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A note on Duchet's conjecture

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Abstract

A kernel in a digraph is a set of vertices that is both an absorbing and independent set. In [4] P. Duchet conjectured that a digraph such that every odd circuit has two short chords is kernel perfect digraph (i.e. every induced subdigraph has a kernel) and he proved the conjecture for digraphs where every 3-circuit is reversal. We give another simple proof of this result by using reorientation and constructive methods. We also show that the conjecture is valid for digraphs D where the underlying undirected spanning subgraph deduced from the spanning subdigraph of D with only pairs of symmetric arcs is a comparability graph.

Keywords: digraph, kernel, kernel-perfect digraph.

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1. Introduction

We consider finite directed graphs with no multiple arcs or loops. Note that any directed graph can be viewed as an orientation of its underlying undirected graph. All terms not defined here may be found in the book of C. Berge [1].

Let $D = (V, U)$ be a directed graph which we call digraph, where V is the vertex-set and U is the arc-set. An arc $u = (x, y)$ in U is called a symmetric arc or a reversible arc if the arc $u^{-1} = (y, x)$ exists in U, otherwise u is said an asymmetric arc. In digraphs considered here, pairs of symmetric arcs (= pairs of opposite arcs) are permitted. $ASY(U)$ and $SY(U)$ denote the set of asymmetric arcs of U and the set of symmetric arcs of U, respectively, so we have $U = ASY(U) \cup SY(U)$. Also we denote by $SY(D)$ and $ASY(D)$ the spanning subdigraph of D whose arcs are symmetric and the spanning subdigraph of D whose arcs are asymmetric, respectively. So, we have $SY(D) = (V, SY(U))$ and $ASY(D) = (V, ASY(U))$.

For a digraph $D = (V, U)$ and $S \subseteq V$ we denote by $D[S]$ the subdigraph induced by S. We set:

$\Gamma^+(x) = \{y \in V : (x, y) \in U\}$

$\Gamma^-(x) = \{y \in V : (y, x) \in U\}$

$\Gamma^-(A) = \underset{a\in A}{\cup} \Gamma^-(a)$

$\Gamma^+(A) = \underset{a\in A}{\cup} \Gamma^+(a).$

A subset A of V is an *absorbing* set if $\Gamma^-(A) \cup A = V$ and is an *independent set* if $(\Gamma^-(A) \cup \Gamma^+(A)) \cap A = \varnothing$. A *kernel* of a digraph D is a subset of V that is both independent and absorbing set. D is said to be a *kernel perfect* if every induced subdigraph has a kernel. A vertex x is a *successor* (resp. *predecessor*) of a vertex y if $x \in \Gamma^+(y)$ (resp. $x \in \Gamma^-(y)$). By a *path* of a digraph $D = (V, U)$, we mean an elementary path that is a sequence of distinct vertices $(x_0, x_1, \ldots, x_p)$ such that $(x_i, x_{i+1}) \in U$ for $0 \le i \le p-1$. A *circuit* is a path $(x_0, x_1, \ldots, x_p, x_{p+1})$ such that $x_0 = x_{p+1}$. The *parity* of path (or circuit) is the parity of the number of its arcs. A 3-circuit is reversal if it is a circuit of length 3 and all its arcs are symmetric. A *chord* of a circuit $C = (x_0, x_1, \ldots, x_p, x_0)$ is an arc (x_i, x_j) with $j \ne i+1$ (modulo p). A *short chord* is a chord of the form (x_i, x_{i+2}) (modulo p). If $C(a, b)$ and $C(b, c)$ represent respectively a path joining a to b and a path joining b to c, then we denote by $C(a, b)$ and $C(b, c)$ the path joining a to c obtained by prolongation of the path $C(a, b)$ by the path $C(b, c)$ and with $C(a, b) \cap C(b, c) = \{b\}$. If c and d are vertices of the path $C(a, b)$, then $C(c, d)$ is the subpath joining c to d deduced from the path $C(a, b)$. A path $C(x_0, x_p)$ of a digraph D is *minimal* if (x_i, x_j) is not an arc of D for every i, j such that $1 \le i+1 < j \le p$. If a path $C(x_0, x_p)$ joining x_0 to x_p is not minimal, then there exists a minimal path $C'(x_0, x_p)$ joining x_0 to x_p (not necessarily unique), using some vertices of $C(x_0, x_p)$, we say in this case that $C'(x_0, x_p)$ is a minimal path induced by the vertices of $C(x_0, x_p)$.

The *underlying undirected graph* $G = (V, E)$ of a digraph $D = (V, U)$ is the simple graph such that the vertex-set is V and two vertices are adjacent in G if and only if there exists at least one arc between them in D (notice that an edge in G may be corresponded to a pair of opposite arcs in D). A simple graph is a *comparability graph* if it admits a transitive asymmetric orientation.

In [4] P. Duchet suggested the following conjecture and proved it for digraphs where every 3-circuit is reversal.

Conjecture 1.1. [4] *If every odd circuit of a digraph D possesses two short chords, then D is kernel-perfect*

Theorem 1.2. [4] *If every 3-circuit of a digraph D is reversal, and every odd circuit possesses two short chords, then D is kernel-perfect.*

In [8] Galeana-Sánchez showed a weaker version of Theorem 1.2 which is consequence of Theorem 1.2.

Theorem 1.3. [8] *If a digraph is without 3-circuit and every odd circuit possesses two short chords, then D is kernel- perfect.*

In this note, we use a reorientation method introduced by P. Duchet [5, 6] and a constructive method to give another simple proof of Theorem 1.2 (Section 4).

First, we prove in Section 2 that a digraph such that every 3-circuit is reversal and every circuit possesses two short chords can be reoriented (i.e. by deleting one sense of each pair of symmetric arcs) to obtain a digraph with no symmetric arcs such that every odd circuit has two short chords and we give a counterexample when we delete the fact that every 3-circuit is reversal. In Section 3 we use a constructive method to prove that every digraph with no symmetric arcs and such that every odd circuit has two short chords is kernel-perfect. Finally in Section 4 we show that if a digraph D is such that every odd circuit possesses two short chords and the underlying undirected spanning subgraph of the spanning subdigraph $SY(D)$ is a comparability graph, then D is kernel-perfect.

2. A Reorientation Method

Definition 2.1. *A reorientation D' of a digraph D is the spanning digraph obtained from D by deleting exactly one arc in each pair of symmetric arcs of D.*

Thus, D' is a spanning digraph of D with no symmetric arcs. So, by reorientation, we mean that we replace the set of symmetric arcs by a set of asymmetric arcs.

Lemma 2.2. *If D is a digraph such that every 3-circuit is reversal, then there exists a reorientation D' of the digraph D without 3-circuit.*

Proof. To obtain the digraph D' from D, we choose any total order of the vertices of $D=(V,U)$. Let $x_1, x_2, \ldots, x_n$ be this order. We delete all the arcs which have the form (x_i, x_j) where $j > i$ and (x_i, x_j) is a symmetric arc. Now, it is clear that the obtained spanning subdigraph is an asymmetric digraph and without 3-circuit. □

Theorem 2.3. *Let D be a digraph such that every 3-circuit is reversal and every odd circuit of D possesses two short chords. Then there exists a reorientation of D which gives a spanning subdigraph D' such that D' is an asymmetric digraph and every odd circuit possesses two short chords.*

Proof. By Lemma 2.2, let D' be an asymmetric digraph without 3-circuit obtained from the digraph D. We prove that D' verifies the condition of this theorem. Assume to the contrary that D' contains an odd circuit C with no two short chords. Since this circuit is also a circuit of D, then it possesses two short chords in D. Let (x_i, x_{i+2}) be any short chord of C in D such that (x_i, x_{i+2}) is not in D'. Then (x_i, x_{i+2}) is a symmetric arc in D and so (x_{i+2}, x_i) must be an arc of D'. Now (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) with (x_{i+2}, x_i) form a 3-circuit in D', a contradiction with D' is without 3-circuit. □

Remark 2.4. *If D is a digraph such that every odd circuit possesses two short chords, then D doesn't necessarily possess a reorientation without 3-circuit. (See Figure 1)*

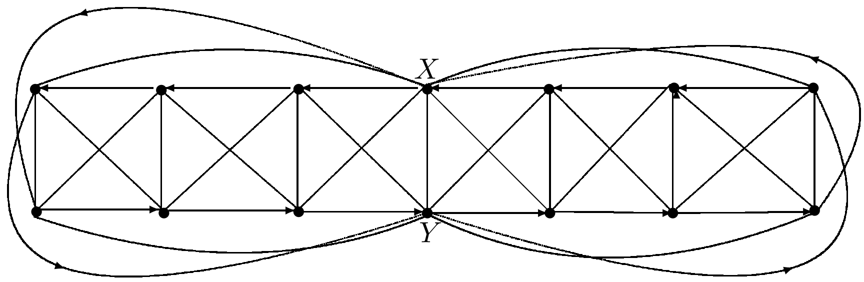

Figure 1

When we orient in one direction the middle symmetric arc (X,Y), we force all other symmetric arcs to be in one direction. Finally, we will create a 3-circuit in every reorientation of this digraph.

3. A Constructive Method

In this section we use a constructive method to prove the existence of a kernel in the case of an asymmetric digraph with the propriety that every odd circuit possesses two short chords. Different constructive methods are used in [7, 2, 3] to determine a kernel in a digraph.

Observation 3.1. *An asymmetric digraph such that every odd circuit possesses two short chords is necessarily without 3-circuit.*

Theorem 3.2. *Let D be an asymmetric digraph such that every odd circuit possesses two short chords, then D is kernel perfect.*

Proof. It is clear that the conditions given in the hypothesis of the theorem are hereditary for the subdigraphs, so it is sufficient to show that D admits a kernel. The proof is by induction on the number of vertices of $D=(V,U)$. The theorem is true for digraphs having less than four vertices. Let D be a digraph of order $n \geq 4$ in which the hypothesis of the theorem are satisfied and suppose that the result is true for all digraphs of order n' smaller than n. Let D be an asymmetric digraph such that every odd circuit possesses two short chords, V the set of the vertices of D and $x \in V$. Since $D-x$ (i.e. $D[V-\{x\}]$) is the subdigraph generated by $V-\{x\}$ with order $n'<n$, it possesses a kernel N. If x has no neighbor in N or x has a successor vertex in N then $N \cup \{x\}$ or N is a kernel of D respectively and we are done. From now on, we may thus assume that x has a least one predecessor, and no successor, in N.

We will define a sequence of sets as follows:

$B_0 = \Gamma^+(x)$

$N_0 = \Gamma^+(B_0) \cap N$

$B_1 = \Gamma^+(N_0) \cap (V - (\{x\} \cup \Gamma^-(x)))$

$N_1 = \Gamma^+(B_1) \cap (N - N_0)$

$B_2 = \Gamma^+(N_1) \cap (V - (\{x\} \cup \Gamma^-(x) \cup B_0 \cup \Gamma^-(N_0))$

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$B_i = \Gamma^+(N_{i-1}) \cap (V - (\{x\} \cup \Gamma^-(x) \cup (\overset{i-2}{\underset{j=0}{\cup}} B_j \cup \Gamma^-(N_j)))$

$N_i = \Gamma^+(B_i) \cap (N - \overset{i-1}{\underset{j=0}{\cup}} N_j)$

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and let k be the smallest integer such that $B_{k+1} = \varnothing$.

We note that $N_0, N_1, ..., N_k$ are subsets of N and $B_0, B_1, ..., B_k$ are subsets of $V - N$. We are going to show that the set $N'' = \{x\} \cup (\overset{k}{\underset{i=0}{\cup}} N_i) \cup N'$ is a kernel of D where N' is a kernel of the subdigraph $D[V - (\{x\} \cup \Gamma^-(x) \cup (\overset{k}{\underset{i=0}{\cup}} (N_i \cup \Gamma^-(N_i))))]$ which exists because $D[V - (\{x\} \cup \Gamma^-(x) \cup (\overset{k}{\underset{i=0}{\cup}} (N_i \cup \Gamma^-(N_i))))]$ is strictly smaller than D and thus possesses a kernel by induction hypothesis.

We break the proof into several steps.

(1) B_0 is not empty.

Because otherwise, we consider a kernel N' in the subdigraph $D[V - (\{x\} \cup \Gamma^-(x))]$ which exists because $D[V - (\{x\} \cup \Gamma^-(x))]$ is strictly smaller than D and thus possesses a kernel by induction hypothesis, and $N'' = \{x\} \cup N'$ would be a kernel of D.

(2) N'' is an absorbing set.

Since by construction, a vertex of V either is in $\{x\} \cup (\overset{k}{\underset{i=0}{\cup}} N_i) \cup N'$ or, it is absorbed by x if it is in $\Gamma^-(x)$ or, by $\overset{k}{\underset{i=0}{\cup}} N_i$ if it is in $\Gamma^-(N_i)$; $i = 0, 1, \ldots, k$. or, by N' if it is in $V - (\{x\} \cup \Gamma^-(x) \cup (\overset{k}{\underset{i=0}{\cup}} (N_i \cup \Gamma^-(N_i))))$.

(3) N' and $\overset{k}{\underset{i=0}{\cup}} N_i$ are independent sets.

By definition of N' and N.

(4) x is not adjacent to vertices of N'.

Since the neighborhood of x are $\Gamma^-(x)$ and B_0 and it is not contained in $D[V-(\{x\} \cup\Gamma^-(x)\cup(\overset{k}{\underset{i=0}{\cup}}(N_i\cup\Gamma^-(N_i))))]$ by construction.

(5) No vertex of N' is adjacent to vertices of $N_i \quad i=0,1,\ldots,k$.

Since by the construction, a vertex adjacent to vertices of N_i belongs necessarily to $\overset{k}{\underset{i=0}{\cup}}\Gamma^-(N_i)$.

It remains to prove the following step.

Claim. x is not adjacent to vertices of $N_i \;; i=0,1,\ldots,k$.

So, with $(1),(2),(3),(4),(5)$ and Claim, N'' will be a kernel of D, which complete the proof of the theorem.

Proof of Claim. We will use induction on i to show that x cannot be adjacent to vertices of $N_i; i=0,1,\ldots,k$.

For $i=0$, x cannot be adjacent to $x'_0\in N_0$, otherwise we obtain 3-circuit, a contradiction to Observation 3.1. Then the statement is true for $i=0$.

Assume that x is not adjacent to vertices of $N_i; 0\le i\le p$ and suppose to the contrary that x is adjacent to $x'_{p+1}\in N_{p+1}$, that is the arc (x'_{p+1},x) exists and by construction there exists a path from x to x'_{p+1} alternately going through vertices of B_i and N_i. Let $C(x,x'_{p+1})=(x,b_0,x'_0,b_1,x'_1,...,b_p,\ \ x'_p,b_{p+1},x'_{p+1})$ be such a path.

By hypothesis of Theorem 3.2, the odd circuit $C(x,x'_{p+1})\&(x'_{p+1},x)$ contains two short chords.

The possible short chords are (x'_{p+1},b_0) or (b_i,b_{i+1}) where $i\in\{0,\ldots,p\}$.

Also there are no arcs of the following form:

$$(x'_i,b_j) \qquad j>i+1 \text{ (also for } x'_i=x\text{) by definition of } B_{i+1} \tag{1}$$

$$(b_j,x'_i) \qquad i>j \quad \text{by definition of } N_i \tag{2}$$

We show that the arc (x'_{p+1},b_0) cannot exist. Suppose to the contrary that (x'_{p+1},b_0) exists, we consider a minimal path $C'(b_0,x'_{p+1})$ from b_0 to x'_{p+1} induced by the vertices of $C(x,x'_{p+1})$. Then $C_1=(x,b_0)\&C'(b_0,x'_{p+1})$ - $\&(x'_{p+1},x)$ or $C_2=C'(b_0,x'_{p+1})\&(x'_{p+1},b_0)$ is an odd circuit having at most one short chord, (x'_{p+1},b_0) in the case of C_1 and (x'_{p+1},b'_1) in the case of C_2 with (b_0,b'_1) is an arc of $C'(b_0,x'_{p+1})$. In any case we obtain a contradiction, so (x'_{p+1},b_0) cannot exist.

By (1), (2) and the minimality of $C'(b_0, x'_{p+1})$, the only possible short chords have the form (b_i, b_{i+1}) where $i \in \{0, ...p\}$.

Now, we show that no chord (b_i, b_{i+1}) with $i \in \{0, \ldots, p\}$ exists.

Suppose that such chord exists. Among all chords (b_i, b_{i+1}) with $i \in \{0, \ldots, p\}$ take one with the smallest index i and consider a minimal path $C'(x, b_i)$ induced by the vertices of $C(x, b_i)$ and a minimal path $C'(b_{i+1}, x'_{p+1})$ induced by the vertices of $C(b_{i+1}, x'_{p+1})$. It is clear that either

$C_1 = C'(x, b_i)\&(b_i, b_{i+1})\&C'(b_{i+1}, x'_{p+1})\&(x'_{p+1}, x)$ or

$C_2 = C'(x, b_i)\&C(b_i, b_{i+1})\&C'(b_{i+1}, x'_{p+1})\&(x'_{p+1}, x)$ (the path $C(b_i, b_{i+1})$ is of the form (b_i, x'_i, b_{i+1})), is an odd circuit. In the first case, if C_1 is an odd circuit, then the only possible short chords are of the form (b'_{i-1}, b_{i+1}) and (b_i, b'_{i+2}), with (b'_{i-1}, b_i) and (b_{i+1}, b'_{i+2}) are the arcs of C_1. Consider the circuit $C'_1 = C'(x, b'_{i-1})$ and $C(b'_{i-1}, b_i)$ and (b_i, b'_{i+2}) and $C'(b'_{i+2}, x'_{p+1})$ and (x'_{p+1}, x). It is easy to see that it has an odd parity and must contain two short chords, by (1), (2) and the minimality of paths, one of the two short chords must be of the form (b_j, b_{j+1}) with b_j and b_{j+1} the vertices of the path $C(b'_{i-1}, b_i)$, a contradiction with the choice of the index i. In the second case, if C_2 is an odd circuit, then by (1), (2) and the minimality of paths, the only possible short chord is (b_i, b_{i+1}). Then the odd circuit $C(x, x'_{p+1})\&(x'_{p+1}, x)$ has no short chords, a contradiction. Therefore the arc (x'_{p+1}, x) cannot exist. □

4. Main Results

By using a constructive method (see Theorem 3.2) applied to the asymmetric digraph D' deduced from Theorem 2.3, we obtain a second proof to Theorem 1.2 [4].

Theorem 4.1. *If every 3-circuit of a digraph D is reversal, and every odd circuit possesses two short chords, then D is kernel-perfect.*

Theorem 4.2. *If D is a digraph such that every odd circuit possesses two short chords and the underlying undirected graph obtained from the spanning subdigraph $SY(D)$ is a comparability graph, then D is kernel-perfect.*

Proof. Consider the subdigraph D' obtained from D as follow: take the arcs of $ASY(U)$ and any transitive orientation of the underlying undirected graph obtained from the spanning subdigraph $SY(D)$, which is a comparability graph. Then D' is an asymmetric digraph D' without 3-circuit and such that every odd circuit possesses two short chords. D' is without 3-circuit, for otherwise we obtain a contradiction with the transitive orientation of the underlying undirected graph deduced from the spanning subdigraph $SY(D)$ or with the fact that every odd circuit in D possesses two short chords. Also D' is such that every odd circuit possesses two short chords, for otherwise by a similar argument to that used in the proof of Lemma 2.2, we have a 3-circuit in D', a contradiction. Now we apply Theorem 3.2 of Section 3 to D' and the result follows. □

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