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Distinguishing and Distinguishing Chromatic Numbers of Generalized Petersen Graphs

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Abstract

Albertson and Collins defined the distinguishing number of a graph to be the smallest number of colors needed to color its vertices so that the coloring is preserved only by the identity automorphism. Collins and Trenk followed by defining the distinguishing chromatic number of a graph to be the smallest size of a coloring that is both proper and distinguishing. We show that, with two exceptions, generalized Petersen graphs are 2-distinguishable and properly 3-distinguishable.

Keywords: graph, graph automorphism, vertex coloring, distinguishing number.

2000 Mathematics Subject Classification: 05C25 (20)

1. Introduction

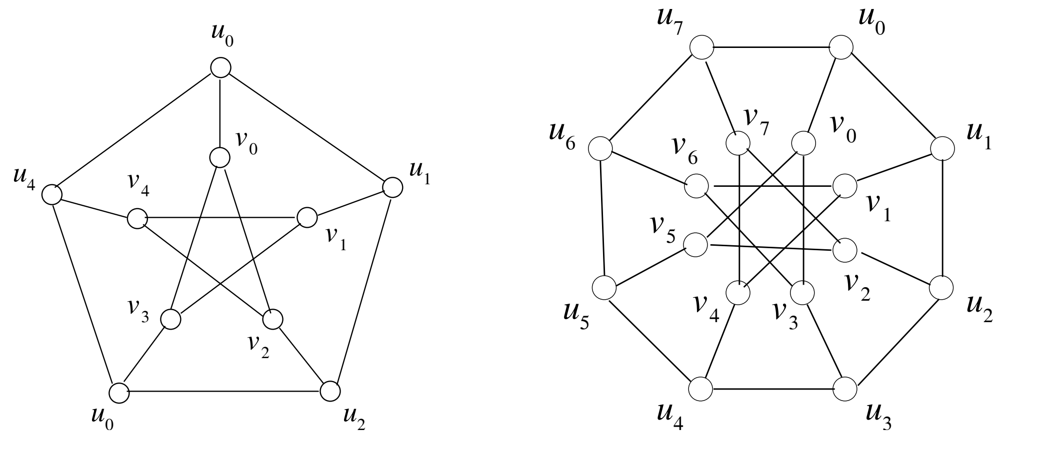
In 1996 Albertson and Collins [3] defined the distinguishing number of a graph to be the minimum number of colors required to color its vertices so that the coloring is preserved only by the identity automorphism. They proved several group theoretic results pertaining to this problem and, as an application, they presented a proof that the classical Petersen graph is 3-distinguishable.

The determination of distinguishing numbers for graphs has developed into an active research area over the last few years. Bogstad and Cowen [4] showed that 2 and 3-cubes are 3-distinguishable and cubes of higher dimension are 2-distinguishable. Albertson and Boutin [1] showed that, with a few exceptions of small order, Kneser graphs have distinguishing number two. Albertson [2] obtained results for Cartesian powers of graphs and this work was followed upon by Imrich and Klavžar [8] and Klavžar and Zhu [9]. Harary and Jacobson [7] introduced a similar concept in 1991 when they considered the problem of orientating some of the edges of a graph so that only the identity automorphism preserves the orientations and the number of oriented edges is minimized.

Collins and Trenk [5] defined the distinguishing chromatic number of a graph to be the smallest size of a coloring that is both proper and distinguishing. They calculated distinguishing and distinguishing chromatic numbers for several graphs including the Peterson graph, which they showed to be properly 4-distinguishable.

We show that, with the exception of the Petersen graph and the 3-cube, which also has distinguishing number three and distinguishing chromatic number four, generalized Petersen graphs are 2-distinguishable and properly 3-distinguishable.

Given integers n and r satisfying $n \geq 3$, $r \geq 1$ and $2r < n$, we define the generalized Petersen graph $P(n, r)$ as follows: We partition the vertex set of $P(n, r)$ into two sets $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$. U induces an n-cycle, for which each u_i has u_{i-1} and u_{i+1} as neighbors. (Throughout this paper, computations involving indices are performed modulo n.) Each v_i has two neighbors in V, v_{i-r} and v_{v+r}. Additionally, u_i and v_i are adjacent for each i. We refer to the members of U (resp. V) as outer (resp. inner) vertices and edges that have both end points in U (V) as outer (inner) edges. Edges that join u_i to v_i are called spokes. We denote the vertex set of $P(n, r)$ by $U \cup V$. The classical Petersen graph $P(5, 2)$ and the generalized Petersen graph $P(8, 3)$ are depicted in Figure 1.



$P(5, 2)$ $P(8, 3)$

Figure 1

2. Automorphism Groups of Generalized Petersen Graphs

We let $A(n, r)$ denote the automorphism group of $P(n, r)$. Frucht, Graver and Watkins [6] exhibited generators and computed orders for each $A(n, r)$. They showed, with the following seven exceptions: $(n, r) = (4, 1)$, $(5, 2)$, $(8, 3)$, $(10, 2)$, $(10, 3)$, $(12, 5)$, $(24, 5)$, that $A(n, r)$ is generated by either the first two or all three of the following:

1. $\rho(u_i) = u_{i+1}$ and $\rho(v_i) = v_{i+1}$.

2. $\delta(u_i) = u_{-i}$ and $\delta(v_i) = v_{-i}$.

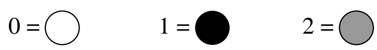
3. $\alpha(u_i) = v_{ri}$ and $\alpha(v_i) = u_{ri}$.

First notice that ρ and δ generate the dihedral group D_n, and this group is contained in each $A(n,r)$. With the exception of $P(10,2)$, they established that equality holds whenever $r^2 \not\equiv \pm 1 \pmod{n}$. When we have equality we say that the graph $P(n,r)$ is of Type I. It is clear that U and V are both invariant for D_n. Albertson and Collins showed that any graph whose automorphism group is a dihedral group must be 2-distinguishable. In this paper, we exhibit such colorings.

In the case that $r^2 \equiv \pm 1 \pmod{n}$, Frucht et al showed that $\langle \rho, \delta, \alpha \rangle \leq A(n,r)$, with equality holding unless $P(n,r)$ is one of the six remaining exceptional graphs. When equality holds we say that such graphs are of Type II. It is clear that U and V are either invariant or interchanged for such graphs. While they presented generators for the seven exceptional cases, for our purposes, it suffices to note that these graphs are characterized by the fact that there exist automorphisms for which the spokes are not invariant. We say that the exceptional graphs are of Type III.

3. The Proof Technique

We let id denote the identity automorphism. A k-coloring of $P(n,r)$ is a function $\phi : U \cup V \to \{0, 1, \ldots, k-1\}$. We may restrict our attention to the cases $2 \leq k \leq 3$ and, when illustrating colorings, we adhere to the following convention:



Assume that $\theta \in A(n,r)$ and ϕ is a coloring of $P(n,r)$. We say that θ preserves ϕ if $\phi(\theta(x)) = \phi(x)$ for each $x \in U \cup V$. We define $F_\theta(n,r) = \{x \in U \cup V \mid \theta(x) = x\}$, and say that members of this set are fixed by θ. The coloring ϕ is said to distinguish $P(n,r)$ if it is preserved only by the identity automorphism. The distinguishing number of $P(n,r)$, denoted $D(P(n,r))$, is defined to be the smallest k for which there exists a k-distinguishing coloring. Following Collins and Trenk, we define the distinguishing chromatic number of $P(n,r)$, denoted $\chi_D(P(n,r))$, to be the smallest size of a coloring that is both proper and distinguishing.

Given $S \subseteq U \cup V$ and a coloring ϕ, we let $\widehat{\phi}(S)$ denote the color multi-set of S. This set consists of the colors assigned to members of S by ϕ, with each color occurring once for each assignment. Clearly, in order for θ to preserve ϕ, it is necessary that $\widehat{\phi}(\theta(S)) = \widehat{\phi}(S)$. The multi-set $\widehat{\phi}(\mathrm{N}(x))$, where $\mathrm{N}(x)$ denotes the open neighborhood of the vertex x, is of special interest.

To show that a particular coloring of $P(n,r)$ is distinguishing, we assume that θ is an arbitrary automorphism and initialize $F_\theta(n,r)$ to the empty set. We then add vertices to $F_\theta(n,r)$ until it equals $U \cup V$. A vertex may be added if it is uniquely identified amongst non-members of $F_\theta(n,r)$ by properties such as color, the color multi-set of its open neighborhood and distances from vertices that are already known to belong to $F_\theta(n,r)$. Since U and V are invariant for automorphisms of Type I graphs and either invariant or interchanged for automorphisms of Type II graphs, known membership in U or V and knowledge of whether or not that set is invariant may also contribute to the establishment of membership in $F_\theta(n,r)$. The following two lemmas provide means for efficiently applying this procedure.

Lemma 1. *Assume that $P(n,r)$ is of Type I or II, $\theta \in A(n,r)$ and there exists i such that $u_i, u_{i+1} \in F_\theta(n,r)$. Then $\theta = id$.*

Proof. We have that $\theta(U) = U$, and u_{i+1} has two neighbors in U, u_i and u_{i+2}, It is now necessary that $u_{i+2} \in F_\theta(n,r)$. Continuing in this manner, we obtain $U \subseteq F_\theta(n,r)$, and it readily follows that $V \subseteq F_\theta(n,r)$. □

Lemma 2. *Assume that $r \geq 2$ and let $\theta \in A(n,r)$. Assume that there exists i such that $\{v_{i+j}\}_{j=0}^{r-1}$, $\{u_{i+l}\}_{l=1}^{r} \subset F_\theta(n,r)$. Then $\theta = id$.*

Proof. Since u_{i+r} and v_i are fixed, their unique common neighbor v_{i+r} must also be fixed. Now, since u_{i+r} and two of its neighbors are fixed, its remaining neighbor u_{i+r+1} must be fixed. Iterate this procedure. □

4. Computation of $D(P(n,r))$

As noted above, Albertson and Collins showed that $D(P(5,2)) = 3$. Since $P(4,1)$ is the 3-cube, Bogstad and Cowen have established that it is 3-distinguishable. We now show that $D(P(n,r)) = 2$ for all other generalized Petersen graphs.

Lemma 3. *Assume that $P(n,r)$ is of Type I or II. Then $D(P(n,r)) = 2$.*

Proof. Define $\phi : U \cup V \to \{0,1\}$ by

$$\theta(x) = \begin{cases} 1 & \text{if } x \in \{u_0, u_1, v_1\} \\ 0 & \text{otherwise} \end{cases} .$$

Let $\theta \in A(n,r)$ and assume that it preserves ϕ. The set of vertices that are assigned color one must be invariant and, since only u_1 neighbors the other two, it must be fixed. Now U must be invariant and this implies that u_0 is also fixed. Apply Lemma 1. □

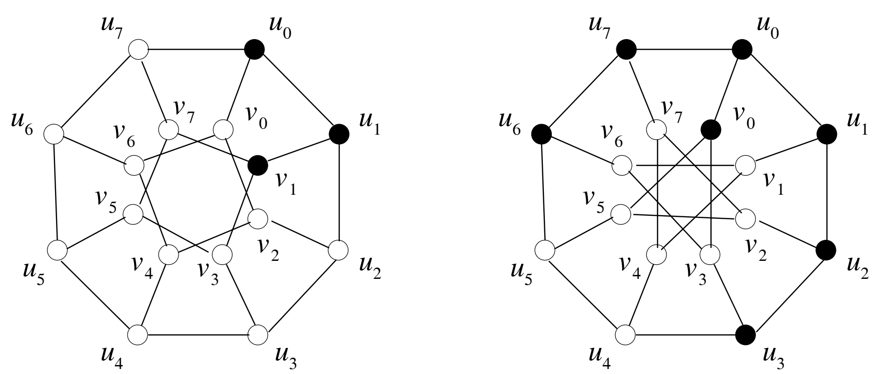
Lemma 4. *Assume that $n \geq 7$. Then $D(P(n,r)) = 2$.*

Proof. Let $m = \max\{r, 3\}$ and $Y = \{u_{n-2}, u_{n-1}, v_0\} \cup \{u_i\}_{i=0}^{m}$. Define $\phi : U \cup V \to \{0, 1\}$ by

$$\theta(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases} .$$

Let $\theta \in A(n, r)$ and assume that it preserves ϕ. Notice that u_0 is fixed because it is the unique vertex whose open neighborhood is contained in Y. Since $n \geq 7$, we have $\phi(u_{n-3}) = 0$. Now u_{n-2}, u_m and v_0 are the only members of Y that have exactly one neighbor in that set and, since no two of them are the same distance from u_0, they are all fixed. It now follows that each remaining member of Y is fixed. Additionally, if $i = n-1$ or $i < m$, then both neighbors of u_i that reside in U are fixed and therefore its remaining neighbor v_i is fixed. Now lemma 2 is applicable. $\square$

Figure 2 illustrates distinguishing colorings of $P(8, 2)$ and $P(8, 3)$.



$P(8, 2)$ $P(8, 3)$

Figure 2

It is of interest that Bogstad and Cowen used a similar approach in their proof that cubes of dimension at least four are 2-distinguishable.

Theorem 5. *If* $(n, r) \neq (4, 1), (5, 2)$, *then* $D(P(n, r)) = 2$.

Proof. At least one of the preceding lemmas is applicable for any $P(n, r)$ that satisfies the hypothesis. $\square$

5. Computation of $\chi_D(P(n, r))$

We now consider proper distinguishing colorings. Collins and Trenk have shown that $\chi_D(P(5, 2)) = 4$, and it is easily verified that $\chi_D(P(4, 1)) = 4$. We next prove that all other generalized Petersen graphs are properly 3-distinguishable.

Theorem 6. *Assume that* $(n, r) \neq (4, 1), (5, 2)$*. Then* $\chi_D(P(n, r)) = 3$.

Proof. First notice that there is no proper 2-coloring of $P(n, r)$ if n is odd or r is even and, when a proper 2-coloring does exist, it is unique and possesses a great deal of symmetry. Therefore we need only exhibit proper distinguishing 3-colorings.

Define $m(i) = \lfloor \frac{i}{r} \rfloor$, $0 \leq i < n$. We define a coloring of the inner vertices as follows:

$$\phi(v_i) = \begin{cases} m(i) \bmod 2 & \text{if } i < n - r \\ 2 & \text{if } i \geq n - r \end{cases} .$$

Notice that color two is assigned to the last r inner vertices even when n is not divisible by r. By doing this we preclude the possibility of assigning color zero to an inner vertex of large index and obtaining an improper coloring. The restriction of ϕ to V is clearly proper.

We extend ϕ to the outer vertices as follows:

$$\phi(u_i) = \begin{cases} 2 - (i \bmod 2) & \text{if } 0 \leq i < r \\ \left\{ \begin{array}{ll} 1 - \phi(v_{m(i)}) & \text{if } 2 \,|\, (i - (m(i)) \\ 2 & \text{otherwise} \end{array} \right\} & \text{if } r \leq i < n - r \\ \left\{ \begin{array}{ll} \phi(v_{n-r-1}) & \text{if } 2 \,|\, (i - n + r)) \\ 1 - \phi(v_{n-r-1}) & \text{otherwise} \end{array} \right\} & \text{if } n - r \leq i < n \end{cases} .$$

Notice that whenever we have a consecutively indexed set of v_i, such that each is assigned the same color, we alternate between the remaining two colors for the corresponding u_i. It is clear that u_i and v_i are never assigned the same color and similarly for u_i and u_{i+1} whenever $\phi(v_i) = \phi(v_{i+1})$. In the case that $\phi(v_i) \neq \phi(v_{i+1})$ we have $\phi(u_{i+1}) = \phi(v_i)$. This completes the proof that ϕ is proper. Figure 3 illustrates the coloring of $P(11, 3)$.

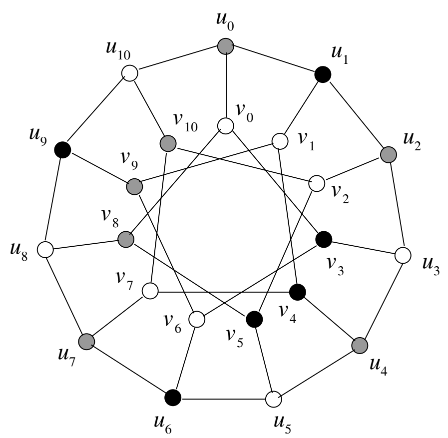


Figure 3. A Proper Distinguishing Coloring of $P(11, 3)$

We first assume that $P(n,r)$ is of Type I or II. Let $\theta \in A(n,r)$ and assume that it preserves the coloring ϕ. Define $W = \{v_i\}_{i=n-r}^{n-1}$ and $Z = \{u_i\}_{i=n-r}^{n-1}$. Assume that $\theta(W) \subset V$. We now have $\theta \in D_n$ and it follows that U, V and Z are also invariant. In the case that $r > 1$, Lemma 1 allows us to assume that $Z \not\subseteq F_\theta(n,r)$. Since Z induces a path, it must be inverted, and this implies $\phi(u_{n-r}) = \phi(u_{n-1})$. This is possible only if r is odd. If we let $s = n - \frac{r+1}{2}$, then θ consists of the transpositions $(u_{s-j}\, u_{s+j})$ and $(v_{s-j}\, v_{s+j})$, $1 \leq j < \frac{n}{2}$. This implies $\phi(v_i) = 0$, $n - 2r \leq i < n - r$, and therefore n is an even multiple of r. We conclude that $\phi(u_{n-r-1}) = 1$ and obtain a contradiction because θ contains the transposition $(u_0\, u_{n-r-1})$ and $\phi(u_0) = 2$. Let $r = 1$. Now u_0 and v_{n-1} are both fixed because they are the only vertices of color two and their open neighborhoods have different color multi-sets. Now u_{n-1} must be fixed because it is the unique neighbor of these two vertices that belongs to U. Apply Lemma 1.

Next assume that $\theta(W) \subseteq U$. In this case each member of U that is assigned color two resides in $\theta(W)$. The distance between any two members of W is at least three, while $\phi(u_0) = \phi(u_2) = 2$ whenever $r \geq 3$. We may therefore assume that $r \leq 2$. Let $r = 2$. If $n \geq 8$, then $\phi(u_3) = \phi(u_5) = 2$ and we again obtain a contradiction. Now two sub cases remain for the case $r = 2$: $(n,r) = (6,2)$, $(7,2)$. Since the proofs are similar, we consider only $P(6,2)$. Now u_0 and v_5 must be transposed, because they are the only vertices of color two whose open neighborhoods have color multi-set $\{0,0,1\}$. This forces u_1 and v_3 to be transposed, but this is impossible because $\widehat{\phi}(\mathrm{N}\,(u_1)) = \{0,0,2\}$ while $\widehat{\phi}(\mathrm{N}\,(u_1)) = \{0,2,2\}$. We may now apply Lemma 1. This coloring is illustrated in Figure 4.

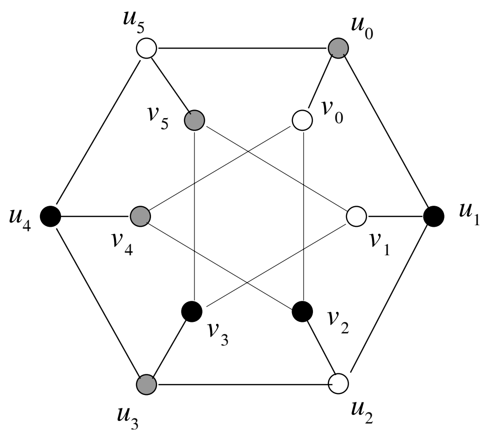


Figure 4. $P(6,2)$ is properly distinguished by ϕ

Finally assume that $r = 1$. Now u_0 and v_{n-1} are the only vertices assigned color two. If n is odd, then these two vertices must be fixed because $\widehat{\phi}(u_0) = \{0,0,1\}$ while $\widehat{\phi}(v_{n-1}) = \{0,1,1\}$. We also have u_{n-1} fixed because it is the unique neighbor of both u_0 and v_{n-1} that resides in U, and U is invariant. This case now follows by Lemma 1. Figure 5 depicts ϕ for $P(5,1)$.

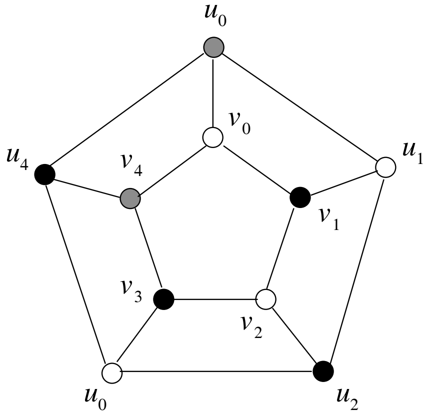


Figure 5. ϕ yields a proper distinguishing coloring for $P(5,1)$

If n is even, then $\widehat{\phi}(u_0) = \widehat{\phi}(v_{n-1}) = \{0,0,0\}$ and it is possible to extend the transposition $(u_0\, v_{n-1})$ to a color preserving automorphism. We introduce a new coloring as follows: Define $\psi : U \cup V \to \{0,1,2\}$ by

$$\psi(u_i) = \begin{cases} 2 & \text{if } i = 0 \\ i \bmod 2 & \text{otherwise} \end{cases} .$$

$$\psi(v_i) = \begin{cases} 2 & \text{if } i = 2 \\ 1 - (i \bmod 2) & \text{otherwise} \end{cases} .$$

This coloring is clearly proper. To see that it is also distinguishing, first notice that u_0 and v_2 are fixed because $\widehat{\psi}(\mathrm{N}(u_0)) = \{0,0,0\}$ while $\widehat{\psi}(\mathrm{N}(v_2)) = \{1,1,1\}$. Now U is invariant and u_1 must be fixed because it is the unique neighbor of u_0 that resides in U and is distance two from v_2. Figure 6 illustrates this coloring for $P(6,1)$.

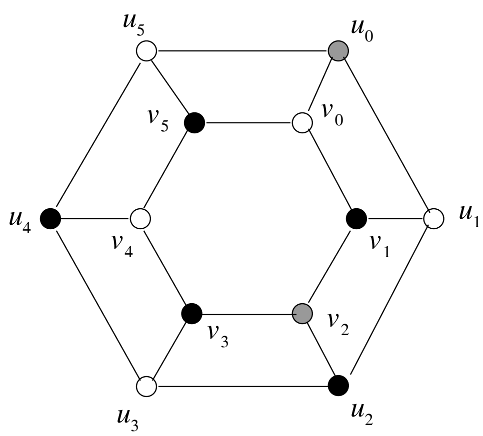


Figure 6. A Proper Distinguishing Coloring for $P(6,1)$

It remains to consider Type III graphs. The coloring ϕ, as defined above, is distinguishing for $(n,r)=(10,2),(12,5),(24,5)$, but there exist nontrivial color preserving automorphisms when $(n,r)=(8,3),(10,3)$. Each of the five graphs under consideration constitutes a separate case. In each case we initialize $F_\theta(n,r)$ to the empty set and add vertices until Lemma 2 is applicable. We present the detailed proof only for $P(8,3)$. For the remaining graphs, we exhibit colorings and a sufficiently large list of fixed vertices to provide for the application of that lemma. The completion of the proofs for these four cases is left to the reader. The coloring ϕ is illustrated for the cases $(n,r)=(10,2),(12,5),(24,5)$, while the the colorings for $P(8,3)$ and $P(10,3)$ are defined by their associated figures.

Let $(n,r)=(8,3)$, and consider Figure 7. We let ζ denote the specified coloring. Assume that θ is a color preserving automorphism of $P(8,3)$, and let $F_\theta(8,3)=\emptyset$.

First u_0 must be fixed because it is uniquely determined by its color and the color multi-set of its open neighborhood. Now the open neighborhood of u_0 is invariant. Each neighbor of u_0 is assigned color one and the color multi-set of each open neighborhood is $\{0,0,2\}$, but we can still distinguish between neighbors of u_0 by considering the color multi-sets of their two neighbors that are distinct from u_0. We have $v_3,v_5\in \mathrm{N}(v_0)$ with $\widehat{\zeta}(\mathrm{N}(v_3))=\{1,1,1\}$, $\widehat{\zeta}(\mathrm{N}(v_5))=\{1,1,2\}$, $v_1,u_2\in \mathrm{N}(u_1)$ with $\widehat{\zeta}(\mathrm{N}(v_1))=\{1,1,2\}$, $\widehat{\zeta}(\mathrm{N}(u_2))=\{1,1,2\}$ and $u_6,v_7\in \mathrm{N}(u_7)$ with $\widehat{\zeta}(\mathrm{N}(u_6))=\{1,1,1\}$, $\widehat{\zeta}(\mathrm{N}(v_7))=\{1,2,2\}$. Therefore all neighbors of u_0 are fixed. Since u_7 and u_0 are fixed, we see that $\{u_6,v_7\}$ is invariant. Now these two vertices must be fixed because the color multi-sets of their open neighborhoods are different. Finally v_6 is fixed because $\{u_5,v_6\}$ is invariant and only v_6 is distance two from u_1. Now apply Lemma 2. This establishes the theorem for $P(8.3)$.

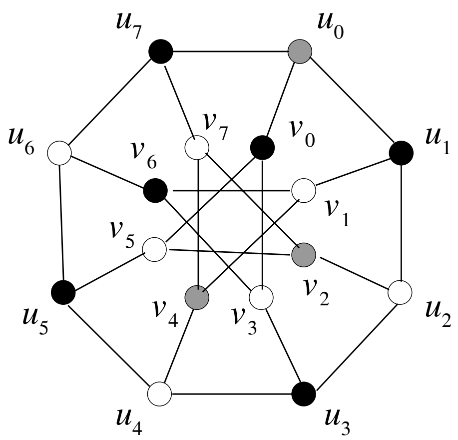


Figure 7. Add vertices to $F_\theta(8,3)$ as follows: u_0, v_0, u_1, u_6, u_7, v_6.

For each of the remaining four cases, vertices are added to $F_\theta(n,r)$ in the orders indicated in the corresponding figures, then Lemma 2 is applied. The theorem follows easily for $P(10,2)$ as is illustrated in Figure 8.

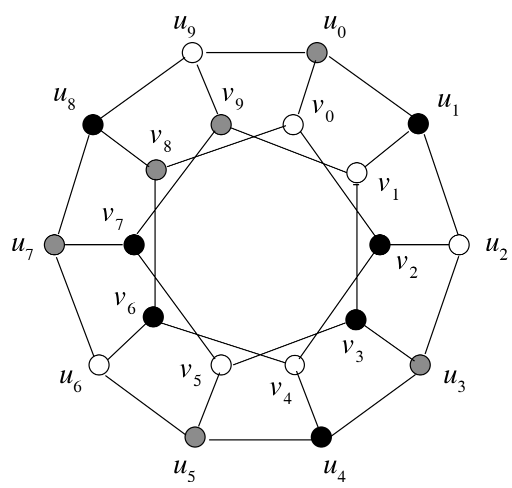


Figure 8. Add vertices to $F_\theta(10,2)$ as follows: v_4, u_4, u_5, u_6, v_5.

In the proof for $P(10,3)$, u_0 and its neighbors are shown to be fixed as in the proof for $P(8,3)$.

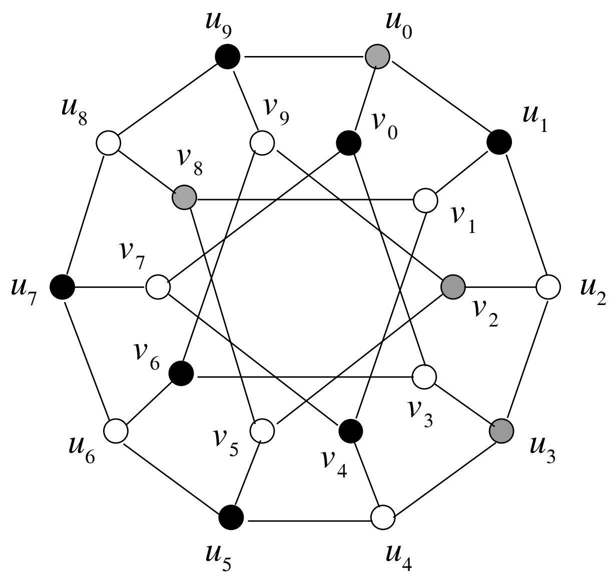


Figure 9. Add vertices to $F_\theta(10,3)$ as follows: u_0, v_0, u_1, u_9, v_1, u_2, v_9.

To begin the proof for $P(12,5)$, notice that v_2 and v_4 must be either fixed or transposed and u_3 is the unique vertex that is distance two from both of them.

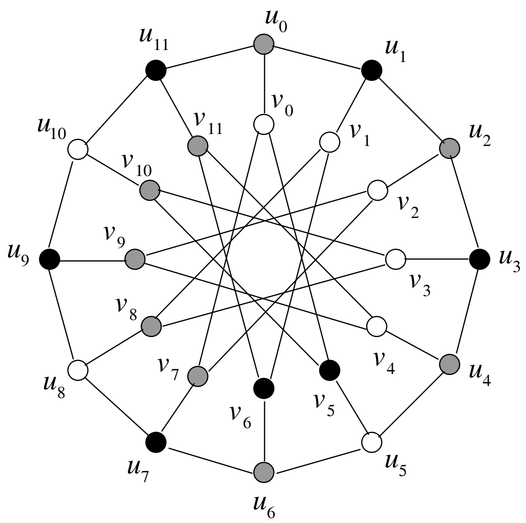


Figure 10. Add vertices to $F_\theta(12,5)$ as follows: u_3, v_3, u_2, v_2, u_1, v_1, u_4, v_4, u_5, v_5, u_6.

We establish that u_{11} is a fixed vertex of $P(24,5)$ by an argument similar to the one used in the preceding case.

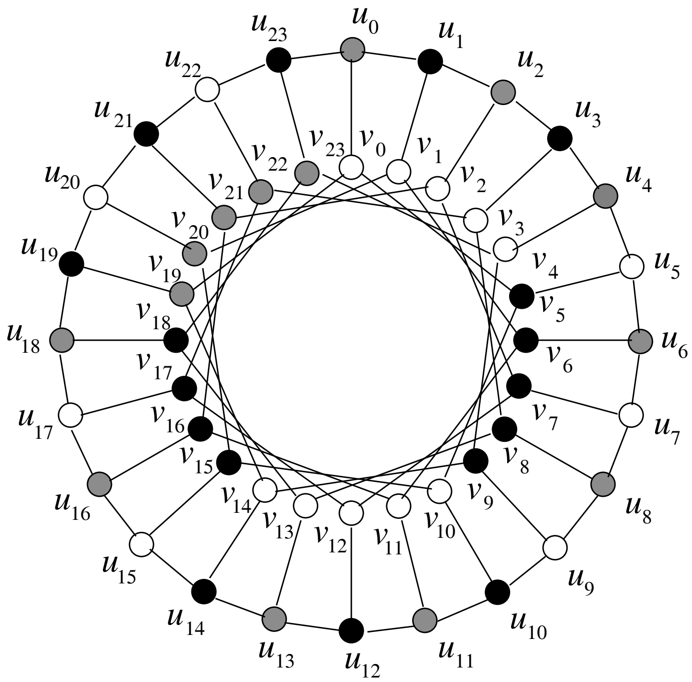


Figure 11. Add vertices to $F_\theta(12,5)$ as follows: $u_{11}, u_{10}, v_{10}, u_9, v_9, u_8, v_8, u_7, v_7, u_6, v_6$.

$\square$

6. Suggestions for Further Research

There has been a substantial amount of work done on the problem of distinguishing Cartesian powers of graphs, but the problem of distinguishing arbitrary Cartesian products remains relatively unexplored, and there has been no published work pertaining to other types of products, such as joins and tensor products.

Proper distinguishing numbers have received comparatively little attention. For example, these numbers have not yet been computed for hypercubes and Kneser graphs.

Most of the work pertaining to symmetry breaking has been done within the context of vertex colorings, but there are clearly other ways to address this issue. The edge orientation approach, as introduced by Harary and Jacobson, provides an alternative means for breaking symmetry. They also considered the problem of minimizing the amount of change needed to break symmetry. Such optimizations yield more information pertaining to the structure of an automorphism group as it relates to the structure of the underlying graph than does the specification of a property that simply suffices to break symmetry. For example, given a 2-distinguishable graph, it is of interest to minimize the size of one of the color classes and characterize all such colorings. Edge colorings and total colorings (colorings of both vertices and edges) provide different perspectives, and comparisons of various results should yield additional insight.

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