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2-DOMINATION SUBDIVISION NUMBER OF GRAPHS

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Abstract

In a graph G, a vertex *dominates* itself and its neighbors. A subset $S \subseteq V(G)$ is a 2-*dominating set* of G if S dominates every vertex of $V(G) \setminus S$ at least twice. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G. The 2-*domination subdivision number* $\mathrm{sd}_{\gamma_2}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-domination number. In this paper, we establish upper bounds on the 2-domination subdivision number for arbitrary graphs in terms of vertex degree and for several graph classes. Then we present some conditions on G which are sufficient to imply that $\mathrm{sd}_{\gamma_2}(G) = 1$.

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1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A vertex v in a graph G is called *simplicial* if the induced subgraph $G[N(v)]$ is a complete graph. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when e and e' are incident in G.

A vertex $v \in V$ *dominates* itself and its neighbors. A subset S of vertices of G is a *dominating set* if $N[S] = V$ (that is, S dominates V). The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, see [7]. The *domination subdivision number* $\mathrm{sd}_\gamma(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number of G, see [6].

Similarly, a subset S of V is a 2-*dominating set* of G if S dominates every vertex of $V(G)\backslash S$ at least twice [3, 4]. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G. A $\gamma_2(G)$-*set* is a 2-dominating set of G with cardinality $\gamma_2(G)$. The 2-*domination subdivision number* $\mathrm{sd}_{\gamma_2}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-domination number of G. For a more thorough treatment of domination parameters and for terminology not presented here, see [7, 8].

The purpose of this paper is to initialize the study of the 2-domination subdivision number $\mathrm{sd}_{\gamma_2}(G)$. Although it may not be immediately obvious that the 2-domination subdivision number is defined for all graphs with a connected component of order at least 3, we will show this shortly. Recall that a vertex cover of a graph G is a set of vertices that contains at least one vertex of every edge. The minimum cardinality of a vertex cover of G is denoted by $\beta(G)$.

We make use of the following results in this paper.

Theorem A. *For each positive integer* $n \geq 3$, $\gamma_2(C_n) = \lceil \frac{n}{2} \rceil$.

Theorem B. [1] *If G is a graph with $\delta(G) \geq 2$, then every vertex cover of G is also a 2-dominating set and thus* $\gamma_2(G) \leq \beta(G)$.

The proofs of the following propositions are straightforward and therefore omitted.

Proposition 1. *If K_n is the complete graph of order n and $n \geq 3$, then* $\mathrm{sd}_{\gamma_2}(K_n) = 2$.

Proposition 2. *If $K_{m,n}$ is the complete bipartite graph and $m, n \geq 4$, then we have* $\mathrm{sd}_{\gamma_2}(K_{m,n}) = 3$.

2. Bounds on the 2-domination subdivision number

In this section, we present some upper bounds on $\mathrm{sd}_{\gamma_2}(G)$ in terms of the vertex degree and the minimum degree of G. We begin this section with two propositions giving some sufficient conditions for a graph to have a small 2-domination subdivision number.

Proposition 3. *Let G be a graph of order $n \geq 3$. If there is a vertex $v \in V(G)$ with at least two neighbors v_1 and v_2 such that all vertices in $N(v)-\{v_1,v_2\}$ are adjacent to both v_1 and v_2, then $\mathrm{sd}_{\gamma_2}(G) \leq 2$.*

Proof. Let G' be the graph that is obtained from G by subdividing the edges vv_1 and vv_2 with vertices x_1 and x_2, respectively. Let S be a minimum 2-dominating set of G'. If $v \notin S$, then $x_1, x_2 \in S$ and it is clear that $(S-\{x_1,x_2\}) \cup \{v\}$ is a 2-dominating set of G with less vertices than S. If $v \in S$, then we can suppose, without loss of generality, that $v_1, v_2 \in S$. Since by the hypothesis every neighbor of v distinct to v_1 and v_2 is adjacent to these both vertices, $S-\{v\}$ is a 2-dominating set of G. Hence, in both cases we obtain that $\gamma_2(G) < \gamma_2(G')$ and thus $\mathrm{sd}_{\gamma_2}(G) \leq 2$. □

Corollary 4. *If a graph G contains a simplicial vertex v of degree at least two, then $\mathrm{sd}_{\gamma_2}(G) \leq 2$.*

The k-trees are the graphs that arise from a k-clique by zero or more iterations of adding a new vertex joined to a k-clique in the old graph.

Corollary 5. *For any k-tree G with $k \geq 3$, $\mathrm{sd}_{\gamma_2}(G) \leq 2$.*

Recall that $\alpha'(G)$ is the maximum number of edges in a matching of G.

Proposition 6. *Let G be a connected graph of order $n \geq 3$ with $\alpha'(G) = 1$. Then $\mathrm{sd}_{\gamma_2}(G) \leq 2$ with equality if and only if $G \simeq K_3$.*

Proof. If $n = 3$, then $G \simeq P_3$ or K_3. We know that $\mathrm{sd}_{\gamma_2}(P_3) = 1$ and $\mathrm{sd}_{\gamma_2}(K_3) = 2$. Let $n \geq 4$. Suppose that v is a vertex of maximum degree Δ. Since G is connected and $\alpha'(G) = 1$, it follows that $\Delta = n-1$ and $N(v)$ is an independent set. Thus, $G \simeq K_{1,n-1}$ and $\mathrm{sd}_{\gamma_2}(G) = 1$. Hence, $\mathrm{sd}_{\gamma_2}(G) \leq 2$ with equality if and only if $G \simeq K_3$. □

Theorem 7. *Let G be a connected graph. If $v \in V(G)$ has degree at least two, then $\mathrm{sd}_{\gamma_2}(G) \leq \deg(v)$.*

Proof. Let $N(v) = \{v_1, v_2, \ldots, v_{\deg(v)}\}$ and let G' be obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_{\deg(v)}$ with vertices $x_1, x_2, \ldots, x_{\deg(v)}$, respectively. Let S be a $\gamma_2(G')$-set. If $S \cap \{x_1, x_2, \ldots, x_{\deg(v)}\} = \emptyset$, then $N[v] \subseteq S$ and $S \setminus \{v\}$ is a 2-dominating set for G. If $|S \cap \{x_1, x_2, \ldots, x_{\deg(v)}\}| \geq 2$, then $(S \setminus \{x_1, x_2, \ldots, x_{\deg(v)}\}) \cup \{v\}$ is a 2-dominating set for G. Let $|S \cap \{x_1, x_2, \ldots, x_{\deg(v)}\}| = 1$. Without loss of generality, we may assume $x_1 \in S$. We note that in order to dominate x_i twice we must have $\{v, v_i\} \subseteq S$

for $i \in \{2, 3, \ldots, \deg(v)\}$. Hence, $S \setminus \{x_1\}$ is a 2-dominating set of G whose size is less than $|S|$. Therefore $\mathrm{sd}_{\gamma_2}(G) \leq \deg(v)$. $\square$

A consequence of Theorem 7 is that $\mathrm{sd}_{\gamma_2}(G)$ is defined for every simple connected graph G of order $n \geq 3$. In addition:

Corollary 8. *For every connected graph G with $\delta \geq 2$, $\mathrm{sd}_{\gamma_2}(G) \leq \delta$.*

Corollary 9. *For every two positive integers m, n for which $m + n \geq 4$,*

$$\mathrm{sd}_{\gamma_2}(P_n \times P_m) \leq 2.$$

Lemma 10. *Let G be a graph of order $n \geq 4$. If there is a vertex $v \in V(G)$ with at least three neighbors v_1, v_2 and v_3 such that for all vertices $x \in N(v) - \{v_1, v_2, v_3\}$ holds that $|N(x) \cap \{v_1, v_2, v_3\}| \geq 2$, then $\mathrm{sd}_{\gamma_2}(G) \leq 3$.*

Proof. Let G' be the graph that is obtained from G by subdividing the edges vv_1, vv_2 and vv_3 with vertices x_1, x_2 and x_3, respectively. Let S be a minimum 2-dominating set of G'. If $v \notin S$, then $x_1, x_2, x_3 \in S$ and it is clear that $(S - \{x_1, x_2, x_3\}) \cup \{v\}$ is a 2-dominating set of G with less vertices than S. If $v \in S$, then we can suppose, without loss of generality, that $v_1, v_2, v_3 \in S$. Since by the hypothesis every neighbor of v distinct to v_1, v_2 and v_3 is adjacent to at least two of these vertices, $S - \{v\}$ is a 2-dominating set of G. Hence, in both cases we obtain that $\gamma_2(G) < \gamma_2(G')$ and thus $\mathrm{sd}_{\gamma_2}(G) \leq 3$. $\square$

Theorem 11. *For any connected graph G with adjacent vertices u and v, each of degree at least two, for which $|N(u) \cap N(v)| \neq 1$,*

$$\mathrm{sd}_{\gamma_2}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 2.$$

Proof. If $N[u] \nsubseteq N[v]$ (respectively, $N[v] \nsubseteq N[u]$), then $\deg(u) \geq |N(u) \cap N(v)| + 2$ (respectively, $\deg(v) \geq |N(u) \cap N(v)| + 2$) and the statement is true by Theorem 7. Let $N[u] = N[v]$ and $N(u) = \{u_1, \ldots, u_t\}$ in which $u_1 = v$. Since $|N(u) \cap N(v)| \neq 1$, we have $t \geq 3$. Let G' be the graph obtained by subdividing the edge uu_i with vertex x_i for $2 \leq i \leq t$ and let S' be a $\gamma_2(G')$-set. We prove that $\gamma_2(G) \leq |S'| - 1$. If $u \notin S'$, then $\{x_2, \ldots, x_t\} \subset S'$ and $(S' \setminus \{x_2, \ldots, x_t\}) \cup \{u\}$ is a 2-dominating set of G. Let $u \in S'$. If $|S' \cap \{x_2, \ldots, x_t\}| \geq 1$, then $S' \setminus \{x_2, \ldots, x_t\}$ is a 2-dominating set of G. If $S' \cap \{x_2, \ldots, x_t\} = \emptyset$, then $S' \setminus \{u\}$ is a 2-dominating set of G. Thus $\mathrm{sd}_{\gamma_2}(G) \leq \deg(u) - 1$. Since $\deg(v) \geq |N(u) \cap N(v)| + 1$,

$$\mathrm{sd}_{\gamma_2}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 2.$$

$\square$

For the graph K_4 this bound is attained by Proposition 1.

Theorem 12. *For every connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_2}(G) \leq \gamma_2(G)$.*

Proof. Define $\ell(G) = \{x \in V(G) \mid \deg(x) = 1\}$. The proof is by induction on $|\ell(G)|$. If $|\ell(G)| = 0$, then $\delta(G) \geq 2$ and, by Theorem B, $\gamma_2(G) \leq \beta(G)$. Let $X = \{v_1, \ldots, v_\beta\}$ be a $\beta(G)$-set. Then for each $v_i \in X$, there exists an edge v_iv_i' such that $v_i' \notin X$. Note that we may have $v_i' = v_j'$ for some $i \neq j$. Let G' be obtained from G by subdividing the edge v_iv_i' with vertex x_i for $i = 1, \ldots, \gamma_2(G)$. Let H be the induced subgraph $G[v_1x_1, \ldots, v_{\gamma_2(G)}x_{\gamma_2(G)}, v_1'x_1, \ldots, v'_{\gamma_2(G)}x_{\gamma_2(G)}]$. Obviously, the components of H are isomorphic to P_3 or subdivided stars. Suppose that S is a $\gamma_2(G')$-set. In order to dominate x_i, we must have $x_i \in S$ or $v_i, v_i' \in S$. First, let $\{x_i \mid 1 \leq i \leq \gamma_2(G)\} \subseteq S$. Now in order to dominate v_1 twice, S must contain a neighbor of v_1 different from x_i, for $1 \leq i \leq \gamma_2(G)$. This implies that $|S| > \gamma_2(G)$. If $x_i \notin S$ for some $1 \leq i \leq \beta$, then $v_i, v_i' \in S$. Thus, $|S| \geq \beta(G) + 1 > \gamma_2(G)$.

Now suppose that $|\ell(G)| \geq 1$ and that the result holds for any connected graph H of order at least 3 and $|\ell(H)| < |\ell(G)|$. Let $u \in \ell(G)$, $uv \in E(G)$ and $H = G - u$. If $\deg(v) = 2$, then $\mathrm{sd}_{\gamma_2}(G) \leq 2 \leq \gamma_2(G)$ by Theorem 7. Assume $\deg(v) \geq 3$. Obviously, $\gamma_2(H) \leq \gamma_2(G) \leq \gamma_2(H) + 1$. Consider two cases.

Case 1. $\gamma_2(G) = \gamma_2(H)$.

Then v does not belong to any $\gamma_2(G)$-set. Let G' be obtained from G by subdividing the edge uv with vertex x and let S be a $\gamma_2(G')$-set. If $x \notin S$, then $v \in S$. If $x \in S$, then $S' = (S \setminus \{x\}) \cup \{v\}$ is a $\gamma_2(G')$-set. It follows that G' has a $\gamma_2(G')$-set S containing v. Now $S \setminus \{u\}$ is a 2-dominating set for H, which implies $|S| \geq \gamma_2(G) + 1$. Therefore $\mathrm{sd}_{\gamma_2}(G) = 1 < \gamma_2(G)$.

Case 2. $\gamma_2(G) = \gamma_2(H) + 1$.

By inductive hypothesis $\mathrm{sd}_{\gamma_2}(H) \leq \gamma_2(H)$. Let $F \subseteq E(H)$ be a set of size $\mathrm{sd}_{\gamma_2}(H)$, which subdividing its edges increases the 2-domination number of H. Let G' be obtained from G by subdividing the edge uv and all the edges in F and let H' be obtained from H by subdividing all the edges in F. It is easy to see that $\gamma_2(G') \geq \gamma_2(H') + 1$. This implies that $\gamma_2(G') \geq \gamma_2(H') + 1 > \gamma_2(H) + 1 = \gamma_2(G)$. Thus $\mathrm{sd}_{\gamma_2}(G) \leq \mathrm{sd}_{\gamma_2}(H) + 1 \leq \gamma_2(H) + 1 = \gamma_2(G)$. This completes the proof. □

Corollary 13. *For every connected graph G with $\delta(G) \geq 2$, $\mathrm{sd}_{\gamma_2}(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. If $\delta < \lceil \frac{n}{2} \rceil$, the statement holds by Theorem 7. Suppose that $\delta \geq \lceil \frac{n}{2} \rceil$. By Dirac's Theorem [2], G has a Hamiltonian cycle C. Thus, with Theorem A we obtain $\gamma_2(G) \leq \gamma_2(C) = \lceil \frac{n}{2} \rceil$ and the result follows by Theorem 12. □

3. Bounds on the 2-domination subdivision number for some graph classes

A graph is a *claw-free graph* if it does not have any induced subgraph isomorphic to the star $K_{1,3}$.

Theorem 14. *For any connected claw-free graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_2}(G) \leq 4.$$

Proof. Let v be a vertex of degree at least two. If $\deg(v) \leq 4$, then the result holds by Theorem 7. Let $\deg(v) \geq 5$ and $N(v) = \{v_1, v_2, \ldots, v_k\}$. We consider two cases.

Case 1. The induced subgraph $G[N(v)]$ has an isolated vertex, say v_1. Since G is claw-free, the induced subgraph $G[N(v) \setminus \{v_1\}]$ is a complete graph. By Lemma 10, it follows directly that $\mathrm{sd}_{\gamma_2}(G) \leq 3$.

Case 2. $G[N(v)]$ has no isolated vertex. If $G[N(v)]$ is a complete graph, then by Corollary 4 we have $\mathrm{sd}_{\gamma_2}(G) \leq 2$. Suppose that $G[N(v)]$ is not a complete graph and let $v_1v_2 \notin E(G)$. If $G[N(v) \setminus \{v_1, v_2\}]$ is a complete graph, then subdivide the edges vv_1, vv_2, vv_3, vv_4 to see that $\mathrm{sd}_{\gamma_2}(G) \leq 4$. Now let $G[N(v) \setminus \{v_1, v_2\}]$ not be a complete graph and let $v_3v_4 \notin E(G)$. Since G is claw-free, every vertex of $N(v) \setminus \{v_1, v_2, v_3, v_4\}$ is adjacent to v_1 or v_2 and v_3 or v_4. Let G' be obtained from G by subdividing the edges vv_1, vv_2, vv_3, vv_4 with vertices x_1, x_2, x_3, x_4, respectively. Let S be a $\gamma_2(G')$-set. If $S \cap \{x_1, x_2, x_3, x_4\} = \emptyset$, then $v \in S$ and $S \setminus \{v\}$ is a 2-dominating set for G. If $|S \cap \{x_1, x_2, x_3, x_4\}| \geq 2$, then $(S \setminus \{x_1, x_2, x_3, x_4\}) \cup \{v\}$ is a 2-dominating set for G. Let $|S \cap \{x_1, x_2, x_3, x_4\}| = 1$. Note that since G is claw-free and $v_1v_2, v_3v_4 \notin E(G)$, without loss of generality, we may assume $v_1v_3, v_2v_4 \in E(G)$. Now we see that $S \setminus \{x_1, x_2, x_3, x_4\}$ is a 2-dominating set for G. Thus $|S| > \gamma_2(G)$. This completes the proof. □

Corollary 15. *For any connected graph G of order $n \geq 4$, $\mathrm{sd}_{\gamma_2}(L(G)) \leq 4$.*

For a graph G, the *inflated* graph G_I is formed by replacing each vertex $v \in V$ with a clique of order $\deg(v)$, each vertex of which is also adjacent to exactly one vertex in a clique corresponding to a neighbor of v in G. Thus, each vertex in such a clique has the same degree in G_I as v has in G. Inflated graphs are examples of claw-free graphs with $\delta(G_I) = \delta(G)$, G_I contains a clique with $\delta(G)$ vertices and each vertex of that clique is of degree $\delta(G)$. Moreover, the inflated graph $G_I = L(S(G))$, where $S(G)$ is the subdivision graph of G obtained from G by subdividing each edge of G exactly once.

Corollary 16. *If G_I is the inflated graph of a graph G with $\delta(G) \geq 2$, then* $\mathrm{sd}_{\gamma_2}(G) \leq 4$.

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block. A *block graph* is a graph whose blocks are all complete graphs. A *cactus graph* is a graph in which every block of the graph is an edge or a cycle. In a *block-cactus graph* the blocks are either cycles or complete graphs.

Theorem 17. *If G is a connected block-cactus graph of order $n \geq 3$, then* $\mathrm{sd}_{\gamma_2}(G) \leq 2$.

Proof. If there is a vertex of degree 2, then, by Theorem 7, we are done. Thus, let us suppose that there is no vertex of degree 2. If there is an end block of order at least 3, then it has to be a clique. This implies that one of its vertices is a simplicial vertex of degree at least 2 and thus Corollary 4 yields the result. Therefore, we may assume that all end blocks of G are isomorphic to K_2. Let $\ell(G)$ be the set of leaves of G. If $G-\ell(G)$ consists of only a block, then set $B=G-\ell(G)$, otherwise let B be an end block of $G-\ell(G)$. Let $v\in V(B)$ be a vertex which is not a cut vertex in $G-\ell(G)$ but which is a support vertex in G. Now we distinguish three cases.

Case 1. Suppose that $V(B)=\{v\}$. Then G is a star $K_{1,n-1}$ and $\mathrm{sd}_{\gamma_2}(G)=1$.

Case 2. Suppose that $V(B)=\{u,v\}$ for a vertex $u\neq v$. Since no vertex has degree 2 in G, v is a support vertex of at least 2 leaves in G. Let $x\in \ell(G)\cap N_G(v)$ and let G' be the graph that is obtained from G by subdividing the edges uv and xv. Let S be a minimum 2-dominating set of G'. Of course, $\ell(G)\subset S$ and, without loss of generality, we can suppose that $u,v\in S$. This implies that $S-\{v\}$ is a 2 dominating set of G and hence $\mathrm{sd}_{\gamma_2}(G)\le 2$.

Case 3. Suppose that $n(B)\ge 3$. Let $v_1,v_2\in V(B)-\{v\}$ be two neighbors of v and let G' be the graph obtained from G by subdividing v_1v and v_2v with vertices x_1 and x_2, respectively. Let S be a minimum 2-dominating set of G'. Obviously, $\ell(G)\subseteq S$. If $v\in S$, then, without loss of generality, we can suppose that $v_1,v_2\in S$ and $x_1,x_2\notin S$. Hence, $S-\{v\}$ is a 2-dominating set of G with less vertices than $\gamma_2(G')$ and thus $\mathrm{sd}_{\gamma_2}(G)\le 2$. If $v\notin S$, then $x_1,x_2\in S$. In this case we obtain that $(S-\{x_1,x_2\})\cup\{v\}$ is a 2-dominating set of G with less vertices than $\gamma_2(G')$ and thus, again, $\mathrm{sd}_{\gamma_2}(G)\le 2$. □

Corollary 18. *If T is a tree of order at least 3, then $\mathrm{sd}_{\gamma_2}(T)\le 2$.*

Corollary 19. *If G is a connected block graph of order $n\ge 3$, then $\mathrm{sd}_{\gamma_2}(G)\le 2$.*

Corollary 20. *If G is a connected cactus graph of order at least 3, then $\mathrm{sd}_{\gamma_2}(G)\le 2$.*

4. Sufficient conditions for $\mathrm{sd}_{\gamma_2}=1$

Theorem 21. *Let G be a graph which is not empty. If all minimum 2-dominating sets of G are independent, then $sd_{\gamma_2}(G)=1$.*

Proof. Suppose that $\mathrm{sd}_{\gamma_2}(G)\ge 2$ and let G' be a graph obtained from G by subdividing an edge $uv\in E(G)$ with a vertex x. Then $\gamma_2(G)=\gamma_2(G')$ holds. Let D' be a minimum 2-dominating set of G'. Then either $x\in D'$ or $x\notin D'$ and $u,v\in D'$.

If $x\in D'$, then either $u\notin D'$ or $v\notin D'$, say $u\notin D'$. Since $\gamma_2(G)=\gamma_2(G')$, it follows that $D=(D'-\{x\})\cup\{u\}$ is a minimum 2-dominating set of G. Since $u\notin D'$, there has to be a neighbor $w\neq x$ of u in G' such that $w\in D'$. But then D contains the two adjacent vertices u and w and hence D is a minimum 2-dominating set of G which is not independent. Thus, suppose that $x\notin D'$ and $u,v\in D'$. Then D' is also a minimum 2-dominating set of G and, since $uv\in E(G)$, it is not independent. It follows that $\mathrm{sd}_{\gamma_2}(G)=1$. □

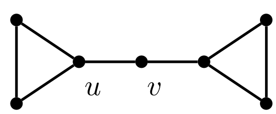
Since a minimum 2-dominating set of a graph with equal 2-domination and domination numbers is always independent (see [5]), we obtain the following corollary.

Corollary 22. *If G is a graph with $\gamma(G) = \gamma_2(G)$, then $\mathrm{sd}_{\gamma_2}(G) = 1$.*

Theorem 23. *Let G be a graph with $\Delta(G) \geq 2$. If G has a unique minimum 2-dominating set, then $sd_{\gamma_2}(G) = 1$.*

Proof. Let D be the unique minimum 2-dominating set of G. Since $\Delta(G) \geq 2$, $V(G) - D \neq \emptyset$. Let $v \in V(G) - D$ and let $u \in N(v) \cap D$. Let G' be the graph obtained from G by subdividing the edge uv with a vertex x. Suppose that $sd_{\gamma_2}(G) \geq 2$. Then $\gamma_2(G) = \gamma_2(G')$ holds. Let D' be a minimum 2-dominating set of G'. Then either $x \in D'$ or $x \notin D'$ and $u, v \in D'$. Assume first that $x \in D'$. This implies that $u \notin D'$ or $v \notin D'$. If $u, v \notin D'$, then both $(D' - \{x\}) \cup \{u\}$ and $(D' - \{x\}) \cup \{v\}$ are minimum 2-dominating sets of G, a contradiction to the hypothesis. Hence, suppose that $u \in D'$ and $v \notin D'$. Now we obtain that $D' - \{x\}$ is a 2-dominating set of G with less vertices than $|D'| = \gamma_2(G') = \gamma_2(G)$, again a contradiction. Thus we have that $x \notin D'$ and $u, v \in D'$. But then D' is a minimum 2-dominating set of G different from D because of $v \notin D$. Thus $\mathrm{sd}_{\gamma_2}(G) = 1$. □

Regarding the graph depicted bellow, it is easy to see that it has no independent minimum 2-dominating set and that there are more than one minimum 2-dominating sets. Subdivision of the edge uv shows that $\mathrm{sd}_{\gamma_2}(G) = 1$. Hence, neither the converse of Theorem 21 nor the converse of Theorem 23 do always hold.



We conclude this paper with the following problems.

Problem 1. *Characterize the trees achieving the bound of Corollary 18.*

Problem 2. *Prove or disprove: If G is a graph of order $n \geq 3$, then $1 \leq \mathrm{sd}_{\gamma_2}(G) \leq 3$.*

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