8-SHREDDERS IN 8-CONNECTED GRAPHS

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Abstract

For a graph $G$, a subset $S$ of $V(G)$ is called a shredder if $G - S$ consists of three or more components. We show that if $G$ is an 8-connected graph of order at least 177, then the number of shredders of cardinality 8 of $G$ is less than or equal to $(2|V(G)| - 10)/3$.

Keywords: Graph; connectivity; shredder; upper bound

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1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to $x$ in $G$. For $A \subseteq V(G)$, we let $N_G(A) = \bigcup_{x \in A} N_G(x) - A$.

A subset $S$ of $V(G)$ is called a shredder if $G - S$ consists of three or more components. A shredder of cardinality $k$ is referred to as a $k$-shredder. The notion of a shredder was introduced by Cheriyan and Thurimella in [1] in connection with the problem of connectivity augmentation. In the algorithm of augmenting the connectivity of a $k$-connected graph which was given in [1], each $k$-shredder makes longer the running time of the algorithm. Thus it is important to obtain an upper bound on the number of $k$-shredders in a $k$-connected graph. In [2;Theorem 1], it is proved that if $k \geq 5$ and $G$ is a $k$-connected graph, then the number of $k$-shredders of $G$ is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible. For $k = 5$, it is shown in [3;Theorem 3] that if $G$ is a 5-connected graph of order at least 135, then the number of 5-shredders of $G$ is less than or equal to $(2|V(G)| - 10)/3$; for $k = 6$, it is shown in [6] that if $G$ is a 6-connected graph of order at least 325, then the number of 6-shredders of $G$ is less than or equal to $(2|V(G)| - 9)/3$; for $k = 7$, it is shown in [5] that if $G$ is a 7-connected graph of order at least 41, then the number of 7-shredders of $G$ is less than or equal to $(2|V(G)| - 8)/3$. It is also shown that each of
these three bounds is attained by infinitely many graphs (for results concerning the case where $1 \leq k \leq 4$, the reader is referred to [4] and [2;Theorem 2]). In this paper, we prove:

**Theorem.** Let $G$ be an 8-connected graph of order at least 177. Then the number of 8-shredders of $G$ is less than or equal to $(2|V(G)| - 10)/3$.

The greater part of our proof of the Theorem consists of modifications of arguments used in [3] and [6]. However, we here mention that the argument used in the proof of Lemma 2.21 and Claim 3.9 is new, and there is no corresponding argument in [3] or [6].

We now construct an infinite family of graphs $G'_m$ which attain the bound $((2|V(G'_m)| - 8)/3$ in the Theorem. Let $m \geq 8$. Define an auxiliary graph $H_m$ of order $m$ by letting

$$
V(H_m) = \{v_i | 1 \leq i \leq m\},
$$
$$
E(H_m) = \{v_i v_{i+3} | 1 \leq i \leq m - 3\}
\cup \{v_1 v_2, v_1 v_3, v_2 v_3, v_{m-2} v_{m-1}, v_{m-2} v_m, v_{m-1} v_m\}.
$$

We define a graph $G_m$ of order $3m - 5$ by adding $m - 5$ vertices to the so-called lexicographic product of $H_m$ and the null graph of order 2. More precisely, we let

$$
V(G_m) = \{x_{i,j} | 1 \leq i \leq m, 1 \leq j \leq 2\} \cup \{\alpha_i | 4 \leq i \leq m - 3\} \cup \{a\},
$$
$$
E(G_m) = \{x_{i,j} x_{i+3,k} | 1 \leq i \leq m - 3, 1 \leq j, k \leq 2\}
\cup \{x_{i-1,j} \alpha_i, x_{i,j} \alpha_i, x_{i+1,j} \alpha_i | 4 \leq i \leq m - 3, 1 \leq j \leq 2\}
\cup \{x_{i,j} a, x_{k,j} a | 1 \leq i \leq 2, m - 1 \leq k \leq m, 1 \leq j \leq 2\}
\cup \{x_{i,j} a, x_{m-3,j} a | 1 \leq j \leq 2\}
\cup \{\alpha_i a | 4 \leq i \leq m - 3\}
\cup \{x_{1,j} x_{2,k}, x_{1,j} x_{3,k}, x_{2,j} x_{3,k} | 1 \leq j, k \leq 2\}
\cup \{x_{m-2,j} x_{m-1,k}, x_{m-2,j} x_{m,k}, x_{m-1,j} x_{m,k} | 1 \leq j, k \leq 2\}.
$$

We define a graph $G'_m$ of order $3m - 4$ by

$$
V(G'_m) = V(G_m) \cup \{b\},
$$
$$
E(G'_m) = E(G_m) \cup \{bv | v \in V(G_m)\}.
$$

Then $G'_m$ is 8-connected, and has $2m - 6$ 8-shredders

$$
\begin{align*}
\{b, x_{i,1}, x_{i,2}, x_{i+1,1}, x_{i+1,2}, x_{i+2,1}, x_{i+2,2}, a\} & \quad (2 \leq i \leq m - 3), \\
\{b, x_{i-3,1}, x_{i-3,2}, x_{i+3,1}, x_{i+3,2}, \alpha_{i-1}, \alpha_i, \alpha_{i+1}\} & \quad (5 \leq i \leq m - 4), \\
\{b, x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{5,1}, x_{5,2}, a\}, \\
\{b, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{6,1}, x_{6,2}, a\}, \\
\{b, x_{1,1}, x_{1,2}, x_{7,1}, x_{7,2}, a, \alpha_{4}, \alpha_5, a\}, \\
\{b, x_{m-6,1}, x_{m-6,2}, x_{m,1}, x_{m,2}, \alpha_{m-4}, \alpha_{m-3}, a\}, \\
\{b, x_{m-5,1}, x_{m-5,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-3}\}, \\
\{b, x_{m-4,1}, x_{m-4,2}, x_{m-2,1}, x_{m-2,2}, x_{m,1}, x_{m,2}, a\}.
\end{align*}
$$
Thus the number of 8-shredders of $G'_m$ is $2m - 6 = (2(3m - 4) - 10)/3 = (2|V(G'_m)| - 10)/3$.

Figure 1: $m = 11$
2. Preliminary Results

Throughout the rest of this paper, let $G$ be an 8-connected graph, and let $\mathcal{S}$ denote the set of 8-shredders of $G$. For each $S \in \mathcal{S}$, we define $\mathcal{K}(S)$, $\mathcal{L}(S)$ and $\mathcal{L}(S)$ as follows. We let $\mathcal{K}(S)$ denote the set of components of $G - S$. Write $\mathcal{K}(S) = \{H_1, \ldots, H_s\}$ $(s = |\mathcal{K}(S)|)$. We may assume $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathcal{L}(S) = \mathcal{K}(S) - \{H_1\}$ and $\mathcal{L}(S) = \bigcup_{2 \leq i \leq s} V(H_i)$; thus $\mathcal{L}(S) = \bigcup_{C \in \mathcal{K}(S)} V(C)$. Now let $\mathcal{H} = \bigcup_{\mathcal{S} \in \mathcal{K}(S)}$. A member $F$ of $\mathcal{H}$ is said to be saturated if there exists a subset $C$ of $\mathcal{H} - \{F\}$ such that $V(F) = \bigcup_{C \in \mathcal{H} - \{F\}} V(C)$. Let $S, T \in \mathcal{S}$ with $S \neq T$. We say that $S$ meshes with $T$ if $S$ intersects with at least two members of $\mathcal{K}(T)$. It is easy to see that if $S$ meshes with $T$, then $S$ intersects with all members of $\mathcal{K}(S)$, and hence $T$ meshes with $S$ and $T$ intersects with all members of $\mathcal{K}(T)$ (see [1;Lemma 4.3 (1)])).

The following two lemmas are proved in [4;Lemmas 2.1 and 3.1] (see also [2;Lemmas 3.2 and 3.4]).

**Lemma 2.1.** Let $S, T \in \mathcal{S}$ with $S \neq T$, and suppose that $S$ does not mesh with $T$. Then one of the following holds:

(i) $L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, and no edge of $G$ joins a vertex in $L(S)$ and a vertex in $L(T)$;

(ii) there exists $C \in \mathcal{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or

(iii) there exists $D \in \mathcal{L}(T)$ such that $V(D) \supseteq L(S)$ (so $L(T) \supseteq L(S)$).

**Lemma 2.2.** Let $S, T \in \mathcal{S}$ with $S \neq T$, and suppose that $S$ meshes with $T$. Then $S \supseteq L(T)$ or $T \supseteq L(S)$.

The following lemma is proved in [2;Lemma 3.6].

**Lemma 2.3.** Let $F \in \mathcal{L}$, and suppose that $F$ is saturated. Then $|V(F)| \geq 4$.

The following lemmas are proved in [3;Lemma 2.8 and 2.9]

**Lemma 2.4.** Let $F \in \mathcal{L}$, and set $\mathcal{T} = \{T \in \mathcal{S} | L(T) \subseteq V(F) \}$. Then the following hold.

(I) $|\mathcal{T}| \leq (2|V(F)| - 2)/3$.

(II) If $|\mathcal{T}| = (2|V(F)| - 2)/3$, then one of the following holds:

(i) $F$ is trivial (i.e., $|V(F)| = 1$); or
(ii) $F$ is saturated, and there exist $T_1, T_2 \in \mathcal{T}$ such that $V(F) = L(T_1) \cup L(T_2)$, $T_1$ meshes with $T_2$, $|\mathcal{L}(T_1)| = |\mathcal{L}(T_2)| = 2$, and $|\{T \in \mathcal{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ for each $i = 1, 2$.

(III) If $|\mathcal{T}| = (2|V(F)| - 3)/3$, then one of the following holds:

(i) $F$ is saturated, and there exist $T_1, T_2 \in \mathcal{T}$ such that $V(F) = L(T_1) \cup L(T_2)$, $T_1$ meshes with $T_2$, $|\mathcal{L}(T_1)| = |\mathcal{L}(T_2)| = 2$, and $|\{T \in \mathcal{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ and $|\{T \in \mathcal{T} | L(T) \subseteq L(T_2)\}| = (2|L(T_2)| - 2)/3$.

(ii) $F$ is saturated, and there exist $T_1, T_2, T_3 \in \mathcal{T}$ such that $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$, $T_3$ meshes with $T_1$ and $T_2$, $|\mathcal{L}(T_1)| = |\mathcal{L}(T_2)| = |\mathcal{L}(T_3)| = 2$, and $|\{T \in \mathcal{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ for each $i = 1, 2, 3$; or

(iii) $F$ is not saturated, and there exists $T_0 \in \mathcal{T}$ such that $|L(T_0)| = |V(F)| - 1$, $|\mathcal{L}(T_0)| = 2$, and $|\{T \in \mathcal{T} | L(T) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$.

Lemma 2.5. Let $S \in \mathcal{T}$, and write $\mathcal{L}(S) = \{F_1, \cdots, F_p\}$ ($p = |\mathcal{L}(S)|$). Set $\mathcal{F} = \{T \in \mathcal{T} | L(T) \subseteq L(S)\}$, and set $\mathcal{F}_i = \{T \in \mathcal{T} | L(T) \subseteq V(F_i)\}$. Then the following hold.

(I) $|\mathcal{F}| \leq (2|L(S)| - 2p + 3)/3 \leq (2|L(S)| - 1)/3$.

(II) If $|\mathcal{F}| = (2|L(S)| - 1)/3$, then $p = 2$ and $|\mathcal{F}_i| = (2|V(F_i)| - 2)/3$ for each $i$.

(III) If $|\mathcal{F}| = (2|L(S)| - 2)/3$, then $p = 2$, and either $|\mathcal{F}_1| = (2|V(F_1)| - 2)/3$ and $|\mathcal{F}_2| = (2|V(F_2)| - 3)/3$, or $|\mathcal{F}_1| = (2|V(F_1)| - 3)/3$ and $|\mathcal{F}_2| = (2|V(F_2)| - 2)/3$.

We define an order relation $\leq$ in $\mathcal{T}$ as follows:

$S \leq T \iff L(S) \subseteq L(T) \ (S, T \in \mathcal{T})$.

The following lemma follows from Lemmas 2.4 and 2.5.

Lemma 2.6. Let $F \in \mathcal{L}$, and suppose that $|V(F)| \neq 1$. Set $\mathcal{F} = \{T \in \mathcal{T} | L(T) \subseteq L(F)\}$, and set $\mathcal{F}_M = \{T \in \mathcal{T} | L(T) \subseteq V(M)\}$ for each $M \in (\bigcup_{T \in \mathcal{T}} \mathcal{L}(T)) \cup \{F\}$ (thus $\mathcal{F} = \mathcal{F}_F$).

(I) Suppose that $|\mathcal{F}| = (2|V(F)| - 2)/3$. Then the following hold.

(i) $|\mathcal{F}_M| = (2|V(M)| - 2)/3$ for each $M \in (\bigcup_{T \in \mathcal{T}} \mathcal{L}(T)) \cup \{F\}$.

(ii) $|\mathcal{L}(T)| = 2$ for each $T \in \mathcal{F}$.

(II) Suppose that $|\mathcal{F}| = (2|V(F)| - 3)/3$. Then the following hold.

(i) $|\mathcal{F}_M| \geq (2|V(M)| - 3)/3$ for each $M \in (\bigcup_{T \in \mathcal{T}} \mathcal{L}(T)) \cup \{F\}$. \


(ii) \(|L(T)| = 2\) for each \(T \in \mathcal{F}\).

**Proof.** We prove (I) and (II) simultaneously. Assume \(|\mathcal{F}| \geq (2|V(F)| - 3)/3\). Let \(T_1, \cdots, T_s\) be the maximal members of \(\mathcal{F}\) (we have \(1 \leq s \leq 3\) by Lemma 2.4). Note that in both (I) and (II), (i) clearly holds for \(M = F\). Hence the lemma will follow if we show that the following holds for each \(T \in \mathcal{F}\):

\[|L(T)| = 2 \text{ and, for each } M \in L(T), |\mathcal{F}_M| \geq (2|V(M)| - 3)/3 \text{ and we have } |\mathcal{F}_M| = (2|V(M)| - 2)/3.\]

We prove (2.1) by backward induction on \(|L(T)|\). First let \(T \in \{T_1, \cdots, T_s\}\). Then it follows from Lemma 2.4 that \(|L(T)| = 2\), \(|R \in \mathcal{F} |L(R) \subseteq L(T)| \geq (2|L(T)| - 2)/3\), and we have \(|\mathcal{F}_M| = (2|V(M)| - 2)/3 \text{ if } |\mathcal{F}| = (2|V(F)| - 2)/3\). Also it follows from Lemma 2.5 that for each \(M \in L(T)\), \(|\mathcal{F}_M| \geq (2|V(M)| - 3)/3\), and we have \(|\mathcal{F}_M| = (2|V(M)| - 2)/3 \text{ if } |\mathcal{F}| = (2|V(F)| - 2)/3\). This shows that (2.1) holds for \(T \in \{T_1, \cdots, T_s\}\). Thus we may assume \(T \notin \{T_1, \cdots, T_s\}\). Then there exists \(S \in \mathcal{F}\) with \(S \neq T\) such that \(T \subseteq S\) (i.e., \(L(T) \subseteq L(S)\)). We choose such \(S\) so that \(S\) is minimal. Then by Lemmas 2.1 and 2.2, \(S\) does not mesh with \(T\) and there exists \(M' \in L(S)\) such that \(L(T) \subseteq V(M')\). By the induction hypothesis, \(|\mathcal{F}_M'| \geq (2|V(M')| - 3)/3\), and we have \(|\mathcal{F}_M'| = (2|V(M')| - 2)/3 \text{ if } |\mathcal{F}| = (2|V(F)| - 2)/3\). Let \(T_1', \cdots, T_s'\) be the maximal members of \(\mathcal{F}_{M'}\). Then \(T \in \{T_1', \cdots, T_s'\}\) by the minimality of \(S\). Consequently, applying Lemma 2.4 to \(M'\), we see that \(|L(T)| = 2\), \(|R \in \mathcal{F} |L(R) \subseteq L(T)| \geq (2|L(T)| - 2)/3\), and we have \(|\{R \in \mathcal{F} |L(R) \subseteq L(T)| = (2|L(T)| - 1)/3 \text{ if } |\mathcal{F}| = (2|V(F)| - 2)/3\). Also it follows from Lemma 2.5 that for each \(M \in L(T)\), \(|\mathcal{F}_M| \geq (2|V(M)| - 3)/3\), and we have \(|\mathcal{F}_M| = (2|V(M)| - 2)/3 \text{ if } |\mathcal{F}| = (2|V(F)| - 2)/3\). Thus (2.1) is proved, which completes the proof of the lemma.

The following lemmas are proved in [3; Lemmas 2.10 through 2.12].

**Lemma 2.7.** Let \(X \subseteq V(G)\). Set \(\mathcal{F} = \{T \in \mathcal{F} |L(T) \subseteq X\}\) and \(\mathcal{F}_0 = \cup_{T \in \mathcal{F}} L(T)\), and suppose that no component in \(\mathcal{F}_0\) is saturated. Then \(|\mathcal{F}| \leq |X|/2\).

**Lemma 2.8.** Let \(S, T \in \mathcal{F}\), and suppose that \(S\) meshes with \(T\) and \(L(S) \nsubseteq T\). Then \(L(T) \subseteq S\) and \(|L(T)| \leq 3\).

**Lemma 2.9.** Suppose that \(|V(G)| \geq 17\). Let \(S, T \in \mathcal{F}\), and suppose that \(S\) meshes with \(T\), \(L(S) \subseteq T\) and \(L(T) \subseteq S\). Then \(|L(S)| + |L(T)| \leq 8\).

The following lemma follows from Lemmas 2.8 and 2.9.

**Lemma 2.10.** Suppose that \(|V(G)| \geq 17\). Let \(S, T \in \mathcal{F}\), and suppose that \(S\) meshes with \(T\) and \(|L(S)| \geq 5\). Then \(L(T) \subseteq S\) and \(|L(T)| \leq 3\).

As an immediate corollary of Lemma 2.10, we obtain the following lemma.
Lemma 2.11. Suppose that $|V(G)| \geq 17$. Let $S,T \in \mathcal{S}$ with $S \neq T$, and suppose that either $|L(S)| \geq 4$ and $|L(T)| \geq 5$, or $|L(S)| \geq 5$ and $|L(T)| \geq 4$. Then $S$ does not mesh with $T$.

Lemma 2.12. Let $S \in \mathcal{S}$, and suppose that $|L(S)| \neq 2$ and $|\{T \in \mathcal{S} | L(T) \subseteq L(S)\}| = (2|L(S)| - 1)/3$. Then $|L(S)| \geq 5$.

Proof. Since $|\{T \in \mathcal{S} | L(T) \subseteq L(S)\}|$ is an integer, we have $|L(S)| \neq 3, 4$, and hence $|L(S)| \geq 5$.

Lemma 2.13. Let $S,T \in \mathcal{S}$ with $S \neq T$, and suppose that $|L(T)| = 2$. Then either $S \supseteq L(T)$ or $S \cap L(T) = \emptyset$.

Proof. If $S$ meshes with $T$, then from the fact that $S$ intersects with each member of $\mathcal{L}(T)$, it follows that $S \supseteq L(T)$. Thus we may assume $S$ does not mesh with $T$. Then (i) or (ii) of Lemma 2.1 holds. If (i) holds, then we clearly have $S \cap L(T) = \emptyset$; if (ii) holds, then $S \cap L(T) \subseteq S \cap L(S) = \emptyset$.

Lemma 2.14. Let $S,T \in \mathcal{S}$ with $S \neq T$, and suppose that $L(S) \supseteq L(T)$. Then $|L(S)| \geq 4$.

Proof. By Lemma 2.2, $S$ does not mesh with $T$, and hence it follows from Lemma 2.1 that there exists $C \in \mathcal{L}(S)$ such that $V(C) \supseteq L(T)$. Since $C$ is connected, $V(C) \neq L(T)$, and hence $|V(C)| \geq |L(T)| + 1 \geq 3$. Consequently $|L(S)| \geq |V(C)| + 1 \geq 4$.

Lemma 2.15. Let $S,T \in \mathcal{S}$, and suppose that $S$ meshes with $T$. Let $F \in \mathcal{S}(S)$, and suppose that $|V(F)| \geq 2$. Then $|T \cap V(F)| \geq 2$.

Proof. If $V(F) \subseteq T$, then we clearly have $|T \cap V(F)| = |V(F)| \geq 2$. Thus we may assume $V(F) \not\subseteq T$. Then there exists $D \in \mathcal{S}(T)$ such that $V(F) \cap V(D) \neq \emptyset$. Since $S$ meshes with $T$, we have $|S - (T \cup V(D))| = \sum_{C \in \mathcal{S}(T) \setminus \{D\}} |S \cap V(C)| \geq 2$. Set $R = (T \cap V(F)) \cup (S \cap (T \cup V(D)))$. Then $R$ separates $V(F) \cap V(D)$ from the rest. This implies $|R| \geq 8$, and hence $|T \cap V(F)| = |R| - |S \cap (T \cup V(D))| \geq 8 - |S \cap (T \cup V(D))| = |S - (T \cup V(D))| \geq 2$.

Lemma 2.16. Let $S,T \in \mathcal{S}$, and suppose that $S$ meshes with $T$ and $|L(S)| \geq 3$. Then $|T \cap L(S)| \geq 3$.

Proof. If $L(S) \subseteq T$, then $|T \cap L(S)| = |L(S)| \geq 3$; if $L(S) \not\subseteq T$, then there exists $F \in \mathcal{L}(S)$ with $|V(F)| \geq 2$, and hence $|T \cap L(S)| \geq |T \cap V(F)| + |S \cap (T \cup V(D))| = |S - (T \cup V(D))| \geq 2 + 1 = 3$ by Lemma 2.15.

Lemma 2.17. Suppose that $|V(G)| \geq 17$. Let $S,T \in \mathcal{S}$, and suppose that $S$ meshes with $T$. Then $|T \cap L(S)| \leq 6$. 
Proof. Suppose that \(|T \cap L(S)| \geq 7\). Write \(\mathcal{X}(S) - \mathcal{L}(S) = \{C\}\). If \(|V(C)| \geq 2\), then \(|T| \geq |T \cap V(C)| + |T \cap L(S)| \geq 2 + 7 = 9\) by Lemma 2.15, which contradicts the fact that \(|T| = 8\). Thus \(|V(C)| = 1\). By the definition of \(\mathcal{L}(S)\), this implies \(|V(F)| = 1\) for all \(F \in \mathcal{X}(S)\), and hence \(V(F) \subseteq T\) for all \(F \in \mathcal{X}(S)\) by the assumption that \(S\) meshes with \(T\). Hence \(|V(G)| = |S| + \sum_{F \in \mathcal{X}(S)} |V(F)| \leq |S| + |T| = 16\), a contradiction. \(\square\)

Lemma 2.18. Suppose that \(|V(G)| \geq 17\). Let \(F \in \mathcal{L}\), and set \(T = \{T \in \mathcal{J} \mid L(T) \subseteq V(F)\}\). Suppose that \(|\mathcal{J}| \geq (2|V(F)| - 3)/3\), and there exist \(T_1, T_2 \in \mathcal{J}\) such that \(V(F) = L(T_1) \cup L(T_2)\) with \(|L(T_1)| \leq |L(T_2)|\). Then \(|L(T_1)| = 2\).

Proof. Set \(\mathcal{J}_1 = \{T \in \mathcal{J} \mid L(T) \subseteq L(T_1)\}\) and \(\mathcal{J}_2 = \{T \in \mathcal{J} \mid L(T) \subseteq L(T_2)\}\). Suppose that \(|L(T_1)| \geq 3\). By Lemma 2.4, \(T_1\) meshes with \(T_2\). Hence \(3 \leq |L(T_1)| \leq 4\) by Lemma 2.11. By Lemma 2.4, we have \(|\mathcal{J}_1| = (2|L(T_1)| - 1)/3\) or \((2|L(T_1)| - 2)/3\) and, if \(|\mathcal{J}_1| = (2|L(T_1)| - 2)/3\), then \(|\mathcal{J}_2| = (2|L(T_2)| - 1)/3\). Since \(|\mathcal{J}_1|\) and \(|\mathcal{J}_2|\) are integers, this forces \(|L(T_1)| = 4\), \(|\mathcal{J}_1| = (2|L(T_1)| - 2)/3\) and \(|\mathcal{J}_2| = (2|L(T_2)| - 1)/3\), and hence \(|L(T_2)| \geq 5\) by Lemma 2.12. But this contradicts Lemma 2.11. \(\square\)

Lemma 2.19. Suppose that \(|V(G)| \geq 17\). Let \(S \in \mathcal{J}\) and \(F \in \mathcal{L}(S)\), and suppose that \(|V(F)| \geq 2\). Set \(\mathcal{J} = \{T \in \mathcal{J} \mid L(T) \subseteq V(F)\}\), and let \(T_1, \ldots, T_s\) be the maximal members of \(\mathcal{J}\) with \(|L(T_1)| \leq |L(T_2)| \leq \cdots \leq |L(T_s)|\). Suppose that \(|\mathcal{J}| \geq (2|V(F)| - 3)/3\), and there exist four members \(P_1, P_2, P_3, P_4\) of \(\mathcal{J}\) which mesh with \(S\). In the case where \(|V(F)| = 3\), suppose further that \(|L(P_i)| = 2\) for each \(1 \leq i \leq 4\). Then the following hold.

(I) \(|L(P_i)| = 2\) for each \(1 \leq i \leq 4\) and \(S = \bigcup_{1 \leq i \leq 4} L(P_i)\).

(II) \(2 \leq s \leq 3\).

(i) \(|L(T_i)| = 2\) for each \(1 \leq i \leq s - 1\).

(ii) There exist four members \(R_1, R_2, R_3, R_4\) of \(\mathcal{J}\) such that \(R_i\) meshes with \(T_s\) and \(|L(R_i)| = 2\) for each \(1 \leq i \leq 4\) and \(T_s = \bigcup_{1 \leq i \leq 4} L(R_i)\).

(iii) \(|T_i \cap L(T_s)|\) is even for each \(1 \leq i \leq s - 1\).

(IV) If \(|\mathcal{J}| = (2|V(F)| - 2)/3\), then \(s = 2\) and there exists \(M \in \mathcal{L}(T_2)\) such that \(|V(M)| = 1\).

Proof. By Lemma 2.4 and the assumption that \(|\mathcal{J}| \geq (2|V(F)| - 3)/3\), we have \(|\mathcal{J}| = (2|V(F)| - 3)/3\) or \((2|V(F)| - 2)/3\). Since \(|\mathcal{J}|\) is an integer, this in particular implies \(|V(F)| \neq 2\). Hence \(|V(F)| \geq 3\). Assume for the moment that \(|V(F)| \geq 4\). Then since \(|\mathcal{L}(S)| \geq 2\), we have \(|L(S)| \geq |V(F)| + 1 \geq 5\). Hence \(L(P_i) \subseteq S\) and \(|L(P_i)| \leq 3\) for
each \(1 \leq i \leq 4\) by Lemma 2.10. By Lemmas 2.1, 2.2 and 2.14, \(L(P_i) \cap L(P_j) = \emptyset\) for each \(1 \leq i < j \leq 4\). Since \(|S| = 8\), this implies
\[
|L(P_i)| = 2 \text{ for each } 1 \leq i \leq 4 \tag{2.2}
\]
and
\[
S = \bigcup_{1 \leq i \leq 4} L(P_i). \tag{2.3}
\]
If \(|V(F)| = 3\), then we have (2.2) by assumption, and hence (2.3) holds. Thus (2.2) and (2.3) hold, and (I) is proved. Now for \(T \in \mathcal{F}\) and \(1 \leq i \leq 4\),
\[
\text{if } T \cap L(P_i) \neq \emptyset, \text{ then } P_i \text{ meshes with } T \text{ and } L(P_i) \subseteq T \tag{2.4}
\]
by Lemma 2.1 (i) and (2.2), because \(L(T) \cap L(P_i) \subseteq L(T) \cap S = \emptyset\). It follows from (2.3) and (2.4) that
\[
|T \cap S| \text{ is even for each } T \in \mathcal{F}. \tag{2.5}
\]
Suppose that \(F\) is not saturated. Then \(s = 1\) and \(|L(T_1)| = |V(F)| - 1\) by Lemma 2.4 (III) (iii). Since \(|V(F)| \geq 2\), we have \(|P_i \cap V(F)| \geq 2\) for each \(1 \leq i \leq 4\) by Lemma 2.15. Hence \(P_i \cap L(T_1) \neq \emptyset\) for each \(1 \leq i \leq 4\). Since \(L(P_i) \cap L(T_1) = \emptyset\) for each \(1 \leq i \leq 4\), it follows from Lemma 2.1 that \(P_i\) meshes with \(T_1\) for each \(1 \leq i \leq 4\). Hence \(L(P_i) \subseteq T_1\) for each \(1 \leq i \leq 4\) by (2.2). Since \(|T_1| = 8 = |S|\), this together with (2.3) implies \(T_1 = S\), a contradiction. Thus \(F\) is saturated, which proves (II).

It follows from (II) that \(V(F) = L(T_1) \cup L(T_2) \cup \cdots \cup L(T_s)\). By Lemma 2.4, we have \(2 \leq s \leq 3\). In proving (III), we consider the cases where \(s = 2\) and \(s = 3\) separately.

Case 1. \(s = 2\).

By Lemmas 2.4 and 2.18, \(T_1\) meshes with \(T_2\) and \(|L(T_1)| = 2\). Since \(N_G(L(T_1)) \cup L(T_2)) = N_G(V(F)) = S\), \(T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1)\). Hence \(T_2 - L(T_1) \subseteq S\). Since \(|T_2 - L(T_1)| = 6\), it follows from (2.3) and (2.4) that three of \(P_1\), \(P_2\), \(P_3\), \(P_4\) mesh with \(T_2\), and hence (III) (ii) holds. Since \(|T_1| = 8\) and \(T_1 = N_G(L(T_1)) \subseteq S \cup L(T_2)\), \(|T_1 \cap L(T_2)| = |T_1| - |T_1 \cap S|\) is even by (2.5).

Case 2. \(s = 3\).

By Lemma 2.4 (III) (iii), \(|\mathcal{L}(T_1)| = |\mathcal{L}(T_2)| = |\mathcal{L}(T_3)| = 2\),
\[
\text{at least one of } T_1 \text{ and } T_2 \text{ meshes with } T_3. \tag{2.6}
\]
and \(|\{T \in \mathcal{F} \mid L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3\) for each \(1 \leq i \leq 3\). Arguing as in the proof of Lemma 2.18, we see from Lemmas 2.11 and 2.12 that
\[
\text{if } T \in \{T_1, T_2\} \text{ and } T \text{ meshes with } T_3, \text{ then } |L(T)| = 2. \tag{2.7}
\]
Also note that

\[
\text{if } T \in \{T_1, T_2\} \text{ and } T \text{ does not mesh with } T_3, \text{ then } T_3 \cap L(T) = \emptyset \quad (2.8)
\]

by Lemma 2.1 (i). Since \( |T_3| = 8 \) and \( T_3 = N_G(L(T_3)) \subseteq S \cup L(T_1) \cup L(T_2) \), it follows from (2.4), (2.7) and (2.8) that (III) (ii) holds, and

\[
\text{at least two of } P_1, P_2, P_3, P_4 \text{ mesh with } T_3. \quad (2.9)
\]

Suppose that \( |L(T_2)| \neq 2 \). Then \( |L(T_2)| \geq 5 \) and \( T_2 \) does not mesh with \( T_3 \) by Lemmas 2.12 and 2.11. Hence it follows from (2.6) and (2.7) that \( T_1 \) meshes with \( T_3 \) and \( |L(T_1)| = 2 \). We also have \( T_2 \cap L(T_3) = \emptyset \) by Lemma 2.1 (i), and hence \( T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1) \). Since \( |T_2| = 8 \), this together with (2.4) implies that at least three of \( P_1, P_2, P_3 \) and \( P_4 \) mesh with \( T_2 \). Hence it follows from (2.9) that there exists \( i \) such that \( P_i \) meshes with \( T_2 \) and \( T_3 \). But then \( |P_i \cap L(S)| \geq |P_i \cap V(F)| + 1 \geq |P_i \cap L(T_2)| + |P_i \cap L(T_3)| + 1 \geq 3 + 3 + 1 = 7 \) by Lemma 2.16, which contradicts Lemma 2.17. Thus \( |L(T_2)| = 2 \). Since \( |L(T_2)| \geq |L(T_1)| \), we also get \( |L(T_1)| = 2 \). Now since \( |L(T_1)| = |L(T_2)| = 2 \) and since \( T_1 = N_G(L(T_1)) \subseteq S \cup L(T_2) \cup L(T_3) \) and \( T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1) \cup L(T_3) \), it follows from (2.5) and Lemma 2.13 that \( |T_1 \cap L(T_3)| \) and \( |T_2 \cap L(T_3)| \) are even. Thus (III) is proved.

Finally we prove (IV). Since \( |\mathcal{S}| = (2|V(F)| - 2)/3 \), it follows from Lemma 2.4 (ii) that \( s = 2 \), \( |\mathcal{S}(T_2)| = 2 \), \( T_1 \) meshes with \( T_2 \), and \( |\{T \in \mathcal{S} \mid L(T) \subseteq L(T_2)\}| = (2|L(T_2)| - 1)/3 \). Write \( \mathcal{S}(T_2) = \{F_1, F_2\} \). By (II), \( |L(T_1)| = 2 \). From the proof of (III) in Case 1, it follows that three of \( P_1, P_2, P_3 \) and \( P_4 \) mesh with \( T_2 \). We may assume that \( P_i \) meshes with \( T_2 \) for each \( 1 \leq i \leq 3 \). Then \( T_2 = L(T_1) \cup (\bigcup_{1 \leq i \leq 3} L(P_i)) \). Suppose that \( |V(F_1)| \neq 1 \) and \( |V(F_2)| \neq 1 \). Since \( \{|T \in \mathcal{S} \mid L(T) \subseteq V(F_1)\}| = (2|V(F_1)| - 2)/3 \) for each \( 1 \leq i \leq 2 \) by Lemma 2.5, it follows from Lemma 2.4 (ii) that \( F_i \) is saturated for each \( 1 \leq i \leq 2 \), and there exist \( Q_1, Q_2, Q_3, Q_4 \in \mathcal{S} \) such that \( V(F_1) = L(Q_1) \cup L(Q_2) \) and \( V(F_2) = L(Q_3) \cup L(Q_4) \). By Lemma 2.18, we may assume \( |L(Q_1)| = |L(Q_2)| = 2 \). Since \( N_G(L(Q_1) \cup L(Q_2)) = N_G(V(F_1)) = T_2 \), \( Q_2 = N_G(L(Q_2)) \subseteq L(Q_1) \cup T_2 = L(Q_1) \cup L(T_1) \cup (\bigcup_{1 \leq i \leq 3} L(P_i)) \). Since \( |Q_2| = 8 \), this together with (2.4) implies

\[
\text{at least two of } P_1, P_2 \text{ and } P_3 \text{ mesh with } Q_2. \quad (2.10)
\]

Similarly,

\[
\text{at least two of } P_1, P_2 \text{ and } P_3 \text{ mesh with } Q_4. \quad (2.11)
\]

Suppose that \( |L(Q_2)| = 2 \). Then arguing as in the proof of (2.10), we see that

\[
\text{at least two of } P_1, P_2 \text{ and } P_3 \text{ mesh with } Q_1. \quad (2.12)
\]

From (2.10) and (2.12), it follows that there exists \( i \) with \( 1 \leq i \leq 3 \) such that \( P_i \) meshes with \( Q_1 \) and \( Q_2 \). But then \( |P_i \cap L(S)| \geq |P_i \cap L(T_2)| + 1 \geq |P_i \cap L(Q_1)| + |P_i \cap L(Q_2)| + |P_i \cap
We first prove (ii) by backward induction on each \(i\), there exist \(L\) with \(|L|≤3\). Note that it follows from (2.10) and (2.11) that there exists \(i\) with \(1≤i≤3\) such that \(P_i\) meshes with \(Q_2\) and \(Q_4\). Now by Lemma 2.16, we obtain \(|P_i∩L(S)|≥|P_i∩L(T_2)|+1≥|P_i∩L(Q_2)|+1≥3+3+1=7\), which contradicts Lemma 2.17. Thus (IV) is proved. □

Applying Lemmas 2.6 and 2.19 repeatedly, we obtain the following lemma.

**Lemma 2.20.** Suppose that \(|V(G)|≥17\). Let \(S∈\mathcal{S}\) and \(F∈\mathcal{L}(S)\), and suppose that \(|V(F)|≥4\). Set \(\mathcal{F} = \{T∈\mathcal{S} | L(T)⊆V(F)\}\), and suppose that \(|\mathcal{F}|=(2|V(F)|−3)/3\). Suppose further that there exist four members of \(\mathcal{S}\) which mesh with \(S\). Then the following hold.

(i) For each \(M∈(\bigcup_{T∈\mathcal{F}}\mathcal{L}(T))∪\{F\}\) with \(|V(M)|≠1\), \(M\) is saturated.

(ii) For each \(T∈\mathcal{F}∪\{S\}\) with \(|L(T)|≠2\), there exist \(R_1,R_2,R_3,R_4∈\mathcal{S}\) such that \(R_i\) meshes with \(T\) and \(|L(R_i)|=2\) for each \(1≤i≤4\) and \(T=\bigcup_{1≤i≤4}L(R_i)\).

**Proof.** We first prove (ii) by backward induction on \(|L(T)|\). Thus let \(T∈\mathcal{F}∪\{S\}\) with \(|L(T)|≠2\). If \(T=S\), the desired conclusion follows from Lemma 2.19 (i). Thus we may assume \(T≠S\). Then there exists \(S'∈\mathcal{F}∪\{S\}\) with \(S'≠T\) such that \(T≤S'\). We choose such \(S'\) so that \(S'\) is minimal, and take \(M∈\mathcal{L}(S')\) such that \(L(T)⊆V(M)\). Then \(M∈(\bigcup_{R∈T}\mathcal{L}(R))∪\{F\}\). By the induction hypothesis, there exist \(Q_1,Q_2,Q_3,Q_4∈\mathcal{S}\) such that \(Q_i\) meshes with \(S'\) and \(|L(Q_i)|=2\) for each \(1≤i≤4\). Since \(|\mathcal{F}|=(2|V(F)|−3)/3\), \(|\{R∈\mathcal{F} | L(R)⊆V(M)\}|≥(2|V(M)|−3)/3\) by Lemma 2.6. Consequently \(S'\) and \(M\) satisfy the assumptions of Lemma 2.19. Let \(T_1,⋯,T_4\) be the maximal members of \(\{R∈\mathcal{F} | L(R)⊆V(M)\}\) with \(|L(T_1)|≤|L(T_2)|≤⋯≤|L(T_4)|\). By the minimality of \(S',\ T∈\{T_1,⋯,T_4\}\). Since \(|L(T)|≠2\), it follows from Lemma 2.19 (III (i)) that \(T=T_4\). Hence the desired conclusion follows from Lemma 2.19 (III (ii)). Thus (ii) is proved. Now to prove (i), let \(M∈(\bigcup_{T∈\mathcal{F}}\mathcal{L}(T))∪\{F\}\) with \(|V(M)|≠1\), and take \(S'∈\mathcal{F}∪\{S\}\) such that \(M∈\mathcal{L}(S')\). Then \(|L(S')|=2\). By (ii), there exist \(Q_1,Q_2,Q_3,Q_4∈\mathcal{S}\) such that \(Q_i\) meshes with \(S'\) and \(|L(Q_i)|=2\) for each \(1≤i≤4\), and \(|\{T∈\mathcal{F} | L(T)⊆V(M)\}|≥(2|V(M)|−3)/3\) by Lemma 2.6. Hence we obtain the desired conclusion by applying Lemma 2.19 (II) to \(S'\) and \(M\). □

**Lemma 2.21.** Suppose that \(|V(G)|≥17\). Let \(S∈\mathcal{S}\), and suppose that there exist four members of \(\mathcal{S}\) which mesh \(S\). Then the following hold.

(i) If \(F∈\mathcal{L}(S)\) and \(|V(F)|≥5\), then \(|\{T∈\mathcal{S} | L(T)⊆V(F)\}|≤(2|V(F)|−3)/3\).

(ii) If \(|L(S)|≥9\), then \(|\{T∈\mathcal{S} | L(T)⊆L(S)\}|≤(2|L(S)|−2)/3\).
Proof. We first prove (i). Thus let \( F \in \mathcal{L}(S) \) with \(|V(F)| \geq 5\). Set \( \mathcal{T} = \{ T \in \mathcal{F} \mid L(T) \subseteq V(F) \} \). Suppose that \(|\mathcal{T}| = (2|V(F)| - 2)/3\). By Lemma 2.4 (II) (ii), there exist \( T_1, T_2 \in \mathcal{T} \) such that \( V(F) = L(T_1) \cup L(T_2) \) with \( |L(T_1)| \leq |L(T_2)| \). By Lemma 2.18, \(|L(T_1)| = 2\). Since \(|V(F)| \geq 5\), \(|L(T_2)| = |V(F)| - |L(T_1)| \geq 5 - 2 = 3 > 2\). Choose \( T_0 \in \mathcal{T} \) with \(|L(T_0)| \neq 2\) so that \(|L(T_0)|\) is as small as possible. Also choose \( S' \in \mathcal{F} \cup \{ S \} \) with \( S' \neq T_0 \) and \( T_0 \leq S' \) so that \( S' \) is minimal, and take \( M \in \mathcal{L}(S') \) such that \( L(T_0) \subseteq V(M) \). Since \(|L(S')| \neq 2\), it follows from Lemma 2.20 (ii) that

\[
\text{there exist } R_1, R_2, R_3, R_4 \in \mathcal{F} \text{ such that } R_i \text{ meshes with } S' \\
\text{and } |L(R_i)| = 2 \text{ for each } 1 \leq i \leq 4. \tag{2.13}
\]

Since \(|\mathcal{T}| = (2|V(F)| - 2)/3\), \(|\{ T \in \mathcal{T} \mid L(T) \subseteq V(M) \}| = (2|V(M)| - 2)/3\) by Lemma 2.6 (I) (i). Hence it follows from Lemma 2.4 (II) (ii) and the minimality of \( S' \) that there exists \( T' \in \mathcal{T} \) such that \( V(M) = L(T') \cup L(T_0) \) and \( T' \) meshes with \( T_0 \), and \(|\mathcal{L}(T_0)| = 2\, and \(|\{ T \in \mathcal{T} \mid L(T) \subseteq L(T_0) \}| = (2|L(T_0)| - 1)/3\). Since \(|L(T_0)| \neq 2\), it follows from (2.13) and (i) and (iii) of Lemma 2.19 (III) that

\[
|T' \cap L(T_0)| \text{ is even.} \tag{2.14}
\]

Write \( \mathcal{L}(T_0) = \{ F_1, F_2 \} \) with \(|V(F_1)| \leq |V(F_2)|\). Since \(|L(T_0)| \neq 2\, we have \(|V(F_2)| \neq 1\). Since \(|\{ T \in \mathcal{T} \mid L(T) \subseteq V(F_2) \}| = (2|V(F_2)| - 2)/3\) by Lemma 2.5 (II), it follows from Lemma 2.4 (II) (ii) that there exist \( Q_1, Q_2 \in \mathcal{T} \) such that \( V(F_2) = L(Q_1) \cup L(Q_2)\). By the minimality of \(|L(T_0)|\), \(|L(Q_1)| = |L(Q_2)| = 2\). By Lemma 2.13, this implies \(|T' \cap V(F_2)|\) is even. On the other hand, \(|V(F_1)| = 1\) by (2.13) and Lemma 2.19 (IV). Since \( T' \) meshes with \( T_0\), this means \(|T' \cap V(F_1)| = 1\). Consequently \(|T' \cap L(T_0)| = |T' \cap V(F_2)| + |T' \cap V(F_1)|\) is odd, which contradicts (2.14). Thus (i) is proved. To prove (ii), assume \(|L(S)| \geq 9\). By way of contradiction, suppose that \(|\{ T \in \mathcal{T} \mid L(T) \subseteq L(S) \}| = (2|L(S)| - 1)/2\). Then by Lemma 2.5 (II), \(|\mathcal{L}(S)| = 2\ and \(|\{ T \in \mathcal{F} \mid L(S) \subseteq V(F) \}| = (2|V(F)| - 2)/3\) for each \( F \in \mathcal{L}(S)\). From \(|\mathcal{L}(S)| = 2\), it follows that there exists \( F \in \mathcal{L}(S)\) such that \(|V(F)| \geq 5\). But then \(|\{ T \in \mathcal{T} \mid L(S) \subseteq V(F) \}| \leq (2|V(F)| - 3)/3\ by (i), a contradiction. \( \square \)

3. Proof of the Theorem

We continue with the notation of the preceding section, and prove the Theorem. Thus let \(|V(G)| \geq 177\ and, by way of contradiction, suppose that

\[
|\mathcal{T}| \geq (2|V(G)| - 9)/3. \tag{3.1}
\]

Let \( S_1, \ldots, S_m \) be the maximal members of \( \mathcal{F} \) with respect to the order relation \( \leq \). We may assume \(|L(S_1)| \geq \cdots \geq |L(S_m)|\). Let \( p_i = |\mathcal{L}(S_i)|\) for each \( i\), and let
$W = V(G) - (L(S_1) \cup \cdots \cup L(S_m))$. Arguing as in [3; Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

**Claim 3.1.**

(i) $m + 2|W| \leq 9$.

(ii) $2p_1 + (m - 1) + 2|W| \leq 12$.

**Sketch of Proof.** By (3.1) and Lemma 2.5 (I), $(2|V(G)| - 9)/3 \leq \sum_{1 \leq i \leq m} (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 9$. Since $p_i \geq 2$ for all $i$, both (i) and (ii) follow from this. □

**Claim 3.2.** $|L(S_1)| \geq 11$.

**Sketch of Proof.** If $|L(S_1)| \leq 10$, then by Claim 3.1 (i), $|V(G)| \leq 10m + |W| \leq 10m + 20|W| \leq 90$, which contradicts the assumption that $|V(G)| \geq 177$. □

**Claim 3.3.** $m \geq 2$ and $|L(S_2)| \geq 9$.

**Sketch of Proof.** Suppose that $m = 1$ or $|L(S_2)| \leq 8$. Then by Claim 3.1 (ii), $|V(G)| - L(S_1)| \leq 8(m - 1) + |W| \leq 96 - 16p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 88 - 16p_1$, which implies $|L(S_1)| \leq p_1(88 - 16p_1)$. Consequently $|V(G)| \leq p_1(88 - 16p_1) + 96 - 16p_1 < 177$ because $p_1$ is an integer, which contradicts the assumption that $|V(G)| \geq 177$. □

By Lemma 2.11, Claims 3.2 and 3.3 imply that $S_1$ does not mesh with $S_2$. Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1. Write $\mathcal{X}(S_1) - \mathcal{L}(S_1) = \{C_1\}$ and $\mathcal{X}(S_2) - \mathcal{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We define $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_{1,1}$, $\mathcal{T}_{1,2}$, $\mathcal{T}_{1,3}$, $\mathcal{T}_{2,1}$, $\mathcal{T}_{2,2}$, $\mathcal{T}_{2,3}$ as follows:

$$
\mathcal{T}_1 = \{T \in \mathcal{X}|L(T) \cap (S_1 \cup S_2) = \emptyset\},
\mathcal{T}_2 = \{T \in \mathcal{X}|L(T) \subseteq S_1 \cup S_2\},
\mathcal{T}_{1,1} = \{T \in \mathcal{X}|L(T) \subseteq L(S_1)\},
\mathcal{T}_{1,2} = \{T \in \mathcal{X}|L(T) \subseteq L(S_2)\},
\mathcal{T}_{1,3} = \{T \in \mathcal{X}|L(T) \subseteq V(C_1) \cap V(C_2)\},
\mathcal{T}_{2,1} = \{T \in \mathcal{X}|L(T) \subseteq S_1 - S_2\},
\mathcal{T}_{2,2} = \{T \in \mathcal{X}|L(T) \subseteq S_2 - S_1\},
\mathcal{T}_{2,3} = \{T \in \mathcal{X}|L(T) \subseteq S_1 \cap S_2\}.
$$

In view of the maximality of $L(S_1)$ and $L(S_2)$ and Claims 3.2 and 3.3, it follows from Lemmas 2.1 and 2.10 that $\mathcal{T}_1$ is the set of those members of $\mathcal{X}$ which mesh with neither
$S_1$ nor $S_2$, and $\mathcal{R}_2$ is the set of those members of $\mathcal{R}$ which mesh with $S_1$ or $S_2$. Thus $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ (disjoint union). Further $\mathcal{R}_1 = \mathcal{R}_{1,1} \cup \mathcal{R}_{1,2} \cup \mathcal{R}_{1,3}$ (disjoint union) and $\mathcal{R}_2 = \mathcal{R}_{2,1} \cup \mathcal{R}_{2,2} \cup \mathcal{R}_{2,3}$ (disjoint union).

The following two claims immediately follow from Lemma 2.5 (I) (see also [3;Claim 3.6]).

**Claim 3.4.** $|\mathcal{R}_{1,i}| \leq (2|L(S_i)| - 1)/3$ ($i = 1, 2$).

**Claim 3.5.** $|\mathcal{R}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

**Proof.** Let $I = \{1 \leq i \leq m|S_i \in \mathcal{R}_{1,3}\}$. Then by Lemma 2.5 (I), $|\mathcal{R}_{1,3}| = \sum_{i \in I} |\{T \in \mathcal{R}_1|L(T) \subseteq L(S_i)\}| \leq \sum_{i \in I} (2|L(S_i)| - 1)/3 \leq (2|V(C_1) \cap V(C_2)| - |I|)/3 \leq 2|V(C_1) \cap V(C_2)|/3$.  

Since $|L(T)| \leq 3$ for each $T \in \mathcal{R}_2$ by Lemma 2.10, the following claim follows from Lemmas 2.3 and 2.7 (see also [3;Claim 3.8]).

**Claim 3.6.**

(i) $|\mathcal{R}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathcal{R}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathcal{R}_{2,3}| \leq |S_1 \cap S_2|/2$.

**Claim 3.7.** $|S_1 \cap S_2|$ is even.

**Proof.** Suppose that $|S_1 \cap S_2|$ is odd. Then it follows from Claim 3.6 that $|\mathcal{R}_2| \leq (|S_1 \cup S_2| - 3)/2$, and it follows from Claims 3.4 and 3.5 that $|\mathcal{R}_1| \leq (2|V(G)| - |S_1 \cup S_2| - 2)/3$. Hence $|\mathcal{R}| \leq (2|V(G)| - (|S_1 \cup S_2| + 13)/2)/3$. Since $|S_1 \cup S_2| \geq 9$, this contradicts (3.1).

Write $|S_1 \cap S_2| = 2x$. Then $|S_1 \cup S_2| = 16 - 2x$. Hence it follows from Claim 3.6 that

$$|\mathcal{R}_2| \leq 8 - x,$$  

and it follows from Claims 3.4 and 3.5 that

$$|\mathcal{R}_1| \leq (2|V(G)| - 34 + 4x)/3.$$  

By (3.2) and (3.3), $|\mathcal{R}| \leq (2|V(G)| - 10 + x)/3$. In view of (3.1), this implies that equality holds in (3.2) (note that $x \leq 3$). Thus it follows from Claim 3.6 that

$$|\mathcal{R}_{2,1}| = 4 - x, |\mathcal{R}_{2,2}| = 4 - x, |\mathcal{R}_{2,3}| = x.$$  

(3.4)
Note that (3.4) implies that

\[
  \text{for each } i = 1, 2, \text{ there exist four members of } \mathcal{S} \text{ which mesh with } S_i. \tag{3.5}
\]

Hence by Claims 3.2, 3.3 and Lemma 2.21 (ii),

\[
|\mathcal{S}_{1,i}| \leq (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.6}
\]

Now it follows from (3.2), (3.6) and Claim 3.5 that $|\mathcal{S}| \leq (2|V(G)| - 12 + x)/3$. In view of (3.1), this implies that $x = 3$ and equality holds in (3.6), i.e.,

\[
|\mathcal{S}_{1,i}| = (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.7}
\]

In what follows, we mainly consider $S_1$. By (3.7), it follows from Lemma 2.5 (III) that $|\mathcal{L}(S_1)| = 2$. Write $\mathcal{L}(S_1) = \{F_1, F_2\}$. By Lemma 2.5 (III), we may assume $|\{T \in \mathcal{S} | L(T) \subseteq V(F_1)\}| = (2|V(F_1)| - 2)/3$ and $|\{T \in \mathcal{S} | L(T) \subseteq V(F_2)\}| = (2|V(F_2)| - 3)/3$.

Then by (3.5) and Lemma 2.21 (i), $|V(F_1)| \leq 4$. Since $|L(S_1)| \geq 11$ by Claim 3.2, this implies $|V(F_2)| \geq 7$. Set $\mathcal{S} = \{T \in \mathcal{S}_{1,i} | L(T) \subseteq V(F_2)\}$, and set $\mathcal{S}_M = \{T \in \mathcal{S} | L(T) \subseteq V(M)\}$ for each $M \in (\bigcup_{T \in \mathcal{S}} \mathcal{L}(T)) \cup \{F_2\}$. Then

\[
|\mathcal{S}_M| \geq (2|V(M)| - 3)/3 \tag{3.8}
\]

for each $M \in (\bigcup_{T \in \mathcal{S}} \mathcal{L}(T)) \cup \{F_2\}$ by Lemma 2.6 (II) (i). Since $|V(F_2)| \geq 7$, it follows from Lemma 2.4 (III) that there exists $T \in \mathcal{S}$ such that $|L(T)| \neq 2$. Choose $T_0 \in \mathcal{S}$ with $|L(T_0)| \neq 2$ so that $|L(T_0)|$ is as small as possible. Since $|\mathcal{S}| = (2|V(F_2)| - 3)/3$, $|\mathcal{L}(T_0)| = 2$ by Lemma 2.6 (II) (ii). Write $\mathcal{L}(T_0) = \{M_1, M_2\}$ with $|V(M_i)| \leq |V(M_2)|$. Let $Q_1, \ldots, Q_l$ be the maximal members of $\mathcal{S}_{M_2}$. By the minimality of $|L(T_0)|$, $|L(Q_i)| = 2$ for each $1 \leq i \leq l$.

**Claim 3.8.** $M_2$ is saturated and $|V(M_2)| = 4$ or 6.

**Proof.** Since $|L(T_0)| \neq 2$, $|V(M_2)| \neq 1$. Hence $M_2$ is saturated by (3.5) and Lemma 2.20 (i). By (3.8) and (II) and (III) of Lemma 2.4, $2 \leq l \leq 3$. Hence $|V(M_2)| = 4$ or 6. \hfill \Box

Choose $R \in \mathcal{S} \cup \{S_1\}$ with $R \neq T_0$ and $T_0 \subseteq R$ so that $R$ is minimal, and take $M \in \mathcal{L}(R)$ such that $L(T_0) \subseteq V(M)$. By (3.5) and Lemma 2.20 (i), $M$ is saturated. Let $T_1, \ldots, T_s$ be the maximal members of $\mathcal{S}_M$. By (3.8) and (II) and (III) of Lemma 2.4,

\[
2 \leq s \leq 3. \tag{3.9}
\]

By the minimality of $R$, $T_0 \in \{T_1, \ldots, T_s\}$. We may assume $T_0 = T_s$. Now since $|L(R)| \neq 2$, applying Lemma 2.20 (ii) with $S = S_1, F = F_2$ and $T = R$, we see from (3.5)

there exist $P_1, P_2, P_3, P_4 \in \mathcal{S}$ such that $P_i$ meshes with $R$

for each $1 \leq i \leq 4 \tag{3.10}$
and such that
\[ |L(P_i)| = 2 \text{ for each } 1 \leq i \leq 4 \text{ and } R = \bigcup_{1 \leq i \leq 4} L(P_i). \]  
(3.11)

**Claim 3.9.** \( |V(M_1)| = 1 \)

*Proof.* Set \( \mathcal{A}_i = \{ T \in \mathcal{J} | L(T) \subseteq L(T_i) \} \) for each \( 1 \leq i \leq s \). Recall that \( T_s = T_0 \) and \( |L(T_0)| \neq 2 \). Thus \( |L(T_i)| = 2 \) for each \( 1 \leq i \leq s - 1 \) by (3.8), (3.10) and Lemma 2.19 (III) (i). Hence \( \mathcal{A}_i = \{ T_i \} \) and
\[ |\mathcal{A}_i| = (2|L(T_i)| - 1)/3 \]  
(3.12)
for each \( 1 \leq i \leq s - 1 \). Since \( |L(T_s)| \neq 2 \), applying Lemma 2.20 (ii) with \( S = S_1, F = F_2 \) and \( T = T_s \), we see from (3.5) that there exist \( R_1, R_2, R_3, R_4 \in \mathcal{J} \) such that \( R_i \) meshes with \( T_s \) for each \( 1 \leq i \leq 4 \)
(3.13)
and such that
\[ |L(R_i)| = 2 \text{ for each } 1 \leq i \leq 4 \text{ and } T_s = \bigcup_{1 \leq i \leq 4} L(R_i). \]  
(3.14)

Note that \( T_s = N_G(L(T_s)) \subseteq R \cup \bigcup_{1 \leq i \leq s-1} L(T_i) \). Thus \( T_s \subseteq \bigcup_{1 \leq i \leq 4} L(P_i) \cup \bigcup_{1 \leq i \leq s-1} L(T_i) \) by (3.11). In view of (3.14), this implies
\[ \{ R_1, R_2, R_3, R_4 \} \subseteq \{ P_1, P_2, P_3, P_4, T_1, T_2, \ldots, T_{s-1} \}. \]  
(3.15)

Suppose that \( |V(M_1)| \neq 1 \). Then by (3.10), it follows from Lemma 2.19 (IV) that \( |\mathcal{J}_M| \leq (2|V(M)| - 3)/3 \). Hence by (3.8),
\[ |\mathcal{J}_M| = (2|V(M)| - 3)/3. \]  
(3.16)

By (3.5) and Lemma 2.20 (i), \( M_1 \) is saturated. Let \( Q'_1, \ldots, Q'_{l'} \) be the maximal members of \( \mathcal{J}_M \). By (3.8) and (II) and (III) of Lemma 2.4, \( 2 \leq l' \leq 3 \). \( |L(Q'_1)| = \cdots = |L(Q'_{l'})| = 2 \). Recall that \( |L(Q_1)| = \cdots = |L(Q_l)| = 2 \), and we have \( 2 \leq l \leq 3 \) by Claim 3.8.

**Case 1.** \( l = 2 \).

Since \( |V(M_1)| \leq |V(M_2)| \), we have \( l' \leq l \), and hence \( l' = 2 \). This implies \( \mathcal{A}_s = \{ T_s, Q_1, Q_2, Q'_1, Q'_2 \} \), and hence \( |\mathcal{A}_s| = (2|L(T_s)| - 1)/3 \). Since \( |\mathcal{J}_M| = |\mathcal{A}_1| + \cdots + |\mathcal{A}_s| \), it now follows from (3.9), (3.12) and (3.16) that \( s = 3 \). For each \( 1 \leq i \leq 2 \), \( |Q_i| = 8 \) and, if we write \( \{ 1, 2 \} = \{ i, j \} \), then \( Q_i = N_G(L(Q_i)) \subseteq T_s \cup L(Q_j) = (\bigcup_{1 \leq h \leq 4} L(R_h)) \cup L(Q_j) \) by (3.14). Hence for each \( 1 \leq i \leq 2 \),
\[ \text{at least three of } R_1, R_2, R_3 \text{ and } R_4 \text{ mesh with } Q_i. \]  
(3.17)
by Lemma 2.1 (see the proof of (2.4) and (2.10)). Similarly, for each $1 \leq i \leq 2$, at least three of $R_1$, $R_2$, $R_3$ and $R_4$ mesh with $Q_i'$.

By (3.17), at least two of $R_1$, $R_2$, $R_3$ and $R_4$ mesh with $Q_1$ and $Q_2$. We may assume that $R_i$ meshes with $Q_1$ and $Q_2$ for each $1 \leq i \leq 2$. If $R_i \in \{P_1, P_2, P_3, P_4\}$ for some $1 \leq i \leq 2$, then $|R_i \cap L(R)| \geq |R_i \cap V(M)| + 1 \geq |R_i \cap L(Q_1)| + |R_i \cap L(Q_2)| + |R_i \cap V(M_1)| + 1 \geq 2 + 2 + 2 + 1 = 7$ by Lemma 2.15, which contradicts Lemma 2.17 (note that $R_i \in \{P_1, P_2, P_3, P_4\}$ meshes with $R$ by (3.10)). Thus

$$R_i \in \{T_1, T_2\} \text{ for each } 1 \leq i \leq 2$$

by (3.15). Arguing similarly by using (3.18) in place of (3.17), we see that there exist $R_1', R_2' \in \{R_1, R_2, R_3, R_4\}$ such that $R_1'$ meshes with $Q_1'$ and $Q_2'$ for each $1 \leq i \leq 2$ and

$$R_i' \in \{T_1, T_2\} \text{ for each } 1 \leq i \leq 2$$

by (3.15). Now by (3.19) and (3.20), $T_1$ meshes with $Q_1$, $Q_2$, $Q_1'$ and $Q_2'$, and hence $|T_1 \cap L(T_3)| = |T_1 \cap V(M_1)| + |T_1 \cap V(M_2)| = 4 + 4 = 8$, which contradicts Lemma 2.17 (note that $T_1 \in \{R_1, R_2, R_3, R_4\}$ meshes with $T_3$ by (3.13)).

**Case 2.** $l = 3$.

Since $|V(M_2)| = 6$ and $\mathcal{M}_2 = \{Q_1, Q_2, Q_3\}$, $|\mathcal{M}_2| = (2|V(M_2)| - 3)/3$. Note that $|\mathcal{A}_s| \geq (2|L(T_s)| - 2)/3$ by (3.16) and Lemma 2.4 (III). We now apply Lemma 2.5 to $\mathcal{A}_s$. Since $|\mathcal{M}_2| = (2|V(M_2)| - 3)/3$, $|\mathcal{A}_s| \neq (2|L(T_s)| - 1)/3$ by Lemma 2.5 (II).

Hence $|\mathcal{A}_s| = (2|L(T_s)| - 2)/3$. Again since $|\mathcal{M}_2| = (2|V(M_2)| - 3)/3$, it follows from Lemma 2.5 (III) that $|\mathcal{M}_1| = (2|V(M_1)| - 2)/3$, and hence $l' = 2$. In view of (3.9), (3.12) and (3.16), we also get $s = 2$. Since $l' = 2$, (3.18) again holds. Since $s = 2$, this together with (3.15) implies that there exists $P \in \{P_1, P_2, P_3, P_4\}$ such that $P$ meshes with $Q_1'$ and $Q_2'$. Now by Lemma 2.15, $|P \cap L(R)| \geq |P \cap V(M)| + 1 \geq |P \cap L(Q_1')| + |P \cap L(Q_2')| + |P \cap V(M_2)| + 1 \geq 2 + 2 + 2 + 1 = 7$, which contradicts Lemma 2.17. This completes the proof of Claim 3.9.

We are now in a position to complete the proof of the Theorem. Recall that $|L(Q_1)| = \cdots = |L(Q_3)| = 2$. By (3.8) and Lemma 2.4, we may assume that $T_1$ meshes with $T_s$. Then $|T_1 \cap V(M_1)| = 1$ by Claim 3.9. On the other hand, $|T_1 \cap V(M_2)|$ is even by Lemma 2.13. Therefore $|T_1 \cap L(T_s)| = |T_1 \cap V(M_2)| + |T_1 \cap V(M_1)|$ is odd. In view of (3.10), this contradicts Lemma 2.19 (III) (iii).

This completes the proof of the Theorem.
References


