a path of length two from (a, b) to a second vertex in S. If $k > 2$ and $b \in J_s$ for some s, $1 < s \leq k-1$, then choose a'' and b'' as above. In addition the vertex a is adjacent to some $a' \in I_s$ since I_s is maximal independent in $G - M_{s-1}$. In this case (a, b) is adjacent to $(a', b) \in I_s \times J_s \subseteq S$ and $(a'', b'') \in S$ is distance two from (a, b). Finally, suppose b is in J_1. With $a'' \in I_1$ adjacent to a as above we see that (a, b) is dominated by $(a'', b) \in S$. Because J_1 is independent and H is connected, $bb' \in E(H)$ for some $b' \in V(H) - J_1$. If there exists such a vertex b' in J_k, then $(a, b') \in I_k \times J_k \subseteq S$, and thus (a, b) has a second neighbor in S. A second possibility is that $b' \in J_s$ for some s, $2 \leq s \leq k-1$. The fact that a is in I_k and that I_s dominates $G - M_{s-1}$ implies that a has a neighbor $a' \in I_s$. The result is that $(a, b), (a, b'), (a', b')$ is a path of length two from (a, b) to a vertex in S. The third possibility in this subcase is that $N(b) \subseteq V(H) - N_k$. Since J_k dominates $H - N_{k-1}$ it follows that there exists $b'' \in J_k \cap N(b')$. The path $(a, b), (a, b'), (a, b'')$ shows that there is a second vertex in S at distance two from (a, b).

Now assume that $k = 1$. Since J_1 is maximal independent in H it follows that there exists $b' \in N(b) \cap J_1$. Hence (a, b) is adjacent to a vertex $(a, b') \in I_1 \times J_1 \subseteq S$. If b belongs to $V(H) - N_t$, then the assumptions that I_1 is independent, that $M_t = V(G)$ and that G is connected imply that a is adjacent to some vertex $a' \in I_s$ for some s, $2 \leq s \leq t$. The set J_s is maximal independent in $H - N_{s-1}$. Hence there exists $b'' \in N(b) \cap J_s$. The distance from (a, b) to (a', b'') is two and $(a', b'') \in S$. Thus we may assume that b belongs to J_n for some n, $2 \leq n \leq t$. Again, I_1 is independent and G is connected and thus a has a neighbor $a' \in V(G) - I_1$. When there is such an a' in I_n, the vertex (a, b) is dominated by a second vertex (a', b) in S. Otherwise a' is in I_m (for some $m \leq n-1$) or a' is in $V(G) - M_n$. In the former case (a', b) is adjacent to some vertex $(a', b'') \in I_m \times J_m$ and thus there is a second vertex in S at distance two from (a, b). In the latter case a' is dominated by the maximal independent set I_n of $G - M_{n-1}$. This implies there is a path $(a, b), (a', b), (a'', b)$ of length two ending at a vertex of S.

We have shown that every (a, b) in $V(G \,\square\, H) - S$ is adjacent to a vertex of S and is within distance two of a second (distinct) vertex of S. Therefore, S is an independent $(1, 2)$-set of $G \,\square\, H$. $\square$

We now show how to construct two classes of graphs that do not have independent $(1, 2)$-sets. The first class can be described as follows. Let T be any tree of order $n \geq 2$. For a vertex $v \in V(T)$ of degree k let C_v denote a complete subgraph of order at least $k+2$. Consider the disjoint union of these n complete subgraphs, and for every edge uv in T, add an edge between a vertex in C_u and one in C_v in such a way that no two edges added between pairs of complete subgraphs have a vertex in common. The class of graphs constructed in this way is denoted $G(T)$.

Theorem 6. *If T is any tree of order $n \geq 2$ and H is any graph in $G(T)$ as constructed above, then H does not have an independent $(1, 2)$-set.*

Proof. It is easy to verify that any graph in $G(P_2)$ does not have an independent $(1,2)$-set. Suppose the theorem holds for some $n \geq 2$. Let T be a tree of order $n+1$, and let H be any graph in $G(T)$. Suppose for the sake of contradiction that H has an independent (1,2)-set S. Remove a leaf v, whose only neighbor is u, from T and denote the resulting tree by T'. Let U denote the set of simplicial vertices in C_v. By construction $|U| \geq 2$.

Let ab be the edge of H between C_v and C_u where $a \in C_v$ and $b \in C_u$. Since S is an independent (1,2)-set it follows that $b \in S$ and $a \notin S$. Hence, $U \cap S \neq \emptyset$. Let $x \in U \cap S$. By its construction we see that the graph $H' = H - C_v$ belongs to $G(T')$. In addition, $S' = S - \{x\}$ is an independent (1,2)-set of H', contradicting the assumption about $G(T')$. □

Notice that all of the graphs in $G(T)$ have bridges, or cut edges, and therefore are not 2-connected. One class of 2-connected graphs not having independent $(1,2)$-sets can be constructed as follows. For positive integers $i, j \geq 4$ let C_i and C_j be vertex disjoint cliques of orders i and j, respectively. For $2 \leq m \leq i-2$ and $2 \leq n \leq j-2$ choose m vertices from C_i and n vertices from C_j and add the edges of a complete bipartite graph $K_{m,n}$ between these vertices. Denote the resulting graph by $C_i K_{m,n} C_j$.

Proposition 7. *No graph of the form $C_i K_{m,n} C_j$, for $i, j \geq 4$, has an independent $(1,2)$-set.*

Proof. Any graph G of the form $C_i K_{m,n} C_j$, for $i, j \geq 4$, has $\beta_0(G) = 2$. Every maximal independent set of G has one vertex chosen from the clique C_i and one vertex chosen from the clique C_j. Such a set is a dominating set but is not a $(1,2)$-set. □

The linear, greedy algorithm designed by Mitchell [6], and later published in [7], for computing the value $\beta_0(T)$ for any tree T, is sufficient to find an independent $(1,2)$-set of cardinality $\beta_0(T)$ in any tree T of order at least three. After rooting a tree T at any non-leaf vertex, this algorithm will find a β_0-set containing all leaves of T. The proof of Proposition 2 guarantees that a maximum independent set containing all leaves of T is a $(1,2)$-set.

Corollary 3 raises the following interesting question. If a graph G has an independent $(1,2)$-set, does it always have one of cardinality $\beta_0(G)$?

For graphs having an independent $(1,2)$-set, one can define the *independent* $(1,2)$-*domination number* $i_{1,2}(G)$ to equal the minimum cardinality of an independent $(1,2)$-set in G, and the $(1,2)$-*independence number* $\beta_{1,2}(G)$ to equal the maximum cardinality of an independent $(1,2)$-set. In this case, we have the following inequality chain

$$\gamma(G) \leq \gamma_{1,2}(G) \leq i_{1,2}(G) \leq \beta_{1,2}(G) \leq \beta_0(G).$$

But we also have the following inequality chain

$$\gamma(G) \leq i(G) \leq i_{1,2}(G) \leq \beta_{1,2}(G) \leq \beta_0(G).$$

It can be see, however, that $i(G)$ and $\gamma_{1,2}(G)$ are not comparable. For example, for the path P_6,

$$\gamma(P_6) = i(P_6) = 2 < 3 = \gamma_{1,2}(P_6).$$

On the other hand, for a double star, for example the tree T obtained by joining the central vertex of $K_{1,3}$ to the central vertex of another $K_{1,3}$,

$$\gamma(T) = \gamma_{1,2}(T) = 2 < i(T) = 4.$$

It is possible for $i_{1,2}(G) < \beta_{1,2}(G)$ to hold. Consider the cycle C_6 of order 6. A maximal independent set of cardinality two can be seen to be an independent $(1,2)$-set, while the maximum independent set of cardinality three in C_6 is also an independent $(1,2)$-set.

Notice that for the path P_5 the unique β_0-set, of cardinality 3, is an independent $(1,1)$-set. Although the path P_6 does not have an independent $(1,1)$-set, it does have an independent $(1,2)$-set. By Corollary 3 every tree of order at least three has a β_0-set that is also a $(1,2)$-set, but we are thus led to ask, which trees have a $(1,1)$ β_0-set? Also, if a tree T has an independent $(1,1)$-set, must its cardinality equal $\beta_0(T)$, or can a tree have two independent $(1,1)$-sets of different cardinalities? Notice, for example, that the complete bipartite graphs $K_{m,n}$, for $2 \leq m < n$, have two independent $(1,1)$-sets of cardinalities m and n. Similarly, the grid graphs $P_{2n+1} \square P_{2n+1}$ have independent $(1,1)$-sets of different cardinalities (the set of red squares of a chessboard or the set of black squares). On the other hand, any odd order path, P_{2n+1} has a unique independent $(1,1)$-set, and it has cardinality $\beta_0(P_{2n+1}) = n+1$.

More generally, we can consider the following decision problem.

INDEPENDENT $(1,1)$-SET
INSTANCE: A graph $G = (V, E)$
QUESTION: Does G have an independent $(1,1)$-set?

In a seemingly unrelated paper, Gunther, Hartnell and Rall [1] investigated this very question. They proved the following theorem that gave a characterization of those trees T that have a maximum independent set that is also a 2-dominating set (i.e., a $(1,1)$-set).

Theorem 8. [1] *The following statements are equivalent for a tree T.*

(i) *For every edge e of T, $\beta_0(T-e) = \beta_0(T)$.*

(ii) *T has a maximum independent set that is a 2-dominating set.*

(iii) *T has a unique maximum independent set.*

Thus, one can run the algorithm of Mitchell [6] to produce a maximum independent set M of a tree T and then test in linear time whether M is a $(1,1)$-set. If it is, then the decision problem has been answered affirmatively for T. On the other hand, if M is found not to be a $(1,1)$-set, then Theorem 8 guarantees that T does not have an independent $(1,1)$-set of cardinality $\beta_0(T)$.

A structural characterization of the class of trees having a $(1,1)$-set that is also a β_0-set is also given in [1]. The following three operations on a graph G with a distinguished subset, M, of vertices to produce a larger graph are needed in the construction of such trees.

1. Type I: Add a new vertex to G and make it adjacent to any vertex $x \in V(G) - M$.
2. Type II: Add a path of order two to G and make one of its vertices adjacent to any $x \in M$.
3. Type III: Add a new path of order three to G and make its center vertex adjacent to any vertex $x \in V(G)$.

It is clear that if M is a maximum independent set in G and a new graph G' is constructed by one of these three operations performed on G, then there is a unique maximum independent set of G' that contains M. Using this and Theorem 8 led to the structural characterization mentioned above.

Theorem 9. [1] *A tree T has a maximum independent set that is a $(1,1)$-set if and only if $T = K_1$ or is constructible from $M = V(K_1)$ by a finite sequence of operations of Type I, II, or III, where after each operation the distinguished set of vertices is the unique β_0-set in the new tree.*

3. $(1,k)$-dominating Sets

In this section we show that in several different contexts, various types of dominating sets also are $(1,k)$-sets for small values of k. In many cases we will be interested in minimum dominating sets that are also $(1,k)$-sets. Thus, throughout the section we will assume that all graphs have domination number at least 2, since this is an obvious necessary condition to have a γ-set that is a $(1,k)$-set.

Let $G = (V,E)$ be any connected graph of diameter m such that $\gamma(G) \geq 2$. Then for every positive integer k, $1 \leq k \leq m$, G has a dominating set that is a $(1,k)$-set, and hence

$$\gamma(G) = \gamma_{1,m}(G) \leq \gamma_{1,m-1}(G) \leq \cdots \leq \gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G)\,. \tag{1}$$

Suppose that G has an independent set that is a $(1,k)$-set. Then we define $i_{1,k}(G)$ and $\beta_{1,k}(G)$ to be the minimum and maximum cardinalities, respectively, of independent $(1,k)$-sets in G. We are interested in determining the smallest such k. It is clear that $k \leq m = \operatorname{diam}(G)$ and then $i(G) = i_{1,m}(G) \leq i_{1,m-1}(G) \leq \cdots \leq i_{1,k}(G) \leq \beta_{1,k}(G) \leq \beta_{1,k+1}(G) \leq \cdots \leq \beta_{1,m}(G) = \beta_0(G)$.

We first present a very general result about arbitrary, not necessarily minimal, dominating sets in connected graphs.

Theorem 10. *Every dominating set of cardinality at least 2 in a connected graph* G *with* $\gamma(G) \geq 2$ *is a* $(1,4)$*-set.*

Proof. Let S be any dominating set in G, and assume that S is not a $(1,4)$-set. This means that there is at least one vertex $v \in V(G) - S$ that is adjacent to some $u \in S$ but is not within distance 4 of a second vertex in S. Consider the eccentricity, $\mathrm{e}(v) = \max_{a \in V} d_G(v,a)$, of vertex v. Suppose $\mathrm{e}(v) \in \{1,2,3\}$. Since $|S| \geq 2$ there exists $w \in S - \{u\}$ and because $\mathrm{e}(v) \leq 3$, $d(v,w) \leq 3$. Otherwise, $\mathrm{e}(v) \geq 4$, and there exists a vertex z whose distance from v is 4. Let v,w,x,y,z be a shortest v,z-path in G. The vertex y cannot be adjacent to u or else v,u,y,z is a v,z-path, which contradicts the assumption that $d(v,z) = 4$. Because S is a dominating set there exists $a \in S$, $a \neq u$, such that a and y are adjacent. But then $d(v,a) \leq 4$. □

An immediate consequence of Theorem 10 is that inequality (1) can be shortened to

$$\gamma(G) = \gamma_{1,4}(G) \leq \gamma_{1,3}(G) \leq \gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G)\,, \tag{2}$$

for any connected graph G with $\gamma(G) \geq 2$. It is important to observe that the first inequality in (2) can indeed be strict. For example, the unique $\gamma(P_6)$-set is not a $(1,3)$-set in P_6.

When restricted to trees, there are at least 2 distinct values in (2).

Theorem 11. *No tree of order at least 3 has a* γ*-set or an* i*-set that is also a* $(1,1)$*-set.*

Proof. Let us assume that we are given a tree T with $\gamma(T) \geq 2$. Assume that S is a $\gamma(T)$-set that is also a $(1,1)$-set. Since every vertex not in S is dominated at least twice, it follows that S must be an independent dominating set. If S was not independent, and had two adjacent vertices x and y, then since S is 2-dominating neither x nor y could have a private neighbor. In addition, since no leaf can be dominated twice, S contains every leaf of T. Let $w \notin S$ be a vertex adjacent to a leaf u. Since w is dominated at least twice by S, there exists $v \in S \cap N(w)$, $v \neq u$. It now follows that $(S - \{u,v\}) \cup \{w\}$ is a dominating set, contradicting the minimality of S.

Assume that S is a $(1,1)$, $i(T)$-set. Thus, S is a minimum cardinality independent dominating set in T and S is a $(1,1)$-set. Since S is a $(1,1)$-set, every leaf in T must be in S. Let $u \in S$ be a leaf, and let $v \notin S$ be the only vertex adjacent to u. Since S is a $(1,1)$-set, vertex v must be adjacent to at least one other vertex in S, say $w \in S$. Let S'' be the set of vertices in $N(v)$ that are in S. That is, $S'' = N(v) \cap S$. Notice that $u, w \in S''$, and therefore $|S''| \geq 2$.

Define a new set $S' = (S - S'') \cup \{v\}$. Notice first that the new set S' is an independent set, and notice that $|S'| < |S|$. Finally, notice that S' is a dominating set. This follows since every vertex $x \in V - S$, where $x \neq v$, must be dominated by two vertices in S. Because T has no cycles it follows that no two vertices in $N(v) \cap S$ can be adjacent to the same vertex in $V - (S \cup \{v\})$, and hence x is adjacent to at least one vertex of S'.

Thus, S' is an independent dominating set of cardinality less than that of S, which contradicts the assumption that S is an $i(T)$-set. □

Therefore, combining this with Theorem 10 yields the following.

Corollary 12. *If D is a γ-set in a tree $T \neq K_{1.n}$, then D is either a $\gamma_{1,2}$-set, a $\gamma_{1,3}$-set or a $\gamma_{1,4}$-set.*

The tree T in Figure 2 demonstrates that (2) can represent 4 distinct values. For this tree it is easy to verify that

$$\gamma(T) = 3 = \gamma_{1,4}(T) < 4 = \gamma_{1,3}(T) < 5 = \gamma_{1,2}(T) < 7 = \gamma_{1,1}(T) = \gamma_2(T)\,.$$

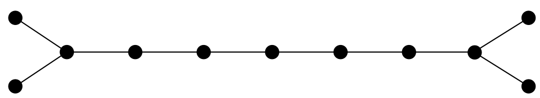

Figure 2: Tree T illustrating (2)

Since every maximal independent set is a dominating set we get the following corollary to Theorem 10.

Corollary 13. *Every maximal independent set in a connected graph G with $\gamma(G) \geq 2$ is a $(1,4)$-set. In particular, $i_{1,4}(G)$ and $\beta_{1,4}(G)$ are defined.*

However, we can improve this result for 2-maximal independent sets. An independent set $S \subseteq V$ is called 2*-maximal independent* if it is maximal independent and it is not possible to remove one vertex from S and then add two other vertices and get a larger independent set. To illustrate this, consider the path P_6, with vertices labeled in order $1,2,3,4,5,6$. It is easy to see that the set $S = \{2,5\}$ is a maximal independent set; in fact, $i(P_6) = 2$. However this set S is not a 2-maximal independent set, since one can remove vertex 2 from S and then add vertices 1 and 3 and get a larger independent set.

Theorem 14. *Every 2-maximal independent set in a connected graph G with $\gamma(G) \geq 2$ is a $(1,3)$-set.*

Proof. Let G be a connected graph with $\gamma(G) \geq 2$ and let S be a 2-maximal independent set of G. The set S is a dominating set and therefore, $|S| \geq 2$. Assume that S is not $(1,3)$-set. This implies that there exists a vertex $v \notin S$ that is adjacent to a vertex $u \in S$ but is not within distance 3 of any other vertex of S.

The distance between v and any vertex of S other than u is at least 4, and consequently $\mathrm{e}(v) \geq 4$. Let v, w, x, y, z be a shortest v, z-path. Then v and x are not adjacent and by assumption $\{t \in S \mid d_G(v,t) \leq 3\} = \{u\}$. This implies that x and u are adjacent, but that x has no other neighbors in S. But now we see that $S' = (S - \{u\}) \cup \{v, x\}$ is independent. This contradicts the assumption that S is a 2-maximal independent set. Hence, S is a $(1,3)$-set. □

The result is that $i_{1,3}(G)$ and $\beta_{1,3}(G)$ are defined. More can be said though. Since any $\beta_0(G)$-set is 2-maximal, the following corollary is immediate.

Corollary 15. *If G is a connected graph with $\gamma(G) \geq 2$, then $\beta_{1,3}(G) = \beta_0(G)$.*

Notice again that for the path P_6 the unique γ-set has cardinality 2, and this set is also an i-set (an independent dominating set of cardinality $i(G)$). This means that i-sets need be not $(1,3)$-sets, although each is a $(1,4)$-set by Theorem 10.

Corollary 16. *Every i-set in a connected graph with domination number at least 2, is an independent $(1,4)$-set.*

Putting this all together we see that the previous inequality chain can be shortened for a connected graph G with $\gamma(G) \geq 2$. Here, we are letting k denote the smallest integer such that G has an independent set that is also a $(1,k)$-set. As shown above, $1 \leq k \leq 3$.

$$i(G) = i_{1,4}(G) \leq i_{1,3}(G) \leq \cdots \leq i_{1,k}(G) \leq \beta_{1,k}(G) \leq \cdots \leq \beta_{1,3}(G) = \beta_0(G)\,.$$

4. Open Problems

In this initial study of secondary domination, many problems have been discovered for which we still have no solutions. Included among these are the following:

1. If a graph G has an independent $(1,2)$-set does it always have one of cardinality $\beta_0(G)$? Note that we have shown that any non-trivial Cartesian product graph has an independent $(1,2)$-set, but is there one that is a maximum independent set?

2. Let $\Gamma(G)$ denote the maximum cardinality of a minimal dominating set in G. Such a set is called a Γ-set. Is every Γ-set of a graph G either a $(1,1)$, a $(1,2)$ or a $(1,3)$-set?

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