AKCE J. Graphs. Combin., 5, No. 2 (2008), pp. 147-164

8-Shredders in 8-Connected Graphs

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Communicated by: Mariko Hagita
Received 30 October 2007; revised 02 May 2008; accepted 29 August 2008

Abstract

For a graph G, a subset S of $V(G)$ is called a shredder if $G-S$ consists of three or more components. We show that if G is an 8-connected graph of order at least 177, then the number of shredders of cardinality 8 of G is less than or equal to $(2|V(G)|-10)/3$.

Keywords: Graph; connectivity; shredder; upper bound

2000 Mathematics Subject Classification: 05C40

1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to x in G. For $A \subseteq V(G)$, we let $N_G(A) = (\bigcup_{x \in A} N_G(x)) - A$.

A subset S of $V(G)$ is called a *shredder* if $G-S$ consists of three or more components. A shredder of cardinality k is referred to as a k-shredder. The notion of a shredder was introduced by Cheriyan and Thurimella in [1] in connection with the problem of connectivity augmentation. In the algorithm of augmenting the connectivity of a k-connected graph which was given in [1], each k-shredder makes longer the running time of the algorithm. Thus it is important to obtain an upper bound on the number of k-shredders in a k-connected graph. In [2;Theorem 1], it is proved that if $k \geq 5$ and G is a k-connected graph, then the number of k-shredders of G is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible. For $k = 5$, it is shown in [3;Theorem 3] that if G is a 5-connected graph of order at least 135, then the number of 5-shredders of G is less than or equal to $(2|V(G)|-10)/3$; for $k = 6$, it is shown in [6] that if G is a 6-connected graph of order at least 325, then the number of 6-shredders of G is less than or equal to $(2|V(G)|-9)/3$; for $k = 7$, it is shown in [5] that if G is a 7-connected graph of order at least 41, then the number of 7-shredders of G is less than or equal to $(2|V(G)|-8)/3$. It is also shown that each of these three bounds is attained by infinitely many graphs (for results concerning the case where $1 \le k \le 4$, the reader is referred to [4] and [2;Theorem 2]). In this paper, we prove:

Theorem. *Let G be an 8-connected graph of order at least 177. Then the number of 8-shredders of G is less than or equal to $(2|V(G)|-10)/3$.*

The greater part of our proof of the Theorem consists of modifications of arguments used in [3] and [6]. However, we here mention that the argument used in the proof of Lemma 2.21 and Claim 3.9 is new, and there is no corresponding argument in [3] or [6].

We now construct an infinite family of graphs G'_m which attain the bound $(2|V(G'_m)|-8)/3$ in the Theorem. Let $m \ge 8$. Define an auxiliary graph H_m of order m by letting

$$\begin{aligned} V(H_m) &= \{v_i | 1 \le i \le m\}, \\ E(H_m) &= \{v_i v_{i+3} | 1 \le i \le m-3\} \\ &\cup \{v_1v_2,\ v_1v_3,\ v_2v_3,\ v_{m-2}v_{m-1},\ v_{m-2}v_m,\ v_{m-1}v_m\}. \end{aligned}$$

We define a graph G_m of order $3m-5$ by adding $m-5$ vertices to the so-called lexicographic product of H_m and the null graph of order 2. More precisely, we let

$$\begin{aligned} V(G_m) &= \{x_{i,j} | 1 \le i \le m,\ 1 \le j \le 2\} \cup \{\alpha_i | 4 \le i \le m-3\} \cup \{a\}, \\ E(G_m) &= \{x_{i,j}x_{i+3,k} | 1 \le i \le m-3,\ 1 \le j,k \le 2\} \\ &\cup \{x_{i-1,j}\alpha_i,\ x_{i,j}\alpha_i,\ x_{i+1,j}\alpha_i | 4 \le i \le m-3,\ 1 \le j \le 2\} \\ &\cup \{x_{i,j}a,\ x_{k,j}a | 1 \le i \le 2,\ m-1 \le k \le m,\ 1 \le j \le 2\} \\ &\cup \{x_{4,j}a,\ x_{m-3,j}a | 1 \le j \le 2\} \\ &\cup \{\alpha_i a | 4 \le i \le m-3\} \\ &\cup \{x_{1,j}x_{2,k},\ x_{1,j}x_{3,k},\ x_{2,j}x_{3,k} | 1 \le j,k \le 2\} \\ &\cup \{x_{m-2,j}x_{m-1,k},\ x_{m-2,j}x_{m,k},\ x_{m-1,j}x_{m,k} | 1 \le j,k \le 2\}. \end{aligned}$$

We define a graph G'_m of order $3m-4$ by

$$\begin{aligned} V(G'_m) &= V(G_m) \cup \{b\} \\ E(G'_m) &= E(G_m) \cup \{bv \mid v \in V(G_m)\}. \end{aligned}$$

Then G'_m is 8-connected, and has $2m-6$ 8-shredders

$\{b, x_{i,1}, x_{i,2}, x_{i+1,1}, x_{i+1,2}, x_{i+2,1}, x_{i+2,2}, a\}$ $(2 \le i \le m-3)$,

$\{b, x_{i-3,1}, x_{i-3,2}, x_{i+3,1}, x_{i+3,2}, \alpha_{i-1}, \alpha_i, \alpha_{i+1}\}$ $(5 \le i \le m-4)$,

$\{b, x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{5,1}, x_{5,2}, a\}$,

$\{b, x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{6,1}, x_{6,2}, \alpha_4\}$,

$\{b, x_{1,1}, x_{1,2}, x_{7,1}, x_{7,2}, \alpha_4, \alpha_5, a\}$,

$\{b, x_{m-6,1}, x_{m-6,2}, x_{m,1}, x_{m,2}, \alpha_{m-4}, \alpha_{m-3}, a\}$,

$\{b, x_{m-5,1}, x_{m-5,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-3}\}$,

$\{b, x_{m-4,1}, x_{m-4,2}, x_{m-2,1}, x_{m-2,2}, x_{m,1}, x_{m,2}, a\}$.

Thus the number of 8-shredders of G'_m is $2m - 6 = (2(3m - 4) - 10)/3 = (2|V(G'_m)| - 10)/3$.

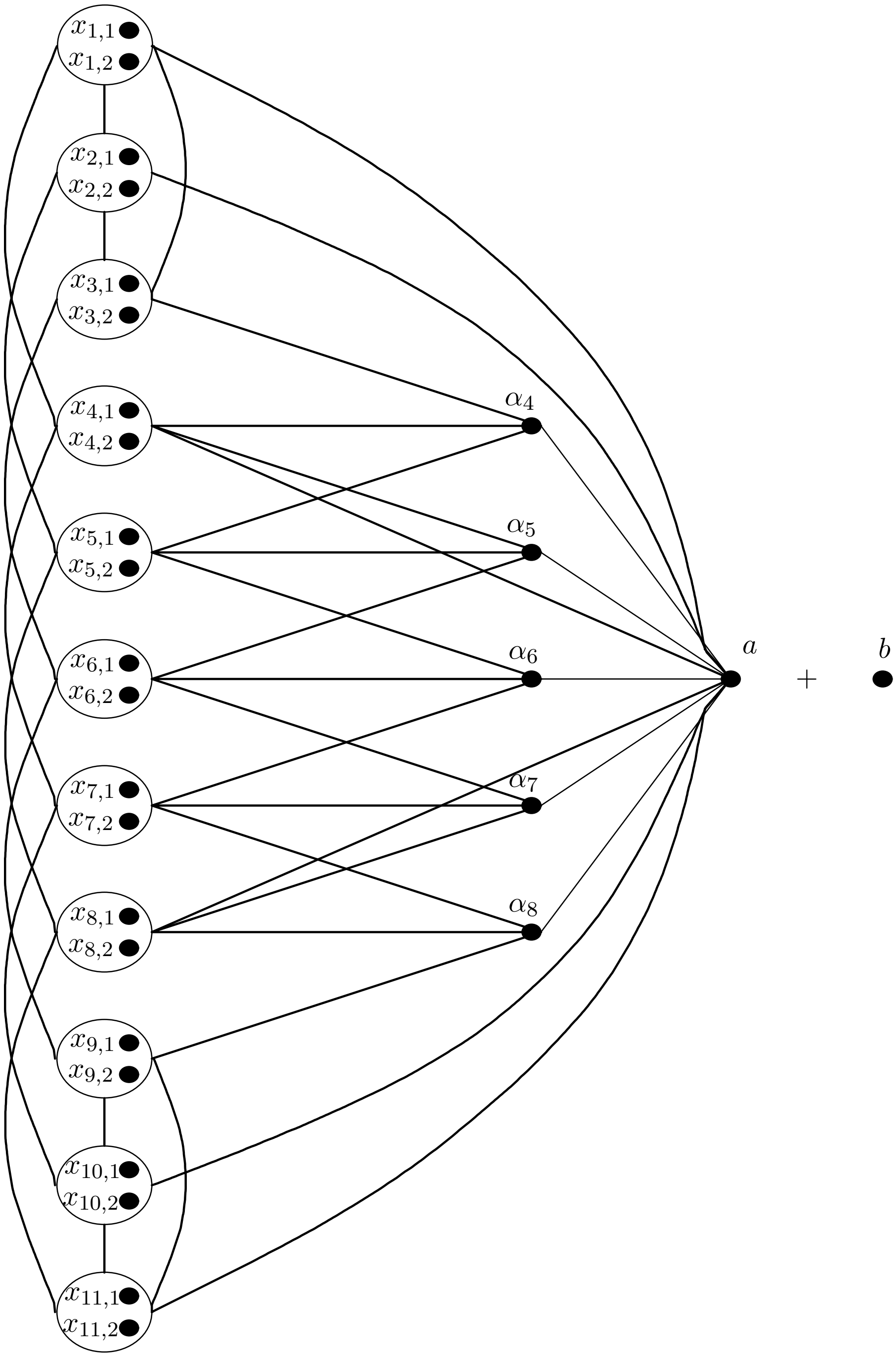

Figure 1: $m = 11$

2. Preliminary Results

Throughout the rest of this paper, let G be an 8-connected graph, and let $\mathscr{S}$ denote the set of 8-shredders of G. For each $S \in \mathscr{S}$, we define $\mathscr{K}(S)$, $\mathscr{L}(S)$ and $L(S)$ as follows. We let $\mathscr{K}(S)$ denote the set of componetns of $G - S$. Write $\mathscr{K}(S) = \{H_1, \cdots, H_s\}$ $(s = |\mathscr{K}(S)|)$. We may assume $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S) = \mathscr{K}(S) - \{H_1\}$ and $L(S) = \bigcup_{2 \leq i \leq s} V(H_i)$; thus $L(S) = \bigcup_{C \in \mathscr{L}(S)} V(C)$. Now let $\mathscr{L} = \bigcup_{S \in \mathscr{S}} \mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L} - \{F\}$ such that $V(F) = \bigcup_{C \in \mathscr{C}} V(C)$.

Let $S, T \in \mathscr{S}$ with $S \neq T$. We say that S *meshes* with T if S intersects with at least two members of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1;Lemma 4.3 (1)]).

The following two lemmas are proved in [4;Lemmas 2.1 and 3.1] (see also [2;Lemmas 3.2 and 3.4]).

Lemma 2.1. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S does not mesh with T. Then one of the following holds:*

(i) $L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, *and no edge of G joins a vertex in $L(S)$ and a vertex in $L(T)$;*

(ii) *there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or*

(iii) *there exists $D \in \mathscr{L}(T)$ such that $V(D) \supseteq L(S)$ (so $L(T) \supseteq L(S)$).*

Lemma 2.2. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S meshes with T. Then $S \supseteq L(T)$ or $T \supseteq L(S)$.*

The following lemma is proved in [2;Lemma 3.6].

Lemma 2.3. *Let $F \in \mathscr{L}$, and suppose that F is saturated. Then $|V(F)| \geq 4$.*

The following lemmas are proved in [3;Lemma 2.8 and 2.9]

Lemma 2.4. *Let $F \in \mathscr{L}$, and set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq V(F)\}$. Then the following hold.*

(I) $|\mathscr{T}| \leq (2|V(F)| - 2)/3$.

(II) *If $|\mathscr{T}| = (2|V(F)| - 2)/3$, then one of the following holds:*

  (i) *F is trivial (i.e., $|V(F)| = 1$); or*

(ii) F *is saturated, and there exist* $T_1, T_2 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2)$, T_1 *meshes with* T_2, $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = 2$, *and* $|\{T \in \mathscr{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ *for each* $i = 1, 2$.

(III) *If* $|\mathscr{T}| = (2|V(F)| - 3)/3$, *then one of the following holds:*

(i) F *is saturated, and there exist* $T_1, T_2 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2)$, T_1 *meshes with* T_2, $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = 2$, *and* $|\{T \in \mathscr{T} | L(T) \subseteq L(T_1)\}| = (2|L(T_1)| - 1)/3$ *and* $|\{T \in \mathscr{T} | L(T) \subseteq L(T_2)\}| = (2|L(T_2)| - 2)/3$.

(ii) F *is saturated, and there exist* $T_1, T_2, T_3 \in \mathscr{T}$ *such that* $V(F) = L(T_1) \cup L(T_2) \cup L(T_3)$, T_3 *meshes with* T_1 *and* T_2, $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = |\mathscr{L}(T_3)| = 2$, *and* $|\{T \in \mathscr{T} | L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ *for each* $i = 1, 2, 3$ *; or*

(iii) F *is not saturated, and there exists* $T_0 \in \mathscr{T}$ *such that* $|L(T_0)| = |V(F)| - 1$, $|\mathscr{L}(T_0)| = 2$, *and* $|\{T \in \mathscr{T} | L(T) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$.

Lemma 2.5. *Let* $S \in \mathscr{S}$, *and write* $\mathscr{L}(S) = \{F_1, \cdots, F_p\}$ $(p = |\mathscr{L}(S)|)$. *Set* $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq L(S)\}$, *and set* $\mathscr{T}_i = \{T \in \mathscr{T} | L(T) \subseteq V(F_i)\}$. *Then the following hold.*

(I) $|\mathscr{T}| \le (2|L(S)| - 2p + 3)/3 \le (2|L(S)| - 1)/3$.

(II) *If* $|\mathscr{T}| = (2|L(S)| - 1)/3$, *then* $p = 2$ *and* $|\mathscr{T}_i| = (2|V(F_i)| - 2)/3$ *for each* i.

(III) *If* $|\mathscr{T}| = (2|L(S)| - 2)/3$, *then* $p = 2$, *and either* $|\mathscr{T}_1| = (2|V(F_1)| - 2)/3$ *and* $|\mathscr{T}_2| = (2|V(F_2)| - 3)/3$, *or* $|\mathscr{T}_1| = (2|V(F_1)| - 3)/3$ *and* $|\mathscr{T}_2| = (2|V(F_2)| - 2)/3$.

We define an order relation $\le$ in $\mathscr{S}$ as follows:

$$S \le T \Longleftrightarrow L(S) \subseteq L(T) \ (S, T \in \mathscr{S}).$$

The following lemma follows from Lemmas 2.4 and 2.5.

Lemma 2.6. *Let* $F \in \mathscr{L}$, *and suppose that* $|V(F)| \ne 1$. *Set* $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq L(F)\}$, *and set* $\mathscr{T}_M = \{T \in \mathscr{T} | L(T) \subseteq V(M)\}$ *for each* $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$ *(thus* $\mathscr{T} = \mathscr{T}_F$ *).*

(I) *Suppose that* $|\mathscr{T}| = (2|V(F)| - 2)/3$. *Then the following hold.*

(i) $|\mathscr{T}_M| = (2|V(M)| - 2)/3$ *for each* $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$.

(ii) $|\mathscr{L}(T)| = 2$ *for each* $T \in \mathscr{T}$.

(II) *Suppose that* $|\mathscr{T}| = (2|V(F)| - 3)/3$. *Then the following hold.*

(i) $|\mathscr{T}_M| \ge (2|V(M)| - 3)/3$ *for each* $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$.

(ii) $|\mathscr{L}(T)| = 2$ *for each* $T \in \mathscr{T}$.

Proof. We prove (I) and (II) simultaneously. Assume $|\mathscr{T}| \geq (2|V(F)|-3)/3$. Let $T_1, \cdots, T_s$ be the maximal members of $\mathscr{T}$ (we have $1 \leq s \leq 3$ by Lemma 2.4). Note that in both (I) and (II), (i) clearly holds for $M = F$. Hence the lemma will follow if we show that the following holds for each $T \in \mathscr{T}$:

$$|\mathscr{L}(T)| = 2 \text{ and, for each } M \in \mathscr{L}(T),\ |\mathscr{T}_M| \geq (2|V(M)|-3)/3 \text{ and we have } |\mathscr{T}_M| = (2|V(M)|-2)/3 \text{ in the case where } |\mathscr{T}| = (2|V(F)|-2)/3. \tag{2.1}$$

We prove (2.1) by backward induction on $|L(T)|$. First let $T \in \{T_1, \cdots, T_s\}$. Then it follows from Lemma 2.4 that $|\mathscr{L}(T)| = 2$, $|\{R \in \mathscr{T} \,|L(R) \subseteq L(T)\}| \geq (2|L(T)|-2)/3$, and we have $|\{R \in \mathscr{T} \,|L(R) \subseteq L(T)\}| = (2|L(T)|-1)/3$ if $|\mathscr{T}| = (2|V(F)|-2)/3$. Also it follows from Lemma 2.5 that for each $M \in \mathscr{L}(T)$, $|\mathscr{T}_M| \geq (2|V(M)|-3)/3$, and we have $|\mathscr{T}_M| = (2|V(M)|-2)/3$ if $|\mathscr{T}| = (2|V(F)|-2)/3$. This shows that (2.1) holds for $T \in \{T_1, \cdots, T_s\}$. Thus we may assume $T \notin \{T_1, \cdots, T_s\}$. Then there exists $S \in \mathscr{T}$ with $S \neq T$ such that $T \leq S$ (i.e., $L(T) \subseteq L(S)$). We choose such S so that S is minimal. Then by Lemmas 2.1 and 2.2, (S does not mesh with T and) there exists $M' \in \mathscr{L}(S)$ such that $L(T) \subseteq V(M')$. By the induction hypothesis, $|\mathscr{T}_{M'}| \geq (2|V(M')|-3)/3$, and we have $|\mathscr{T}_{M'}| = (2|V(M')|-2)/3$ if $|\mathscr{T}| = (2|V(F)|-2)/3$. Let $T_1', \cdots, T_{s'}'$ be the maximal members of $\mathscr{T}_{M'}$. Then $T \in \{T_1', \cdots, T_{s'}'\}$ by the minimality of S. Consequently, applying Lemma 2.4 to M', we see that $|\mathscr{L}(T)| = 2$, $|\{R \in \mathscr{T} \,|L(R) \subseteq L(T)\}| \geq (2|L(T)|-2)/3$, and we have $|\{R \in \mathscr{T} \,|L(R) \subseteq L(T)\}| = (2|L(T)-1)/3$ if $|\mathscr{T}| = (2|V(F)|-2)/3$. Also it follows from Lemma 2.5 that for each $M \in \mathscr{L}(T)$, $|\mathscr{T}_M| \geq (2|V(M)|-3)/3$, and we have $|\mathscr{T}_M| = (2|V(M)|-2)/3$ if $|\mathscr{T}| = (2|V(F)|-2)/3$. Thus (2.1) is proved, which completes the proof of the lemma. □

The following lemmas are proved in [3;Lemmas 2.10 through 2.12].

Lemma 2.7. *Let* $X \subseteq V(G)$. *Set* $\mathscr{T} = \{T \in \mathscr{S} \,|L(T) \subseteq X\}$ *and* $\mathscr{L}_0 = \cup_{T \in \mathscr{T}} \mathscr{L}(T)$, *and suppose that no component in* $\mathscr{L}_0$ *is saturated. Then* $|\mathscr{T}| \leq |X|/2$.

Lemma 2.8. *Let* $S, T \in \mathscr{S}$, *and suppose that* S *meshes with* T *and* $L(S) \nsubseteq T$. *Then* $L(T) \subseteq S$ *and* $|L(T)| \leq 3$.

Lemma 2.9. *Suppose that* $|V(G)| \geq 17$. *Let* $S, T \in \mathscr{S}$, *and suppose that* S *meshes with* T, $L(S) \subseteq T$ *and* $L(T) \subseteq S$. *Then* $|L(S)| + |L(T)| \leq 8$.

The following lemma follows from Lemmas 2.8 and 2.9.

Lemma 2.10. *Suppose that* $|V(G)| \geq 17$. *Let* $S, T \in \mathscr{S}$, *and suppose that* S *meshes with* T *and* $|L(S)| \geq 5$. *Then* $L(T) \subseteq S$ *and* $|L(T)| \leq 3$.

As an immediate corollary of Lemma 2.10, we obtain the following lemma.

Lemma 2.11. *Suppose that* $|V(G)| \geq 17$*. Let* $S, T \in \mathscr{S}$ *with* $S \neq T$*, and suppose that either* $|L(S)| \geq 4$ *and* $|L(T)| \geq 5$*, or* $|L(S)| \geq 5$ *and* $|L(T)| \geq 4$*. Then* S *does not mesh with* T*.*

Lemma 2.12. *Let* $S \in \mathscr{S}$*, and suppose that* $|L(S)| \neq 2$ *and* $|\{T \in \mathscr{S} \mid L(T) \subseteq L(S)\}| = (2|L(S)| - 1)/3$*. Then* $|L(S)| \geq 5$*.*

Proof. Since $|\{T \in \mathscr{S} \,|\, L(T) \subseteq L(S)\}|$ is an integer, we have $|L(S)| \neq 3, 4$, and hence $|L(S)| \geq 5$. □

Lemma 2.13. *Let* $S, T \in \mathscr{S}$ *with* $S \neq T$*, and suppose that* $|L(T)| = 2$*. Then either* $S \supseteq L(T)$ *or* $S \cap L(T) = \emptyset$*.*

Proof. If S meshes with T, then from the fact that S intersects with each member of $\mathscr{L}(T)$, it follows that $S \supseteq L(T)$. Thus we may assume S does not mesh with T. Then (i) or (ii) of Lemma 2.1 holds. If (i) holds, then we clearly have $S \cap L(T) = \emptyset$; if (ii) holds, then $S \cap L(T) \subseteq S \cap L(S) = \emptyset$. □

Lemma 2.14. *Let* $S, T \in \mathscr{S}$ *with* $S \neq T$*, and suppose that* $L(S) \supseteq L(T)$*. Then* $|L(S)| \geq 4$*.*

Proof. By Lemma 2.2, S does not mesh with T, and hence it follows from Lemma 2.1 that there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$. Since C is connected, $V(C) \neq L(T)$, and hence $|V(C)| \geq |L(T)| + 1 \geq 3$. Consequently $|L(S)| \geq |V(C)| + 1 \geq 4$. □

Lemma 2.15. *Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T*. Let* $F \in \mathscr{K}(S)$*, and suppose that* $|V(F)| \geq 2$*. Then* $|T \cap V(F)| \geq 2$*.*

Proof. If $V(F) \subseteq T$, then we clearly have $|T \cap V(F)| = |V(F)| \geq 2$. Thus we may assume $V(F) \not\subseteq T$. Then there exists $D \in \mathscr{K}(T)$ such that $V(F) \cap V(D) \neq \emptyset$. Since S meshes with T, we have $|S - (T \cup V(D))| = \sum_{C \in \mathscr{K}(T) - \{D\}} |S \cap V(C)| \geq 2$. Set $R = (T \cap V(F)) \cup (S \cap (T \cup V(D)))$. Then R separates $V(F) \cap V(D)$ from the rest. This imlies $|R| \geq 8$, and hence $|T \cap V(F)| = |R| - |S \cap (T \cup V(D))| \geq 8 - |S \cap (T \cup V(D))| = |S - (T \cup V(D))| \geq 2$. □

Lemma 2.16. *Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T *and* $|L(S)| \geq 3$*. Then* $|T \cap L(S)| \geq 3$*.*

Proof. If $L(S) \subseteq T$, then $|T \cap L(S)| = |L(S)| \geq 3$; if $L(S) \not\subseteq T$, then there exists $F \in \mathscr{L}(S)$ with $|V(F)| \geq 2$, and hence $|T \cap L(S)| \geq |T \cap V(F)| + 1 \geq 2 + 1 = 3$ by Lemma 2.15. □

Lemma 2.17. *Suppose that* $|V(G)| \geq 17$*. Let* $S, T \in \mathscr{S}$*, and suppose that* S *meshes with* T*. Then* $|T \cap L(S)| \leq 6$*.*

Proof. Suppose that $|T \cap L(S)| \geq 7$. Write $\mathscr{K}(S) - \mathscr{L}(S) = \{C\}$. If $|V(C)| \geq 2$, then $|T| \geq |T \cap V(C)| + |T \cap L(S)| \geq 2 + 7 = 9$ by Lemma 2.15, which contradicts the fact that $|T| = 8$. Thus $|V(C)| = 1$. By the definition of $\mathscr{L}(S)$, this implies $|V(F)| = 1$ for all $F \in \mathscr{K}(S)$, and hence $V(F) \subseteq T$ for all $F \in \mathscr{K}(S)$ by the assumption that S meshes with T. Hence $|V(G)| = |S| + \sum_{F \in \mathscr{K}(S)} |V(F)| \leq |S| + |T| = 16$, a contradiction. □

Lemma 2.18. *Suppose that $|V(G)| \geq 17$. Let $F \in \mathscr{L}$, and set $\mathscr{T} = \{T \in \mathscr{S} \,|\, L(T) \subseteq V(F)\}$. Suppose that $|\mathscr{T}| \geq (2|V(F)| - 3)/3$, and there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$ with $|L(T_1)| \leq |L(T_2)|$. Then $|L(T_1)| = 2$.*

Proof. Set $\mathscr{T}_1 = \{T \in \mathscr{S} \mid L(T) \subseteq L(T_1)\}$ and $\mathscr{T}_2 = \{T \in \mathscr{S} \mid L(T) \subseteq L(T_2)\}$. Suppose that $|L(T_1)| \geq 3$. By Lemma 2.4, T_1 meshes with T_2. Hence $3 \leq |L(T_1)| \leq 4$ by Lemma 2.11. By Lemma 2.4, we have $|\mathscr{T}_1| = (2|L(T_1)| - 1)/3$ or $(2|L(T_1)| - 2)/3$ and, if $|\mathscr{T}_1| = (2|L(T_1)| - 2)/3$, then $|\mathscr{T}_2| = (2|L(T_2)| - 1)/3$. Since $|\mathscr{T}_1|$ and $|\mathscr{T}_2|$ are integers, this forces $|L(T_1)| = 4$, $|\mathscr{T}_1| = (2|L(T_1)| - 2)/3$ and $|\mathscr{T}_2| = (2|L(T_2)| - 1)/3$, and hence $|L(T_2)| \geq 5$ by Lemma 2.12. But this contradicts Lemma 2.11. □

Lemma 2.19. *Suppose that $|V(G)| \geq 17$. Let $S \in \mathscr{S}$ and $F \in \mathscr{L}(S)$, and suppose that $|V(F)| \geq 2$. Set $\mathscr{T} = \{T \in \mathscr{S} \,|\, L(T) \subseteq V(F)\}$, and let $T_1, \cdots, T_s$ be the maximal members of $\mathscr{T}$ with $|L(T_1)| \leq |L(T_2)| \leq \cdots \leq |L(T_s)|$. Suppose that $|\mathscr{T}| \geq (2|V(F)| - 3)/3$, and there exist four members P_1, P_2, P_3, P_4 of $\mathscr{S}$ which mesh with S. In the case where $|V(F)| = 3$, suppose further that $|L(P_i)| = 2$ for each $1 \leq i \leq 4$. Then the following hold.*

(I) $|L(P_i)| = 2$ *for each* $1 \leq i \leq 4$ *and* $S = \bigcup_{1 \leq i \leq 4} L(P_i)$.

(II) F *is saturated (so* $|V(F)| \geq 4$ *by Lemma 2.3).*

(III) $2 \leq s \leq 3$.

- (i) $|L(T_i)| = 2$ *for each* $1 \leq i \leq s - 1$.
- (ii) *There exist four members* R_1, R_2, R_3, R_4 *of* $\mathscr{S}$ *such that* R_i *meshes with* T_s *and* $|L(R_i)| = 2$ *for each* $1 \leq i \leq 4$ *and* $T_s = \bigcup_{1 \leq i \leq 4} L(R_i)$.
- (iii) $|T_i \cap L(T_s)|$ *is even for each* $1 \leq i \leq s - 1$.

(IV) *If* $|\mathscr{T}| = (2|V(F)| - 2)/3$, *then* $s = 2$ *and there exists* $M \in \mathscr{L}(T_2)$ *such that* $|V(M)| = 1$.

Proof. By Lemma 2.4 and the assumption that $|\mathscr{T}| \geq (2|V(F)| - 3)/3$, we have $|\mathscr{T}| = (2|V(F)| - 3)/3$ or $(2|V(F)| - 2)/3$. Since $|\mathscr{T}|$ is an integer, this in particular implies $|V(F)| \neq 2$. Hence $|V(F)| \geq 3$. Assume for the moment that $|V(F)| \geq 4$. Then since $|\mathscr{L}(S)| \geq 2$, we have $|L(S)| \geq |V(F)| + 1 \geq 5$. Hence $L(P_i) \subseteq S$ and $|L(P_i)| \leq 3$ for each $1 \leq i \leq 4$ by Lemma 2.10. By Lemmas 2.1, 2.2 and 2.14, $L(P_i) \cap L(P_j) = \emptyset$ for each $1 \leq i < j \leq 4$. Since $|S| = 8$, this implies

$$|L(P_i)| = 2 \text{ for each } 1 \leq i \leq 4 \tag{2.2}$$

and

$$S = \bigcup_{1 \leq i \leq 4} L(P_i). \tag{2.3}$$

If $|V(F)|$ =3, then we have (2.2) by assumption, and hence (2.3) holds. Thus (2.2) and (2.3) hold, and (I) is proved. Now for $T \in \mathscr{T}$ and $1 \leq i \leq 4$,

$$\text{if } T \cap L(P_i) \neq \emptyset, \text{ then } P_i \text{ meshes with } T \text{ and } L(P_i) \subseteq T \tag{2.4}$$

by Lemma 2.1 (i) and (2.2), because $L(T) \cap L(P_i) \subseteq L(T) \cap S = \emptyset$. It follows from (2.3) and (2.4) that

$$|T \cap S| \text{ is even for each } T \in \mathscr{T}. \tag{2.5}$$

Suppose that F is not saturated. Then $s = 1$ and $|L(T_1)| = |V(F)| - 1$ by Lemma 2.4 (III) (iii). Since $|V(F)| \geq 2$, we have $|P_i \cap V(F)| \geq 2$ for each $1 \leq i \leq 4$ by Lemma 2.15. Hence $P_i \cap L(T_1) \neq \emptyset$ for each $1 \leq i \leq 4$. Since $L(P_i) \cap L(T_1) = \emptyset$ for each $1 \leq i \leq 4$, it follows from Lemma 2.1 that P_i meshes with T_1 for each $1 \leq i \leq 4$. Hence $L(P_i) \subseteq T_1$ for each $1 \leq i \leq 4$ by (2.2). Since $|T_1| = 8 = |S|$, this together with (2.3) implies $T_1 = S$, a contradiction. Thus F is saturated, which proves (II).

It follows from (II) that $V(F) = L(T_1) \cup L(T_2) \cup \cdots \cup L(T_s)$. By Lemma 2.4, we have $2 \leq s \leq 3$. In proving (III), we consider the cases where $s = 2$ and $s = 3$ separately.

Case 1. $s = 2$.

By Lemmas 2.4 and 2.18, T_1 meshes with T_2 and $|L(T_1)| = 2$. Since $N_G(L(T_1) \cup L(T_2)) = N_G(V(F)) = S$, $T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1)$. Hence $T_2 - L(T_1) \subseteq S$. Since $|T_2 - L(T_1)| = 6$, it follows from (2.3) and (2.4) that three of P_1, P_2, P_3, P_4 mesh with T_2, and hence (III) (ii) holds. Since $|T_1| = 8$ and $T_1 = N_G(L(T_1)) \subseteq S \cup L(T_2)$, $|T_1 \cap L(T_2)| = |T_1| - |T_1 \cap S|$ is even by (2.5).

Case 2. $s = 3$.

By Lemma 2.4 (III) (ii), $|\mathscr{L}(T_1)| = |\mathscr{L}(T_2)| = |\mathscr{L}(T_3)| = 2$,

$$\text{at least one of } T_1 \text{ and } T_2 \text{ meshes with } T_3, \tag{2.6}$$

and $|\{T \in \mathscr{T} \mid L(T) \subseteq L(T_i)\}| = (2|L(T_i)| - 1)/3$ for each $1 \leq i \leq 3$. Arguing as in the proof of Lemma 2.18, we see from Lemmas 2.11 and 2.12 that

$$\text{if } T \in \{T_1, T_2\} \text{ and } T \text{ meshes with } T_3, \text{ then } |L(T)| = 2. \tag{2.7}$$

Also note that

$$\text{if } T \in \{T_1, T_2\} \text{ and } T \text{ does not mesh with } T_3, \text{ then } T_3 \cap L(T) = \emptyset \tag{2.8}$$

by Lemma 2.1 (i). Since $|T_3| = 8$ and $T_3 = N_G(L(T_3)) \subseteq S \cup L(T_1) \cup L(T_2)$, it follows from (2.4), (2.7) and (2.8) that (III) (ii) holds, and

$$\text{at least two of } P_1,\ P_2,\ P_3,\ P_4 \text{ mesh with } T_3. \tag{2.9}$$

Suppose that $|L(T_2)| \neq 2$. Then $|L(T_2)| \geq 5$ and T_2 does not mesh with T_3 by Lemmas 2.12 and 2.11. Hence it follows from (2.6) and (2.7) that T_1 meshes with T_3 and $|L(T_1)| = 2$. We also have $T_2 \cap L(T_3) = \emptyset$ by Lemma 2.1 (i), and hence $T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1)$. Since $|T_2| = 8$, this together with (2.4) implies that at least three of P_1, P_2, P_3 and P_4 mesh with T_2. Hence it follows from (2.9) that there exists i such that P_i meshes with T_2 and T_3. But then $|P_i \cap L(S)| \geq |P_i \cap V(F)| + 1 \geq |P_i \cap L(T_2)| + |P_i \cap L(T_3)| + 1 \geq 3 + 3 + 1 = 7$ by Lemma 2.16, which contradicts Lemma 2.17. Thus $|L(T_2)| = 2$. Since $|L(T_2)| \geq |L(T_1)|$, we also get $|L(T_1)| = 2$. Now since $|L(T_1)| = |L(T_2)| = 2$ and since $T_1 = N_G(L(T_1)) \subseteq S \cup L(T_2) \cup L(T_3)$ and $T_2 = N_G(L(T_2)) \subseteq S \cup L(T_1) \cup L(T_3)$, it follows from (2.5) and Lemma 2.13 that $|T_1 \cap L(T_3)|$ and $|T_2 \cap L(T_3)|$ are even. Thus (III) is proved.

Finally we prove (IV). Since $|\mathscr{T}| = (2|V(F)| - 2)/3$, it follows from Lemma 2.4 (II) (ii) that $s = 2$, $|\mathscr{L}(T_2)| = 2$, T_1 meshes with T_2, and $|\{T \in \mathscr{T} \mid L(T) \subseteq L(T_2)\}| = (2|L(T_2)| - 1)/3$. Write $\mathscr{L}(T_2) = \{F_1, F_2\}$. By (II), $|L(T_1)| = 2$. From the proof of (III) in Case 1, it follows that three of P_1, P_2, P_3 and P_4 mesh with T_2. We may assume that P_i meshes with T_2 for each $1 \leq i \leq 3$. Then $T_2 = L(T_1) \cup (\bigcup_{1 \leq i \leq 3} L(P_i))$. Suppose that $|V(F_1)| \neq 1$ and $|V(F_2)| \neq 1$. Since $|\{T \in \mathscr{T} \mid L(T) \subseteq V(F_i)\}| = (2|V(F_i)| - 2)/3$ for each $1 \leq i \leq 2$ by Lemma 2.5, it follows from Lemma 2.4 (II) (ii) that F_i is saturated for each $1 \leq i \leq 2$, and there exist $Q_1, Q_2, Q_3, Q_4 \in \mathscr{S}$ such that $V(F_1) = L(Q_1) \cup L(Q_2)$ and $V(F_2) = L(Q_3) \cup L(Q_4)$. By Lemma 2.18, we may assume $|L(Q_1)| = |L(Q_3)| = 2$. Since $N_G(L(Q_1) \cup L(Q_2)) = N_G(V(F_1)) = T_2$, $Q_2 = N_G(L(Q_2)) \subseteq L(Q_1) \cup T_2 = L(Q_1) \cup L(T_1) \cup (\bigcup_{1 \leq i \leq 3} L(P_i))$. Since $|Q_2| = 8$, this together with (2.4) implies

$$\text{at least two of } P_1,\ P_2 \text{ and } P_3 \text{ mesh with } Q_2. \tag{2.10}$$

Similarly,

$$\text{at least two of } P_1,\ P_2 \text{ and } P_3 \text{ mesh with } Q_4. \tag{2.11}$$

Suppose that $|L(Q_2)| = 2$. Then arguing as in the proof of (2.10), we see that

$$\text{at least two of } P_1,\ P_2 \text{ and } P_3 \text{ mesh with } Q_1. \tag{2.12}$$

From (2.10) and (2.12), it follows that there exists i with $1 \leq i \leq 3$ such that P_i meshes with Q_1 and Q_2. But then $|P_i \cap L(S)| \geq |P_i \cap L(T_2)| + 1 \geq |P_i \cap L(Q_1)| + |P_i \cap L(Q_2)| + |P_i \cap$

$V(F_2)|+1 \geq 2+2+2+1 = 7$ by Lemma 2.15, which contradicts Lemma 2.17. Consequently $|L(Q_2)| \geq 3$. Similarly $|L(Q_4)| \geq 3$. Note that it follows from (2.10) and (2.11) that there exists i with $1 \leq i \leq 3$ such that P_i meshes with Q_2 and Q_4. Now by Lemma 2.16, we obtain $|P_i \cap L(S)| \geq |P_i \cap L(T_2)| + 1 \geq |P_i \cap L(Q_2)| + |P_i \cap L(Q_4)| + 1 \geq 3 + 3 + 1 = 7$, which contradicts Lemma 2.17. Thus (IV) is proved. □

Applying Lemmas 2.6 and 2.19 repeatedly, we obtain the following lemma.

Lemma 2.20. *Suppose that $|V(G)| \geq 17$. Let $S \in \mathscr{S}$ and $F \in \mathscr{L}(S)$, and suppose that $|V(F)| \geq 4$. Set $\mathscr{T} = \{T \in \mathscr{S} \mid L(T) \subseteq V(F)\}$, and suppose that $|\mathscr{T}| \geq (2|V(F)| - 3)/3$. Suppose further that there exist four members of $\mathscr{S}$ which mesh with S. Then the following hold.*

(i) *For each $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$ with $|V(M)| \neq 1$, M is saturated.*

(ii) *For each $T \in \mathscr{T} \cup \{S\}$ with $|L(T)| \neq 2$, there exist $R_1, R_2, R_3, R_4 \in \mathscr{S}$ such that R_i meshes with T and $|L(R_i)| = 2$ for each $1 \leq i \leq 4$ and $T = \bigcup_{1 \leq i \leq 4} L(R_i)$.*

Proof. We first prove (ii) by backward induction on $|L(T)|$. Thus let $T \in \mathscr{T} \cup \{S\}$ with $|L(T)| \neq 2$. If $T = S$, the desired conclusion follows from Lemma 2.19 (I). Thus we may assume $T \neq S$. Then there exists $S' \in \mathscr{T} \cup \{S\}$ with $S' \neq T$ such that $T \leq S'$. We choose such S' so that S' is minimal, and take $M \in \mathscr{L}(S')$ such that $L(T) \subseteq V(M)$. Then $M \in (\bigcup_{R \in \mathscr{T}} \mathscr{L}(R)) \cup \{F\}$. By the induction hypothesis, there exist $Q_1, Q_2, Q_3, Q_4 \in \mathscr{S}$ such that Q_i meshes with S' and $|L(Q_i)| = 2$ for each $1 \leq i \leq 4$. Since $|\mathscr{T}| \geq (2|V(F)| - 3)/3$, $|\{R \in \mathscr{T} | L(R) \subseteq V(M)\}| \geq (2|V(M)| - 3)/3$ by Lemma 2.6. Consequently S' and M satisfy the assumptions of Lemma 2.19. Let $T_1, \cdots, T_s$ be the maximal members of $\{R \in \mathscr{T} | L(R) \subseteq V(M)\}$ with $|L(T_1)| \leq |L(T_2)| \leq \cdots \leq |L(T_s)|$. By the minimality of S', $T \in \{T_1, \cdots, T_s\}$. Since $|L(T)| \neq 2$, it follows from Lemma 2.19 (III) (i) that $T = T_s$. Hence the desired conclusion follows from Lemma 2.19 (III) (ii). Thus (ii) is proved. Now to prove (i), let $M \in (\bigcup_{T \in \mathscr{T}} \mathscr{L}(T)) \cup \{F\}$ with $|V(M)| \neq 1$, and take $S' \in \mathscr{T} \cup \{S\}$ such that $M \in \mathscr{L}(S')$. Then $|L(S')| \neq 2$. By (ii), there exist $Q_1, Q_2, Q_3, Q_4 \in \mathscr{S}$ such that Q_i meshes with S' and $|L(Q_i)| = 2$ for each $1 \leq i \leq 4$, and $|\{T \in \mathscr{T} | L(T) \subseteq V(M)\}| \geq (2|V(M)| - 3)/3$ by Lemma 2.6. Hence we obtain the desired conclusion by applying Lemma 2.19 (II) to S' and M. □

Lemma 2.21. *Suppose that $|V(G)| \geq 17$. Let $S \in \mathscr{S}$, and suppose that there exist four members of $\mathscr{S}$ which mesh S. Then the following hold.*

(i) *If $F \in \mathscr{L}(S)$ and $|V(F)| \geq 5$, then $|\{T \in \mathscr{S} \mid L(T) \subseteq V(F)\}| \leq (2|V(F)| - 3)/3$.*

(ii) *If $|L(S)| \geq 9$, then $|\{T \in \mathscr{S} \mid L(T) \subseteq L(S)\}| \leq (2|L(S)| - 2)/3$.*

Proof. We first prove (i). Thus let $F \in \mathscr{L}(S)$ with $|V(F)| \geq 5$. Set $\mathscr{T} = \{T \in \mathscr{S} \mid L(T) \subseteq V(F)\}$. Suppose that $|\mathscr{T}| = (2|V(F)| - 2)/3$. By Lemma 2.4 (II) (ii), there exist $T_1, T_2 \in \mathscr{T}$ such that $V(F) = L(T_1) \cup L(T_2)$ with $|L(T_1)| \leq |L(T_2)|$. By Lemma 2.18, $|L(T_1)| = 2$. Since $|V(F)| \geq 5$, $|L(T_2)| = |V(F)| - |L(T_1)| \geq 5 - 2 = 3 > 2$. Choose $T_0 \in \mathscr{T}$ with $|L(T_0)| \neq 2$ so that $|L(T_0)|$ is as small as possible. Also choose $S' \in \mathscr{T} \cup \{S\}$ with $S' \neq T_0$ and $T_0 \leq S'$ so that S' is minimal, and take $M \in \mathscr{L}(S')$ such that $L(T_0) \subseteq V(M)$. Since $|L(S')| \neq 2$, it follows from Lemma 2.20 (ii) that

$$\text{there exist } R_1, R_2, R_3, R_4 \in \mathscr{S} \text{ such that } R_i \text{ meshes with } S' \text{ and } |L(R_i)| = 2 \text{ for each } 1 \leq i \leq 4. \tag{2.13}$$

Since $|\mathscr{T}| = (2|V(F)| - 2)/3$, $|\{T \in \mathscr{T} \mid L(T) \subseteq V(M)\}| = (2|V(M)| - 2)/3$ by Lemma 2.6 (I) (i). Hence it follows from Lemma 2.4 (II) (ii) and the minimality of S' that there exists $T' \in \mathscr{T}$ such that $V(M) = L(T') \cup L(T_0)$ and T' meshes with T_0, and $|\mathscr{L}(T_0)| = 2$, and $|\{T \in \mathscr{T} \,|L(T) \subseteq L(T_0)\}| = (2|L(T_0)| - 1)/3$. Since $|L(T_0)| \neq 2$, it follows from (2.13) and (i) and (iii) of Lemma 2.19 (III) that

$$|T' \cap L(T_0)| \text{ is even.} \tag{2.14}$$

Write $\mathscr{L}(T_0) = \{F_1, F_2\}$ with $|V(F_1)| \leq |V(F_2)|$. Since $|L(T_0)| \neq 2$, we have $|V(F_2)| \neq 1$. Since $|\{T \in \mathscr{T} \,|L(T) \subseteq V(F_2)\}| = (2|V(F_2)| - 2)/3$ by Lemma 2.5 (II), it follows from Lemma 2.4 (II) (ii) that there exist $Q_1, Q_2 \in \mathscr{T}$ such that $V(F_2) = L(Q_1) \cup L(Q_2)$. By the minimality of $|L(T_0)|$, $|L(Q_1)| = |L(Q_2)| = 2$. By Lemma 2.13, this implies $|T' \cap V(F_2)|$ is even. On the other hand, $|V(F_1)| = 1$ by (2.13) and Lemma 2.19 (IV). Since T' meshes with T_0, this means $|T' \cap V(F_1)| = 1$. Consequently $|T' \cap L(T_0)| = |T' \cap V(F_2)| + |T' \cap V(F_1)|$ is odd, which contradicts (2.14). Thus (i) is proved. To prove (ii), assume $|L(S)| \geq 9$. By way of contradiction, suppose that $|\{T \in \mathscr{S} \,|L(T) \subseteq L(S)\}| = (2|L(S)| - 1)/2$. Then by Lemma 2.5 (II), $|\mathscr{L}(S)| = 2$ and $|\{T \in \mathscr{S} \,|L(S) \subseteq V(F)\}| = (2|V(F)| - 2)/3$ for each $F \in \mathscr{L}(S)$. From $|\mathscr{L}(S)| = 2$, it follows that there exists $F \in \mathscr{L}(S)$ such that $|V(F)| \geq 5$. But then $|\{T \in \mathscr{S} \,|L(S) \subseteq V(F)\}| \leq (2|V(F)| - 3)/3$ by (i), a contradiction. □

3. Proof of the Theorem

We continue with the notation of the preceding section, and prove the Theorem. Thus let $|V(G)| \geq 177$ and , by way of contradiction, suppose that

$$|\mathscr{S}| \geq (2|V(G)| - 9)/3. \tag{3.1}$$

Let $S_1, \cdots, S_m$ be the maximal members of $\mathscr{S}$ with respect to the order relation $\leq$. We may assume $|L(S_1)| \geq \cdots \geq |L(S_m)|$. Let $p_i = |\mathscr{L}(S_i)|$ for each i, and let

$W = V(G) - (L(S_1) \cup \cdots \cup L(S_m)\,)$. Arguing as in [3; Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 3.1.

(i) $m + 2|W| \leq 9$.

(ii) $2p_1 + (m-1) + 2|W| \leq 12$.

Sketch of Proof. By (3.1) and Lemma 2.5 (I), $(2|V(G)| - 9)/3 \leq \sum_{1 \leq i \leq m} \ (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 9$. Since $p_i \geq 2$ for all i, both (i) and (ii) follow from this. □

Claim 3.2. $|L(S_1)| \geq 11$.

Sketch of Proof. If $|L(S_1)| \leq 10$, then by Claim 3.1 (i), $|V(G)| \leq 10m + |W| \leq 10m + 20|W| \leq 90$, which contradicts the assumption that $|V(G)| \geq 177$. □

Claim 3.3. $m \geq 2$ *and* $|L(S_2)| \geq 9$.

Sketch of Proof. Suppose that $m = 1$ or $|L(S_2)| \leq 8$. Then by Claim 3.1 (ii), $|V(G) - L(S_1)| \leq 8(m-1) + |W| \leq 96 - 16p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 88 - 16p_1$, which implies $|L(S_1)| \leq p_1(88 - 16p_1)$. Consequently $|V(G)| \leq p_1(88 - 16p_1) + 96 - 16p_1 < 177$ because p_1 is an integer, which contradicts the assumption that $|V(G)| \geq 177$. □

By Lemma 2.11, Claims 3.2 and 3.3 imply that S_1 does not mesh with S_2. Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1. Write $\mathscr{K}(S_1) - \mathscr{L}(S_1) = \{C_1\}$ and $\mathscr{K}(S_2) - \mathscr{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We define $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_{1,1}$, $\mathscr{T}_{1,2}$, $\mathscr{T}_{1,3}$, $\mathscr{T}_{2,1}$, $\mathscr{T}_{2,2}$, $\mathscr{T}_{2,3}$ as follows:

$$\begin{aligned}
\mathscr{T}_1 &= \{T \in \mathscr{S} | L(T) \cap (S_1 \cup S_2) = \emptyset\}, \\
\mathscr{T}_2 &= \{T \in \mathscr{S} | L(T) \subseteq S_1 \cup S_2\}, \\
\mathscr{T}_{1,1} &= \{T \in \mathscr{T}_1 | L(T) \subseteq L(S_1)\}, \\
\mathscr{T}_{1,2} &= \{T \in \mathscr{T}_1 | L(T) \subseteq L(S_2)\}, \\
\mathscr{T}_{1,3} &= \{T \in \mathscr{T}_1 | L(T) \subseteq V(C_1) \cap V(C_2)\}, \\
\mathscr{T}_{2,1} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 - S_2\}, \\
\mathscr{T}_{2,2} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_2 - S_1\}, \\
\mathscr{T}_{2,3} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 \cap S_2\}.
\end{aligned}$$

In view of the maximality of $L(S_1)$ and $L(S_2)$ and Claims 3.2 and 3.3, it follows from Lemmas 2.1 and 2.10 that $\mathscr{T}_1$ is the set of those members of $\mathscr{S}$ which mesh with neither

S_1 nor S_2, and $\mathscr{T}_2$ is the set of those members of $\mathscr{S}$ which mesh with S_1 or S_2. Thus $\mathscr{S} = \mathscr{T}_1 \cup \mathscr{T}_2$ (disjoint union). Further $\mathscr{T}_1 = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2} \cup \mathscr{T}_{1,3}$ (disjoint union) and $\mathscr{T}_2 = \mathscr{T}_{2,1} \cup \mathscr{T}_{2,2} \cup \mathscr{T}_{2,3}$ (disjoint union).

The following two claims immediately follow from Lemma 2.5 (I) (see also [3;Claim 3.6]).

Claim 3.4. $|\mathscr{T}_{1,i}| \leq (2|L(S_i)| - 1)/3$ $(i = 1, 2)$.

Claim 3.5. $|\mathscr{T}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

Proof. Let $I = \{1 \leq i \leq m | S_i \in \mathscr{T}_{1,3}\}$. Then by Lemma 2.5 (I), $|\mathscr{T}_{1,3}| = \sum_{i \in I} |\{T \in \mathscr{S}_i \,|\, L(T) \subseteq L(S_i)\}| \leq \sum_{i \in I} (2|L(S_i)| - 1)/3 \leq (2|V(C_1) \cap V(C_2)| - |I|)/3 \leq 2|V(C_1) \cap V(C_2)|/3$. □

Since $|L(T)| \leq 3$ for each $T \in \mathscr{T}_2$ by Lemma 2.10, the following claim follows from Lemmas 2.3 and 2.7 (see also [3;Claim 3.8]).

Claim 3.6.

(i) $|\mathscr{T}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathscr{T}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathscr{T}_{2,3}| \leq |S_1 \cap S_2|/2$.

Claim 3.7. $|S_1 \cap S_2|$ *is even.*

Proof. Suppose that $|S_1 \cap S_2|$ is odd. Then it follows from Claim 3.6 that $|\mathscr{T}_2| \leq (|S_1 \cup S_2| - 3)/2$, and it follows from Claims 3.4 and 3.5 that $|\mathscr{T}_1| \leq (2(|V(G)| - |S_1 \cup S_2|) - 2)/3$. Hence $|\mathscr{S}| \leq (2|V(G)| - (|S_1 \cup S_2| + 13)/2)/3$. Since $|S_1 \cup S_2| \geq 9$, this contradicts (3.1). □

Write $|S_1 \cap S_2| = 2x$. Then $|S_1 \cup S_2| = 16 - 2x$. Hence it follows from Claim 3.6 that

$$|\mathscr{T}_2| \leq 8 - x, \tag{3.2}$$

and it follows from Claims 3.4 and 3.5 that

$$|\mathscr{T}_1| \leq (2|V(G)| - 34 + 4x)/3. \tag{3.3}$$

By (3.2) and (3.3), $|\mathscr{S}| \leq (2|V(G)| - 10 + x)/3$. In view of (3.1), this implies that equality holds in (3.2) (note that $x \leq 3$). Thus it follows from Claim 3.6 that

$$|\mathscr{T}_{2,1}| = 4 - x, |\mathscr{T}_{2,2}| = 4 - x, |\mathscr{T}_{2,3}| = x. \tag{3.4}$$

Note that (3.4) implies that

$$\text{for each } i = 1, 2, \text{ there exist four members of } \mathscr{S} \text{ which mesh with } S_i. \tag{3.5}$$

Hence by Claims 3.2, 3.3 and Lemma 2.21 (ii),

$$|\,\mathscr{T}_{1,i}\,| \le (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.6}$$

Now it follows from (3.2), (3.6) and Claim 3.5 that $|\,\mathscr{S}\,| \le (2|V(G)| - 12 + x)/3$. In view of (3.1), this implies that $x = 3$ and equality holds in (3.6), i.e.,

$$|\,\mathscr{T}_{1,i}\,| = (2|L(S_i)| - 2)/3 \text{ for each } i = 1, 2. \tag{3.7}$$

In what follows, we mainly consider S_1. By (3.7), it follows from Lemma 2.5 (III) that $|\,\mathscr{L}(S_1)| = 2$. Write $\mathscr{L}(S_1) = \{F_1, F_2\}$. By Lemma 2.5 (III), we may assume $|\{T \in \mathscr{S}\,|L(T) \subseteq V(F_1)\}| = (2|V(F_1)|-2)/3$ and $|\{T \in \mathscr{S}\,|L(T) \subseteq V(F_2)\}| = (2|V(F_2)|-3)/3$. Then by (3.5) and Lemma 2.21 (i), $|V(F_1)| \le 4$. Since $|L(S_1)| \ge 11$ by Claim 3.2, this implies $|V(F_2)| \ge 7$. Set $\mathscr{T} = \{T \in \mathscr{T}_{1,1}\,|L(T) \subseteq V(F_2)\}$, and set $\mathscr{T}_M = \{T \in \mathscr{T}\,|L(T) \subseteq V(M)\}$ for each $M \in (\bigcup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F_2\}$. Then

$$|\,\mathscr{T}_M\,| \ge (2|V(M)| - 3)/3 \tag{3.8}$$

for each $M \in (\bigcup_{T\in\mathscr{T}} \mathscr{L}(T)) \cup \{F_2\}$ by Lemma 2.6 (II) (i). Since $|V(F_2)| \ge 7$, it follows from Lemma 2.4 (III) that there exists $T \in \mathscr{T}$ such that $|L(T)| \ne 2$. Choose $T_0 \in \mathscr{T}$ with $|L(T_0)| \ne 2$ so that $|L(T_0)|$ is as small as possible. Since $|\,\mathscr{T}\,| = (2|V(F_2)| - 3)/3$, $|\,\mathscr{L}(T_0)| = 2$ by Lemma 2.6 (II) (ii). Write $\mathscr{L}(T_0) = \{M_1, M_2\}$ with $|V(M_1)| \le |V(M_2)|$. Let $Q_1, \cdots, Q_l$ be the maximal members of $\mathscr{T}_{M_2}$. By the minimality of $|L(T_0)|$, $|L(Q_i)| = 2$ for each $1 \le i \le l$.

Claim 3.8. *M_2 is saturated and $|V(M_2)| = 4$ or 6.*

Proof. Since $|L(T_0)| \ne 2$, $|V(M_2)| \ne 1$. Hence M_2 is saturated by (3.5) and Lemma 2.20 (i). By (3.8) and (II) and (III) of Lemma 2.4, $2 \le l \le 3$. Hence $|V(M_2)| = 4$ or 6. □

Choose $R \in \mathscr{T} \cup \{S_1\}$ with $R \ne T_0$ and $T_0 \le R$ so that R is minimal, and take $M \in \mathscr{L}(R)$ such that $L(T_0) \subseteq V(M)$. By (3.5) and Lemma 2.20 (i), M is saturated. Let $T_1, \cdots, T_s$ be the maximal members of $\mathscr{T}_M$. By (3.8) and (II) and (III) of Lemma 2.4,

$$2 \le s \le 3. \tag{3.9}$$

By the minimality of R, $T_0 \in \{T_1, \cdots, T_s\}$. We may assume $T_0 = T_s$. Now since $|L(R)| \ne 2$, applying Lemma 2.20 (ii) with $S = S_1$, $F = F_2$ and $T = R$, we see from (3.5)

$$\begin{aligned}&\text{there exist } P_1, P_2, P_3, P_4 \in \mathscr{S} \text{ such that } P_i \text{ meshes with } R\\ &\text{for each } 1 \le i \le 4\end{aligned} \tag{3.10}$$

and such that

$$|L(P_i)| = 2 \text{ for each } 1 \leq i \leq 4 \text{ and } R = \bigcup_{1 \leq i \leq 4} L(P_i). \tag{3.11}$$

Claim 3.9. $|V(M_1)| = 1$

Proof. Set $\mathscr{A}_i = \{T \in \mathscr{T} \mid L(T) \subseteq L(T_i)\}$ for each $1 \leq i \leq s$. Recall that $T_s = T_0$ and $|L(T_0)| \neq 2$. Thus $|L(T_i)| = 2$ for each $1 \leq i \leq s-1$ by (3.8), (3.10) and Lemma 2.19 (III) (i). Hence $\mathscr{A}_i = \{T_i\}$ and

$$|\,\mathscr{A}_i\,| = (2|L(T_i)| - 1)/3 \tag{3.12}$$

for each $1 \leq i \leq s-1$. Since $|L(T_s)| \neq 2$, applying Lemma 2.20 (ii) with $S = S_1$, $F = F_2$ and $T = T_s$, we see from (3.5) that

$$\text{there exist } R_1, R_2, R_3, R_4 \in \mathscr{S} \text{ such that } R_i \text{ meshes with } T_s \text{ for each } 1 \leq i \leq 4 \tag{3.13}$$

and such that

$$|L(R_i)| = 2 \text{ for each } 1 \leq i \leq 4 \text{ and } T_s = \bigcup_{1 \leq i \leq 4} L(R_i). \tag{3.14}$$

Note that $T_s = N_G(L(T_s)) \subseteq R \cup (\bigcup_{1 \leq i \leq s-1} L(T_i))$. Thus $T_s \subseteq (\bigcup_{1 \leq i \leq 4} L(P_i)) \cup (\bigcup_{1 \leq i \leq s-1} L(T_i))$ by (3.11). In view of (3.14), this implies

$$\{R_1, R_2, R_3, R_4\} \subseteq \{P_1, P_2, P_3, P_4, T_1, T_2, \cdots, T_{s-1}\}. \tag{3.15}$$

Suppose that $|V(M_1)| \neq 1$. Then by (3.10), it follows from Lemma 2.19 (IV) that $|\,\mathscr{T}_M\,| \leq (2|V(M)| - 3)/3$. Hence by (3.8),

$$|\,\mathscr{T}_M\,| = (2|V(M)| - 3)/3. \tag{3.16}$$

By (3.5) and Lemma 2.20 (i), M_1 is saturated. Let $Q'_1, \cdots, Q'_{l'}$ be the maximal members of $\mathscr{T}_{M_1}$. By (3.8) and (II) and (III) of Lemma 2.4, $2 \leq l' \leq 3$. $|L(Q'_1)| = \cdots = |L(Q'_{l'})| = 2$. Recall that $|L(Q_1)| = \cdots = |L(Q_l)| = 2$, and we have $2 \leq l \leq 3$ by Claim 3.8.

Case 1. $l = 2$.

Since $|V(M_1)| \leq |V(M_2)|$, we have $l' \leq l$, and hence $l' = 2$. This implies $\mathscr{A}_s = \{T_s, Q_1, Q_2, Q'_1, Q'_2\}$, and hence $|\,\mathscr{A}_s\,| = (2|L(T_s)| - 1)/3$. Since $|\,\mathscr{T}_M\,| = |\,\mathscr{A}_1\,| + \cdots + |\,\mathscr{A}_s\,|$, it now follows from (3.9), (3.12) and (3.16) that $s = 3$. For each $1 \leq i \leq 2$, $|Q_i| = 8$ and, if we write $\{1, 2\} = \{i, j\}$, then $Q_i = N_G(L(Q_i)) \subseteq T_s \cup L(Q_j) = (\bigcup_{1 \leq h \leq 4} L(R_h)) \cup L(Q_j)$ by (3.14). Hence for each $1 \leq i \leq 2$,

$$\text{at least three of } R_1,\ R_2,\ R_3 \text{ and } R_4 \text{ mesh with } Q_i \tag{3.17}$$

by Lemma 2.1 (see the proof of (2.4) and (2.10)). Similarly, for each $1 \le i \le 2$,

$$\text{at least three of } R_1,\ R_2,\ R_3 \text{ and } R_4 \text{ mesh with } Q_i'. \tag{3.18}$$

By (3.17), at least two of R_1, R_2, R_3 and R_4 mesh with Q_1 and Q_2. We may assume that R_i meshes with Q_1 and Q_2 for each $1 \le i \le 2$. If $R_i \in \{P_1, P_2, P_3, P_4\}$ for some $1 \le i \le 2$, then $|R_i \cap L(R)| \ge |R_i \cap V(M)| + 1 \ge |R_i \cap L(Q_1)| + |R_i \cap L(Q_2)| + |R_i \cap V(M_1)| + 1 \ge 2+2+2+1 = 7$ by Lemma 2.15, which contradicts Lemma 2.17 (note that $R_i \in \{P_1, P_2, P_3, P_4\}$ meshes with R by (3.10)). Thus

$$R_i \in \{T_1, T_2\} \text{ for each } 1 \le i \le 2 \tag{3.19}$$

by (3.15). Arguing similarly by using (3.18) in place of (3.17), we see that there exist $R_1', R_2' \in \{R_1, R_2, R_3, R_4\}$ such that R_i' meshes with Q_1' and Q_2' for each $1 \le i \le 2$ and

$$R_i' \in \{T_1, T_2\} \text{ for each } 1 \le i \le 2 \tag{3.20}$$

by (3.15). Now by (3.19) and (3.20), T_1 meshes with Q_1, Q_2, Q_1' and Q_2', and hence $|T_1 \cap L(T_3)| = |T_1 \cap V(M_1)| + |T_1 \cap V(M_2)| = 4 + 4 = 8$, which contradicts Lemma 2.17 (note that $T_1 \in \{R_1, R_2, R_3, R_4\}$ meshes with T_3 by (3.13)).

Case 2. $l = 3$.

Since $|V(M_2)| = 6$ and $\mathscr{T}_{M_2} = \{Q_1, Q_2, Q_3\}$, $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 3)/3$. Note that $|\mathscr{A}_s| \ge (2|L(T_s)| - 2)/3$ by (3.16) and Lemma 2.4 (III). We now apply Lemma 2.5 to $\mathscr{A}_s$. Since $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 3)/3$, $|\mathscr{A}_s| \ne (2|L(T_s)| - 1)/3$ by Lemma 2.5 (II). Hence $|\mathscr{A}_s| = (2|L(T_s)| - 2)/3$. Again since $|\mathscr{T}_{M_2}| = (2|V(M_2)| - 3)/3$, it follows from Lemma 2.5 (III) that $|\mathscr{T}_{M_1}| = (2|V(M_1)| - 2)/3$, and hence $l' = 2$. In view of (3.9), (3.12) and (3.16), we also get $s = 2$. Since $l' = 2$, (3.18) again holds. Since $s = 2$, this together with (3.15) implies that there exists $P \in \{P_1, P_2, P_3, P_4\} \cap \{R_1, R_2, R_3, R_4\}$ such that P meshes with Q_1' and Q_2'. Now by Lemma 2.15, $|P \cap L(R)| \ge |P \cap V(M)| + 1 \ge |P \cap L(Q_1')| + |P \cap L(Q_2')| + |P \cap V(M_2)| + 1 \ge 2 + 2 + 2 + 1 = 7$, which contradicts Lemma 2.17. This completes the proof of Claim 3.9. □

We are now in a position to complete the proof of the Theorem. Recall that $|L(Q_1)| = \cdots = |L(Q_l)| = 2$. By (3.8) and Lemma 2.4, we may assume that T_1 meshes with T_s. Then $|T_1 \cap V(M_1)| = 1$ by Claim 3.9. On the other hand, $|T_1 \cap V(M_2)|$ is even by Lemma 2.13. Therefore $|T_1 \cap L(T_s)| = |T_1 \cap V(M_2)| + |T_1 \cap V(M_1)|$ is odd. In view of (3.10), this contradicts Lemma 2.19 (III) (iii).

This completes the proof of the Theorem.

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