AKCE J. Graphs. Combin., 5, No. 2 (2008), pp. 135-145

The k-fold list coloring of cycles with Hall's condition

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Communicated by: S. Arumugam
Received 30 November 2007; accepted 09 June 2008

Abstract

We prove that any cycle C_n, $n \geq 4$, with list assignment L, has a k-fold list coloring from the given lists if (i) each list contains at least $2k$ colors and (ii) C_n and L satisfy Hall's condition for k-fold list colorings. Further, $2k$ in (i) cannot be replaced by $2k-1$ if either n is odd, or n is even and $n \geq k+2$. In other words, if $n \geq 4$, the k-fold Hall number of a cycle C_n satifies $h^{(k)}(C_n) \leq 2k$, with equality if n is odd, or n is even and $n \geq k+2$.

Keywords: Hall's condition, Hall number, list coloring, list assignment, k-fold choice number, k-fold chromatic number.

2000 Mathematics Subject Classification: 05C69

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1. Introduction

A *list assignment,* or a *color supply,* for a graph $G = (V, E)$ is an assignment to the vertices of G of finite subsets ("lists") of a set C of colors. A *color demand* for G is an assignment of positive integers to the vertices of G. If L is a color supply and w is a color demand for G, an (L, w)-coloring of G is a function φ which assigns to each $v \in V$ a subset $\varphi(v) \subseteq L(v)$, with $|\varphi(v)| = w(v)$, such that $\varphi(u) \cap \varphi(v) = \emptyset$ whenever uv is an edge in G. When w is a constant function, say $w = k$, we let the value k stand for the function w and we may refer to an (L, k)-coloring of G as a k-fold list coloring from L.

For any positive integer k the *k-fold chromatic number* of G, denoted $\chi^{(k)}(G)$, is the smallest integer m such that there is a k-fold list coloring from the constant color supply $L \equiv \{1, \dots, m\}$. The *k-fold choice number* of G, denoted $ch^{(k)}(G)$, is the smallest integer m such that there is a k-fold list coloring from any color supply L satisfying $|L(v)| \geq m$ for all $v \in V$.

The idea of coloring the vertices of a graph with subsets of a fixed set arguably originated with Hilton, Rado, Scott, or Stahl ([14],[17],[19]). The k-fold coloring of cycles is a particular case of edge coloring of multicycles which was studied by Kostochka and Woodall in [15]. It turns out that for cycles the k-fold chromatic and choice numbers are the same: Tuza and Voigt in [20] and Gutner and Tarsi in [6] proved that $ch^{(k)}(C_{2m}) = 2k$, $m = 1, 2, \dots,$ $k = 1, 2, \dots,$ and Slivnik [18] used measure theoretic methods to show that $ch^{(k)}(C_{2m+1}) = 2k + \lceil \frac{k}{m} \rceil$; these values had long been known for $\chi^{(k)}(G)$. Our aim here is to give an upper bound for cycles, with equality in many cases, on another k-fold list coloring parameter whose definition involves a fairly well-known necessary condition for the existence of an (L, w)-coloring of G called Hall's condition. Our main result will imply the results above about $ch^{(k)}$ as a corollary.

Given G, L, w, an induced subgraph H of G, and a color $x \in C$, let $H(x, L)$ denote the subgraph of H induced by $\{v \in V(H) | x \in L(v)\}$ and let $\alpha(H(x, L))$ denote the vertex independence number of this subgraph. [If x does not appear on any lists $L(v), v \in V(H)$, set $\alpha(H(x, L)) = 0$.] If there is an (L, w)-coloring φ of G then the set of vertices $v \in V(H)$ such that $x \in \varphi(v)$ is an independent set of vertices in $H(x, L)$. Therefore, there can be no more than $\alpha(H(x, L))$ appearances of x in the color sets for vertices in H. Since the total number of appearances of all symbols in those color sets is $\sum_{x \in V(H)} |\varphi(v)| = \sum_{v \in V(H)} w(v)$, we have

$$(*) \qquad \sum_{x \in C} \alpha(H(x, L)) \geq \sum_{v \in V(H)} w(v).$$

If (*) holds for every induced subgraph H of G, then G, L and w satisfy *Hall's condition.* (NOTE: If G, L and w satisfy Hall's condition, then (*) holds for every subgraph H of G, induced or not.) Observe that when $w = k$, a constant, the right hand side of (*) is $k|V(H)|$.

Hall's condition is so named because it was inspired by Hall's theorem on systems of distinct representatives [7], which can be viewed as a list coloring theorem about the special case when G is a clique and $w = 1$ [10]. Further, the improvement of Hall's theorem, in which the requirement that $w = 1$ is removed, due to Rado [16] and, independently, to Halmos and Vaughan [8], can be stated thus: when G is a clique, Hall's condition suffices for the existence of an (L, w)-coloring of G.

For each positive integer k, the k-fold Hall number of G, denoted $h^{(k)}(G)$, is the smallest integer $m \geq k$ such that there is an (L, w)-coloring of G whenever G, L and k satisfy Hall's condition and $|L(v)| \geq m$ for all $v \in V$. Note that $h^{(k)}(G) \geq k$ for every G and k and the Hall-Rado-Halmos-Vaughan theorem refereed to above implies equality, for every k, when G is a clique. It follows from results in [2],[5] and [3] that $h^{(k)}(G) = k$ for $k = 1, 2, \ldots$ if and only if G is the line graph of a forest.

Clearly $h^{(k)}(G) \leq ch^{(k)}(G)$ for all k and G, but $h^{(k)}(G) < \chi^{(k)}(G)$, $h^{(k)}(G) = \chi^{(k)}(G)$, and $h^{(k)}(G) > \chi^{(k)}(G)$ are all possible [5]. In [5] it is shown that $h^{(k)}(G) = ch^{(k)}(G)$ if either is larger than $ch^{(k)}(G)$. This result may have its uses, since both $ch^{(k)}(G)$ and $h^{(k)}(G)$ are difficult to determine.

For $k \geq 2$, the problem of determining the k-fold Hall numbers of trees is solved in [3]:

If $r \geq 3$, $h^{(k)}(K_{1,r}) = 2k - \left\lfloor \frac{k}{r} \right\rfloor$; if $r < 3$, $h^{(k)}(K_{1,r}) = k$; and if T is a tree which is not a star, $h^{(k)}(T) = 2k$. It follows from the main result in [10] that $h^{(1)}(T) = 1$ for all trees T.

Regarding cycles, since $C_3 = K_3$ we have $h^{(k)}(C_3) = k$ for all $k = 1, 2, \ldots$. In [4], we proved that $h^{(k)}(C_4) = \left\lceil \frac{5k}{3} \right\rceil$ for all $k = 1, 2, \ldots$. From [13] we have $h^{(1)}(C_n) = 2$ for all $n > 3$. Our main result is the following:

Theorem 1.1. *For any integers $n \geq 4$ and $k \geq 1$, $h^{(k)}(C_n) \leq 2k$, with equality if n is odd, or if n is even and $n \geq k + 2$.*

For n even the inequality $h^{(k)}(C_n) \leq 2k$ follows from $h^{(k)} \leq ch^{(k)}$ and the previously cited result from [6] and [20] that $ch^{(k)}(C_n) = 2k$. But our proof will be independent of all direct proofs of this fact, and that allows us to turn the tables and give a new proof of both old results about $ch^{(k)}(C_n)$, n even or odd. The following is a straightforward consequence of the previously mentioned result in [5] that $ch^{(k)} = \max[\chi^{(k)}, h^{(k)}]$, of the well known values of $\chi^{(k)}(C_n), k \geq 1$, $n \geq 3$, and of the theorem.

Corollary 1.2. *For any integers $k \geq 1$, $m \geq 2$, $ch^{(k)}(C_{2m}) = 2k$ and $ch^{(k)}(C_{2m+1}) = 2k + \left\lceil \frac{k}{m} \right\rceil$.*

In the case of C_{2m+1} the proof of the corollary constitutes a purely combinatorial proof of the result proved by Slivnik [18] with the involvement of Lebesgue measure.

There is an open problem in [5] that our results bear on: does $\lim_{k\to\infty} \frac{h^{(k)}(G)}{k}$ exist for every graph G? To the two classes of graphs for which the answer was known to be yes–line graphs of forests and trees–we can now add: odd cycles. It feels strange to wrestle with a list coloring problem that is apparently harder for even cycles than for odd cycles! We throw further light on $h^{(k)}(C_{2m})$ when $2m \leq k+1$ in [4].

2. Proof of the Theorem

The proof that $h^{(k)}(C_n) \leq 2k$ will use the following, a special case of the main result in [2], first proved in [1].

Path Lemma. For any path P with color demand w and color supply L, P is (L, w)-colorable if and only if P, L and w satisfy Hall's condition.

It may be worth mentioning that in [1] there is an efficient algorithm for either coloring P or discovering that Hall's condition is not satisfied.

Our strategy for proving that $h^{(k)}(C_n) \leq 2k$ will use the Path Lemma thus: we will show that when C_n, L and k satisfy Hall's condition and $|L(v)| \geq 2k$ for all $v \in V(C_n)$, then for some $v \in V(C_n)$ there is a k-set $S \subseteq L(v)$ such that if L' is defined on the path $C_n - v$ by removing all of the elements of S from the lists on the neighbors of v, and otherwise putting $L' = L$, then $C_n - v$, L' and k satisfy Hall's condition. Coloring v with S and putting this with an (L', k)-coloring of $C_n - v$ then produces an (L, k)-coloring of C_n.

It will be useful to note as in [1] that for any list assignment L to C_n and any $x \in C$, we may as well suppose that $C_n(x, L)$ is connected. The reason: for any graph G, color supply L, and color demand w, if $G(x, L)$ is disconnected for some x in C, we can make a new lists assignment $\widehat{L}$ by replacing x on the lists on each component of $G(x, L)$ by a new symbol, so that x's replacements on those components are different from each other and from all the other symbols appearing on lists on G. It is straightforward to see that G, L and w satisfy Hall's condition if and only if $G, \widehat{L}$ and w do, and that there is an (L, w)-coloring of G if and only if there is an $(\widehat{L}, w)$-coloring of G.

Given a vertex v of $G = C_n$ and a color $x \in L(v)$, we say that x is *bad at* v if and only if no maximum independent set of vertices of $G(x, L)$ contains v. Under the assumption that $G(x, L)$ is connected, this means that $G(x, L)$ is a path of odd order and v is one of the *even* numbered vertices, if we count along the path with the count starting with the number 1 (since the path has odd order, you can start from either end).

For each vertex $v \in V(G)$ we partition its supply, $L(v)$, as follows:

$B(v) = \{x \in L(v) : x \text{ is bad at } v\}$

$O(v) = \{x \in L(v) : G(x, L) \text{ is a path of odd order }\} \backslash B(v)$

$E(v) = L(v) \backslash (B(v) \cup O(v))$.

If $G(x, L)$ is simply a vertex v then $x \in O(v)$. If $G(x, L) = G$ then $x \in E(v)$. We now assume that G, L and k satisfy Hall's condition, and that $|L(v)| \geq 2k$ for all $v \in V(G)$, and set about showing that there is an (L, k)-coloring, by the strategy described earlier.

If $x \in B(v)$ then $x \in O(u)$ for each neighbor u of v. Therefore, if $|O(v) \cup E(v)| < k$ then $|O(u)| \geq |B(v)| > k$ for each neighbor u of v, since $|L(u)| \geq 2k$ for every vertex. It follows that there is a vertex v_0 such that $|O(v_0) \cup E(v_0)| \geq k$. Let the vertices of G be $v_0, v_1, \ldots, v_{n-1}$, one way or the other around the cycle.

Let $X_0 \subseteq O(v_0) \cup E(v_0)$ be of size k and such that $|X_0 \cap O(v_0)|$ is as large as possible. We intend to color v_0 with X_0. Define L' on $G - v_0$ by

$$L'(v_i) = \begin{cases} L(v_i) \backslash X_0 & \text{if } i = 1, n-1 \\ L(v_i) & \text{otherwise.} \end{cases}$$

We shall finish the proof that $h^{(k)}(C_n) \leq 2k$ by showing that $G - v_0$, L' and k satisfy Hall's condition.

Let H be an induced subgraph of $G - v_0$. To verify (*) for H, L' and $w = k$, it suffices to verify it for every connected component of H, so we may as well consider H to be connected; that is, H is a path. If H is a single vertex v_i, $\sum_{x \in C} \alpha(H(x, L')) = |L'(v_i)|$, which is either at least $2k$, if $1 < i < n-1$, or is $|L(v_i) \backslash X_0| \geq 2k - k = k$, if $i \in \{1, n-1\}$. In any case, $\sum_{x \in C} \alpha(H(x, L')) \geq k = k|V(H)|$, so (*) holds and we assume that $|V(H)| > 1$. If H is a path containing neither v_1 nor v_{n-1} then (*) holds because $L = L'$ on $V(H)$ and G, L and k are assumed to satisfy Hall's condition. Therefore, we need only consider the case that H is a path containing either v_1 or v_{n-1}, or both.

Suppose that H contains v_1 but not v_{n-1}. (Disposing of this case will also take care of the case when H contains v_{n-1} but not v_1.) Then, using self-explanatory notation for paths, $H = (v_1, v_2, \ldots, v_t)$, for some $t, 1 < t < n-1$. Since $v_t, v_{t-2}, \ldots, v_{t-2r}$, where $r = \left\lfloor \frac{t-1}{2} \right\rfloor$, are independent vertices in H, we have that

$$\begin{aligned} \sum_{x \in C} \alpha(H(x, L')) &\geq \sum_{i=0}^{r} |L'(v_{t-2i})| \\ &\geq \begin{cases} 2rk + k & \text{if } t \text{ is odd} \\ 2(r+1)k & \text{if } t \text{ is even} \end{cases} \\ &= tk = k|V(H)|. \end{aligned}$$

Now suppose that $H = (v_1, v_2, \ldots, v_{n-1}) = G - v_0$.

If $x \in C \backslash X_0$ then $\alpha(H(x, L')) = \alpha(H(x, L))$. Also, the definition of $O(v_0)$ implies that, for every $x \in O(v_0)$, $\alpha(H(x, L')) = \alpha(H(x, L)) = \alpha(G(x, L)) - 1$. If $|O(v_0)| \geq k$ then X_0 is a subset of $O(v_0)$ so we have that $\alpha(H(x, L')) = \alpha(H(x, L))$ for all x, and we are done by the assumption that G, L and k satisfy Hall's condition.

Therefore, we may assume that $|O(v_0)| < k$ and, consequently, by the choice of X_0, that $O(v_0) \subseteq X_0$. As noted above, if $x \in O(v_0)$, $\alpha(H(x, L')) = \alpha(H(x, L)) = \alpha(G(x, L)) - 1$. For $x \in X_0 \backslash O(v_0) \subseteq E(v_0)$, we also have that

$\alpha(H(x,L')) \geq \alpha(G(x,L)) - 1$, by the definition of $E(v_0)$ and of L'. Thus $\alpha(H(x,L')) \geq \alpha(G(x,L)) - 1$ for all $x \in X_0$. If $x \in C \backslash X_0$ then either $x \notin L(v_0)$, in which case $\alpha(H(x,L')) = \alpha(G(x,L))$, or $x \in B(v_0) \cup (E(v_0) \backslash X_0)$, in which case $\alpha(H(x,L')) = \alpha(G(x,L))$, as well. Consequently,

$$\begin{aligned}
\sum_{x \in C} \alpha(H(x,L')) &= \sum_{x \in X_0} \alpha(H(x,L')) + \sum_{x \in C \backslash X_0} \alpha(H(x,L')) \\
&\geq \sum_{x \in X_0} \alpha(G(x,L)) - |X_0| + \sum_{x \in C \backslash x_0} \alpha(G(x,L)) \\
&= \sum_{x \in C} \alpha(G(x,L)) - k \\
&\geq kn - k \\
&= k|V(H)|.
\end{aligned}$$

This completes the proof $h^{(k)}(C_n) \leq 2k$.

Next we show that $h^{(k)}(C_{2m+1}) = 2k$ for all $k \geq 1$ and $m \geq 2$ by exhibiting a color supply L satisfying Hall's condition with $G = C_{2m+1}$ and $w = k$ and $|L(v)| \geq 2k-1$ for all v, such that there is no (L,k)-coloring of G. Let $v_0, v_1, \ldots, v_{2m}$ be the vertices of G around the cycle. For any positive integer z, let $[z] = \{1, 2, \ldots, z\}$ and define $L(v_0) = [2k-1]$, $L(v_1) = L(v_{2m}) = [2k]$ and $L(v_j) = [2k] + (2k-1)$ for $2 \leq j \leq 2m-1$, where $[z] + x = \{1+x, 2+x, \ldots, z+x\}$.

In any (L,k)-coloring of G, because the lists $L(v_j)$, $2 \leq j \leq 2m-1$, have only $2k$ elements, color $2k$ would have to appear on either v_2 or v_{2m-1}. On the other hand, in every (L,k)-coloring of (v_1, v_0, v_{2m}) color $2k$ must be used on both v_1 and v_{2m}. Therefore, no such coloring of the cycle exists. It is straightforward to verify that there is an (L,k)-coloring of $G - v$ for every $v \in V$; therefore (*) holds for every proper induced subgraph H of G. Thus we need only to verify (*) for $H = G$. The following are easily seen: for $x \in [2k-1]$, $\alpha(G(x,L)) = 2$; $\alpha(G(2k,L)) = m$ and for $x \in [2k-1] + 2k$, $\alpha(G(x,L)) = m-1$. Therefore,

$$\begin{aligned}
\sum_{x \in C} \alpha(G(x,L)) &= 2(2k-1) + m + (m-1)(2k-1) \\
&= k(2m+1) + k - 1 \\
&\geq k(2m+1) \\
&= k|V(G)|.
\end{aligned}$$

It may be worth pointing out that the list assignment given to show that $h^{(k)}(C_{2m+1}) \geq 2k$ does not do the same job for $C_{2m}, m \geq 2$, for the reason that there is an (L,k)-coloring of C_{2m}, if L is defined as above.

To finish the proof of the theorem, we suppose that $m \geq \lceil \frac{k+2}{2} \rceil$, so that $n = 2m > k+1$, and we give a list assignment L to $G = C_{2m}$ such that G, L and k satisfy Hall's condition, and $|L(v)| \geq 2k-1$ for all $v \in V$, yet there is no (L,k)-coloring of G. Let the vertices of G be $v_0, v_1, \dots, v_{2m-1}$, around the cycle.

Case 1. $m \leq k$: set $L(v_0) = [2k]$, $L(v_j) = [2k-1]+j$, $1 \leq j \leq 2m-3$, $L(v_{2m-2}) = [2k]+(2m-3)$ and $L(v_{2m-1}) = \{1,3,5,\dots,2m-3\} \cup \{2k+1, 2k+3, 2k+5, \dots, 2k+2m-3\} \cup ([k-1]+(2k+2m-3))$.

Case 2. $m \geq k+1$: set $L(v_0) = [2k]$, $L(v_j) = [2k-1]+j$, $1 \leq j \leq 2k-1$, $L(v_j) = [2k]+(2k-1)$, $2k \leq j \leq 2m-2$ and $L(v_{2m-1}) = \{1,3,5,\dots,2k-1\} \cup \{2k+1, 2k+3, 2k+5, \dots, 4k-1\} \cup ([k-1]+(4k-1))$.

First note that all lists have size at least $2k-1$. The only unobvious case is that of $L(v_{2m-1})$ in Case 1, where we have $|L(v_{2m-1})| = m-1+m-1+k-1 = 2m+k-3 \geq k+2+k-3 = 2k-1$.

Next we shall see that there is no (L,k)-coloring of G. If φ were such a coloring, then, because $|L(v_j) \cup L(v_{j+1})| = 2k$, $0 \leq j \leq 2m-3$, it must be that $\varphi(v_j) = L(v_j) \backslash \varphi(v_{j+1})$, $0 \leq j \leq 2m-3$ and $\varphi(v_j) = L(v_j) \backslash \varphi(v_{j-1})$, $1 \leq j \leq 2m-2$. Related observations: for $0 \leq j \leq 2m-4$ in Case 1 and for $0 \leq j \leq 2k-2$ in Case 2, it must be that $j+1 \in \varphi(v_j)$. Otherwise, $\varphi(v_j) \subseteq L(v_{j+1})$ and we would have $|\varphi(v_{j+1})| = |L(v_{j+1}) \backslash \varphi(v_j)| = 2k-1-k = k-1$. By the same argument, slightly modified, $2m-2 \in \varphi(v_{2m-3})$ in Case 1 and $2k \in \varphi(v_{2k-1})$ in Case 2. Similarly, for $2 \leq j \leq 2m-2$ in Case 1 and for $2 \leq j \leq 2k$ in Case 2, it must be that $2k-1+j \in \varphi(v_j)$.

Using these observations with $\varphi(v_j) = L(v_j) \backslash \varphi(v_{j\pm 1})$ for various values of j, it can be seen that, in Case 1, $\varphi(v_0)$ must contain $1, 3, \dots, 2m-3$ and $\varphi(v_{2m-2})$ must contain $2m+2k-3, 2m+2k-5, \dots, 2k+1$. But this leaves only $k-1$ colors in $L(v_{2m-1})$ eligible for $\varphi(v_{2m-1})$. So there is no such φ in Case 1. In Case 2, something similar happens: $\varphi(v_0)$ must contain $1, 3, \dots, 2k-1$ and $\varphi(v_{2m-2})$ must contain $4k-1, 4k-3, \dots, 2k+1$, which leaves only $k-1$ elements in $L(v_{2m-1})$ eligible for $\varphi(v_{2m-1})$.

To show that G, L and $w = k$ satisfy Hall's condition is straightforward, but enough of a chore that we shall bear some of the burden here. First we show that $G - v$ has an (L,k)-coloring for each $v \in V$. From the discussion of the non-existence of φ, above, it is clear that this holds for $v = v_{2m-1}$, in both cases, but for other $v \in V$ there is work to be done. We start by explicitly defining a "near" k-fold coloring of G from the list assignment L, the very φ whose values on $v_0, v_1, \dots, v_{2m-2}$ are forced in Case 2 and partially in Case 1. For each $v \in V$, let $odd(v)$ and $even(v)$ denote respectively the subsets of odd and even elements in $L(v)$, and for $0 \leq j \leq 2m-2$,

$$\varphi(v_j) = \begin{cases} odd(v_j) & \text{if } j \text{ is even,} \\ even(v_j) & \text{if } j \text{ is odd;} \end{cases}$$

further,

$$\varphi(v_{2m-1}) = \begin{cases} [k-1]+(2k+2m-3) & \text{in Case 1} \\ [k-1]+(4k-1) & \text{in Case 2.} \end{cases}$$

If there were any doubt in the reader's mind about $G - v_{2m-1}$, then the reader could observe that φ restricted to $V\backslash\{v_{2m-1}\}$ is an (L,k)-coloring of that graph. Further, φ can be modified to an (L,k)-coloring of $G - v_0$ by adding 1 to $\varphi(v_{2m-1})$ and to an (L,k)-coloring of $G - v_{2m-2}$ by adding $2k+2m-3$ in Case 1, and $4k-1$ in Case 2, to $\varphi(v_{2m-1})$. For $i \in [2m-3]$ we modify φ to an (L,k)-coloring φ^* on $G - v_i$ as follows:

In Case 1, if i is even, then for $0 \le j < i$,

$$\varphi^*(v_j) = \begin{cases} (\varphi(v_j)\backslash\{i+1\}) \cup \{i\} & \text{if } j \text{ is even} \\ (\varphi(v_j)\backslash\{i\}) \cup \{i+1\} & \text{if } j \text{ is odd,} \end{cases}$$

and for $i < j \le 2m-2$, $\varphi^*(v_j) = \varphi(v_j)$ and $\varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i+1\}$; if i is odd then for $0 \le j < i$,

$$\varphi^*(v_j) = \begin{cases} (\varphi(v_j)\backslash\{i+1\}) \cup \{i\} & \text{if } j \text{ is odd} \\ (\varphi(v_j)\backslash\{i\}) \cup \{i+1\} & \text{if } j \text{ is even,} \end{cases}$$

and for $i < j \le 2m-2$, $\varphi^*(v_j) = \varphi(v_j)$ and $\varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i\}$.

In Case 2, for $i \in [2k-1]$, if i is even then for $0 \le j < i$,

$$\varphi^*(v_j) = \begin{cases} (\varphi(v_j)\backslash\{i+1\}) \cup \{i\} & \text{if } j \text{ is even} \\ (\varphi(v_j)\backslash\{i\}) \cup \{i+1\} & \text{if } j \text{ is odd,} \end{cases}$$

for $i < j \le 2m-2$, $\varphi^*(v_j) = \varphi(v_j)$, and $\varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i+1\}$; if i is odd then for $0 \le j < i$,

$$\varphi^*(v_j) = \begin{cases} (\varphi(v_j)\backslash\{i+1\}) \cup \{i\} & \text{if } j \text{ is odd} \\ (\varphi(v_j)\backslash\{i\}) \cup \{i+1\} & \text{if } j \text{ is even,} \end{cases}$$

for $i < j \le 2m-2$, $\varphi^*(v_j) = \varphi(v_j)$, and $\varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i\}$. If $2k \le i \le 2m-2$, for $0 \le j < i$, set $\varphi^*(v_j) = \varphi(v_j)$, and for $i < j \le 2m-2$,

$$\varphi^*(v_j) = \begin{cases} (\varphi(v_j)\backslash\{4k-2\}) \cup \{4k-1\} & \text{if } j \text{ is odd} \\ (\varphi(v_j)\backslash\{4k-1\}) \cup \{4k-2\} & \text{if } j \text{ is even,} \end{cases}$$

and $\varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{4k-1\}$.

The proof will be complete when we verify that (*) holds for G, that is $\sum\limits_{x \in C} \alpha(G(x,L)) \ge nk = 2mk$.

Case 1. For $1 \le x \le 2m-3$, $G(x,L)$ is a path of order $x+1$ if x is odd and x if x is even, so $\alpha(G(x,L)) = \left\lceil \frac{x}{2} \right\rceil$. For $2m-2 \le x \le 2k$, $G(x,L) = G - v_{2m-1}$, so $\alpha(G(x,L)) = m$. For $2k+1 \le x \le 2m+2k-3$, $G(x,L)$ is a path of order $2m+2k-x-2$ if x is even, and of order $2m+2k-x-1$ if x is odd, so $\alpha(G(x,L)) = m+k-1-\left\lfloor \frac{x}{2} \right\rfloor$. Finally, for $x \in [k-1] + (2k+2m-3)$, $\alpha(G(x,L)) = 1$. Therefore,

$$\begin{aligned}
\sum_{x \in C} \alpha(G(x,L)) &= \sum_{x=1}^{2m-3} \left\lceil \frac{x}{2} \right\rceil + \sum_{x=2m-2}^{2k} m + \sum_{x=2k+1}^{2m+2k-3} \left(m+k-1-\left\lfloor \frac{x}{2} \right\rfloor \right) + (k-1) \\
&= (m-1)^2 + m(2k-2m+3) + (m+k-1)(2m-3) \\
&\qquad - ((m+k-1)(m+k-2) - k^2) + k - 1 \\
&= 2mk + k - m + 1 \\
&\geq 2mk \\
&= kn.
\end{aligned}$$

Case 2. For $1 \leq x \leq 2k-1$, $G(x,L)$ is a path of order $x+1$ if x is odd and x if x is even; thus $\alpha(G(x,L)) = \left\lceil \frac{x}{2} \right\rceil$. $G(2k,L) = G - v_{2m-1}$, so $\alpha(G(2k,L)) = m$. For $x = 2k+r$, $1 \leq r \leq 2k-2$, $G(x,L)$ is a path of order $2m-2-r$ if r (and thus x) is even, and of order $2m-1-r$ if r (and thus x) is odd; $\alpha(G(x,L)) = m-1-\left\lfloor \frac{r}{2} \right\rfloor$. Clearly $\alpha(G(4k-1,L)) = m-k$ and $\alpha(G(x,L)) = 1$ for $x \in [k-1]+(4k-1)$. Therefore,

$$\begin{aligned}
\sum_{x \in C} \alpha(G(x,L)) &= \sum_{x=1}^{2k-1} \left\lceil \frac{x}{2} \right\rceil + m + \sum_{r=1}^{2k-2} \left(m-1-\left\lfloor \frac{r}{2} \right\rfloor \right) + (m-k) + (k-1) \\
&= k^2 + m + (m-1)(2k-2) - (k-1)^2 + (m-1) \\
&= 2mk \\
&= kn.
\end{aligned}$$

□

Acknowledgement

The authors thank the Vernon Wilson Endowment of Eastern Kentucky University for the support that allowed this collaboration.

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