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A Note on Ternary Sequences of Strings of 0 and 1

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Abstract

B. D. Acharya has conjectured that if $\left(A_i : i = 1, 2, \ldots, 2^{|X|} - 1\right)$ is a permutation of all nonempty subsets of a set X with at least two elements such that for each even positive integer $j < 2^{|X|} - 1$, $A_{j-1} \triangle A_j \triangle A_{j+1} = \emptyset$, then $|X| = 2$. In this article, we show that if the cardinality of a set X is more than four, then a permutation as described above indeed exists.

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Throughout this article, $\mathbb{G}$ is the group defined on $\{0, 1\}$. Let n be any positive integer. We denote the identity of $\mathbb{G}^n$ usually by 0 and sometimes by z_n to avoid ambiguity. Let $\mathbb{F}^n$ be the set of all nonzero elements in $\mathbb{G}^n$; when $n \neq 1$, a permutation $(v_i : i = 1, 2, \ldots, 2^n - 1)$ of $\mathbb{F}^n$ is called *ternary* if for each $i \in \{2, 4, \ldots, 2^n - 2\}$, $v_{i-1} + v_i + v_{i+1} = 0$; when such a permutation exists, $\mathbb{G}^n$ is called *sequentially ternary.* In this article, elements in $\mathbb{G}^n$ are represented by strings of 'symbols' where each symbol is an element in a power of $\mathbb{G}$. For example, if $\alpha = 01$ and $\beta = 10$, then $1\alpha 0\beta$ is the element $(1, 0, 1, 0, 1, 0)$ in $\mathbb{G}^6$. .

In [1], B. D. Acharya has observed that $\mathbb{G}^3$ is not sequentially ternary and conjectured that if n is an integer which is larger than 2, then $\mathbb{G}^n$ is *not* sequentially ternary. This article is an outcome of settling this conjecture. For basic group theoretic results needed in this connection, we rely on [2].

Proposition 1. *For any integer $n > 2$, if $\mathbb{G}^n$ is sequentially ternary, then $\mathbb{G}^{n+2}$ is also sequentially ternary.*

Proof. Let $k = 2^n - 1$. Define sequences $\big(a_i\big)_{i=1}^{k}$, $\big(b_i\big)_{i=1}^{k}$, $\big(c_i\big)_{i=1}^{k}$ and $\big(d_i\big)_{i=1}^{k}$ in $\mathbb{G}^2$ as described below.

$$a_i = \begin{cases} 00, & \text{if } i \equiv 1 \pmod 4; \\ 01, & \text{otherwise.} \end{cases}$$

$$b_i = \begin{cases} 10, & \text{if } i \equiv 1 \pmod 2; \\ 00, & \text{otherwise.} \end{cases}$$

$$c_i = \begin{cases} 00, & \text{if } i \equiv 3 \pmod 4; \\ 11, & \text{otherwise.} \end{cases}$$

$$d_i = \begin{cases} 01, & \text{if } i \equiv 1 \pmod 4; \\ 11, & \text{if } i \equiv 3 \pmod 4; \\ 10, & \text{otherwise.} \end{cases}$$

It is easy to see that

(1) for each $i \in \{1, 2, \ldots, k\}$ the strings a_i, b_i, c_i and d_i are distinct and

(2) for each $i \in \{2, 4, \ldots, k-1\}$,

$$a_{i-1} + a_i + a_{i+1} = b_{i-1} + b_i + b_{i+1} = c_{i-1} + c_i + c_{i+1} = d_{i-1} + d_i + d_{i+1} = z_2.$$

Let $\big(v_i : i = 1, \ldots, k\big)$ be a ternary permutation of $\mathbb{F}^n$. Define a sequence $\big(w_i\big)_{i=1}^{4k+3}$ in $\mathbb{F}^{n+2}$ as described below.

For each $i \in \{1, 2, \ldots, k\}$, $w_i = v_{k-i+1}a_{k-i+1}$ and $w_{k+1} = z_n10$.

For each $i \in \{k+2, k+3, \ldots, 2k-1\}$, $w_i = v_{i-k-1}b_{i-k-1}$.

$w_{2k} = v_k b_k$, $w_{2k+1} = v_{k-1}b_{k-1}$, $w_{2k+2} = z_n11$, $w_{2k+3} = v_{k-1}c_{k-1}$ and $w_{2k+4} = v_k c_k$.

For each $i \in \{2k+5, 2k+6, \ldots, 3k\}$, $w_i = v_{3k+3-i}c_{3k+3-i}$.

$w_{3k+1} = v_1c_1$, $w_{3k+2} = v_2c_2$, $w_{3k+3} = z_n01$, $w_{3k+4} = v_2d_2$ and $w_{3k+5} = v_1d_1$.

For each $i \in \{3k+6, 3k+7, \ldots, 4k+3\}$, $w_i = v_{i-3k-3}d_{i-3k-3}$.

It can be verified that $\{w_i : i = 1, 2, \ldots, 4k+3\} = \{v_ia_i, v_ib_i, v_ic_i, v_id_i : i = 1, \ldots, k\} \cup \{z_n10, z_n11, z_n01\}$. Therefore by (1), all terms of $\big(w_i\big)_{i=1}^{4k+3}$ are distinct. From the definition of this sequence we have the following.

$$\begin{aligned} w_k + w_{k+1} + w_{k+2} &= v_1a_1 + z_n10 + v_1b_1; \\ w_{2k+1} + w_{2k+2} + w_{2k+3} &= v_{k-1}b_{k-1} + z_n11 + v_{k-1}c_{k-1}; \\ \text{and } w_{3k+2} + w_{3k+3} + w_{3k+4} &= v_2c_2 + z_n01 + v_2d_2. \end{aligned}$$

It is easy to see that for each of the above three equations, its right side is zero; therefore for all $i \in \{k+1, 2k+2, 3k+3\}$, $w_{i-1} + w_i + w_{i+1} = 0$; by using (2) it can be easily verified that for each $i \in \{2, 4, 6, \ldots, 4k+2\} \setminus \{k+1, 2k+2, 3k+3\}$ also, the just mentioned equality holds. Therefore, the permutation $\big(w_i : i = 1, \ldots, 4k+3\big)$ is ternary. $\square$

v_{01}	00001	01	$v_{01}a_{01}$	$z_5$10	$v_{01}b_{01}$		$v_{02}c_{02}$	$z_5$01	$v_{02}d_{02}$
v_{02}	00010	02	$v_{02}a_{02}$		$v_{02}b_{02}$		$v_{01}c_{01}$		$v_{01}d_{01}$
v_{03}	00011	03	$v_{03}a_{03}$		$v_{03}b_{03}$		$v_{03}c_{03}$		$v_{03}d_{03}$
v_{04}	00100	04	$v_{04}a_{04}$		$v_{04}b_{04}$		$v_{04}c_{04}$		$v_{04}d_{04}$
v_{05}	00111	07	$v_{05}a_{05}$		$v_{05}b_{05}$		$v_{05}c_{05}$		$v_{05}d_{05}$
v_{06}	01000	08	$v_{06}a_{06}$		$v_{06}b_{06}$		$v_{06}c_{06}$		$v_{06}d_{06}$
v_{07}	01111	15	$v_{07}a_{07}$		$v_{07}b_{07}$		$v_{07}c_{07}$		$v_{07}d_{07}$
v_{08}	00101	05	$v_{08}a_{08}$		$v_{08}b_{08}$		$v_{08}c_{08}$		$v_{08}d_{08}$
v_{09}	01010	10	$v_{09}a_{09}$		$v_{09}b_{09}$		$v_{09}c_{09}$		$v_{09}d_{09}$
v_{10}	10000	16	$v_{10}a_{10}$		$v_{10}b_{10}$		$v_{10}c_{10}$		$v_{10}d_{10}$
v_{11}	11010	26	$v_{11}a_{11}$		$v_{11}b_{11}$		$v_{11}c_{11}$		$v_{11}d_{11}$
v_{12}	00110	06	$v_{12}a_{12}$		$v_{12}b_{12}$		$v_{12}c_{12}$		$v_{12}d_{12}$
v_{13}	11100	28	$v_{13}a_{13}$		$v_{13}b_{13}$		$v_{13}c_{13}$		$v_{13}d_{13}$
v_{14}	01001	09	$v_{14}a_{14}$		$v_{14}b_{14}$		$v_{14}c_{14}$		$v_{14}d_{14}$
v_{15}	10101	21	$v_{15}a_{15}$		$v_{15}b_{15}$		$v_{15}c_{15}$		$v_{15}d_{15}$
v_{16}	11000	24	$v_{16}a_{16}$		$v_{16}b_{16}$		$v_{16}c_{16}$		$v_{16}d_{16}$
v_{17}	01101	13	$v_{17}a_{17}$		$v_{17}b_{17}$		$v_{17}c_{17}$		$v_{17}d_{17}$
v_{18}	10011	19	$v_{18}a_{18}$		$v_{18}b_{18}$		$v_{18}c_{18}$		$v_{18}d_{18}$
v_{19}	11110	30	$v_{19}a_{19}$		$v_{19}b_{19}$		$v_{19}c_{19}$		$v_{19}d_{19}$
v_{20}	10010	18	$v_{20}a_{20}$		$v_{20}b_{20}$		$v_{20}c_{20}$		$v_{20}d_{20}$
v_{21}	01100	12	$v_{21}a_{21}$		$v_{21}b_{21}$		$v_{21}c_{21}$		$v_{21}d_{21}$
v_{22}	11011	27	$v_{22}a_{22}$		$v_{22}b_{22}$		$v_{22}c_{22}$		$v_{22}d_{22}$
v_{23}	10111	23	$v_{23}a_{23}$		$v_{23}b_{23}$		$v_{23}c_{23}$		$v_{23}d_{23}$
v_{24}	11001	25	$v_{24}a_{24}$		$v_{24}b_{24}$		$v_{24}c_{24}$		$v_{24}d_{24}$
v_{25}	01110	14	$v_{25}a_{25}$		$v_{25}b_{25}$		$v_{25}c_{25}$		$v_{25}d_{25}$
v_{26}	10001	17	$v_{26}a_{26}$		$v_{26}b_{26}$		$v_{26}c_{26}$		$v_{26}d_{26}$
v_{27}	11111	31	$v_{27}a_{27}$		$v_{27}b_{27}$		$v_{27}c_{27}$		$v_{27}d_{27}$
v_{28}	10100	20	$v_{28}a_{28}$		$v_{28}b_{28}$		$v_{28}c_{28}$		$v_{28}d_{28}$
v_{29}	01011	11	$v_{29}a_{29}$		$v_{29}b_{29}$		$v_{29}c_{29}$		$v_{29}d_{29}$
v_{30}	10110	22	$v_{30}a_{30}$		$v_{31}b_{31}$		$v_{31}c_{31}$		$v_{30}d_{30}$
v_{31}	11101	29	$v_{31}a_{31}$		$v_{30}b_{30}$	$z_5$11	$v_{30}c_{30}$		$v_{31}d_{31}$

117	*089*	044	*081*	125	*069*	056	*101*	093	*109*	048	*073*	121	*077*	052
														097
004	*009*	013	*017*	028	*033*	061	*021*	040	*065*	105	*025*	112	*037*	085
002														
006	*008*	014	*016*	030	*032*	062	*020*	042	*064*	106	*024*	114	*036*	086
														096
088	*118*	046	*080*	126	*068*	058	*100*	094	*108*	050	*072*	122	*076*	054
003														
091	*116*	047	*083*	124	*071*	059	*103*	092	*111*	051	*075*	120	*079*	055
														099
011	*007*	012	*019*	031	*035*	060	*023*	043	*067*	104	*027*	115	*039*	084
001														
010	*005*	015	*018*	029	*034*	063	*022*	041	*066*	107	*026*	113	*038*	087
														098
119	*090*	045	*082*	127	*070*	057	*102*	095	*110*	049	*074*	123	*078*	053

Figure 1

Remark 2. *Figure* 1 *illustrates the method described by Proposition 1 for* $n = 5$*. It can be verified that the sequence of strings of length 5 which is displayed in this figure—for each* $k \in \{1, \dots, 31\}$*,* k*-th string is labelled by* v_{0k} *or* v_k *and treating each string as a binary number, its value in the decimal form is written on its right—is a ternary permutation of* $\mathbb{F}^5$*. The ordering of the other two sequences in the figure are indicated by directed lines. (Note that* $v_{31}b_{31}$*,* $v_{31}c_{31}$*,* $v_{01}c_{01}$ *and* $v_{01}d_{01}$ *appear in between* $v_{29}b_{29}$ *and* $v_{30}b_{30}$*,* $v_{30}c_{30}$ *and* $v_{29}c_{29}$*,* $v_{03}c_{03}$ *and* $v_{02}c_{02}$*, and* $v_{02}d_{02}$ *and* $v_{03}d_{03}$ *respectively.) The ternary permutation of* $\mathbb{F}^7$ *thus obtained is represented by a sequence of decimal numbers. (The even numbered terms of the sequence are italicized.)*

Let n be an integer which exceeds 1. Let u be any element in $\mathbb{F}^n$. It is easy to see that $\mathbb{G}^n$ can be partitioned into 2^{n-1} pairs such that sum of the elements of each pair is u; therefore the sum of all elements in $\mathbb{G}^n$ is 0. We use this (well known) fact to settle the first part of the next result.

Proposition 3. *Neither* $\mathbb{G}^3$ *nor* $\mathbb{G}^4$ *is sequentially ternary.*

Proof. Let $\big(a_i : i = 1, \dots, 7\big)$ be a permutation of $\mathbb{F}^3$. Since $a_1 + a_2 + \cdots + a_7 = 0$ either $a_1 + a_2 + a_3$ or $a_5 + a_6 + a_7$ is nonzero; therefore this permutation is not ternary.

Next, suppose that $\big(a_i : i = 1, \dots, 15\big)$ is a ternary permutation of $\mathbb{F}^4$. Let H be the subgroup of $\mathbb{G}^4$ which is generated by $\{a_2, a_3, a_4\}$. Obviously, $a_1, a_5 \in H$. Let us find the other two nonzero elements in H. Noting that if K is a subgroup of cardinality 4, then $|H \cap K| \geqslant 2$, because

$$|H \cap K| = \frac{|H| \times |K|}{|HK|} \geqslant \frac{|H| \times 4}{|\mathbb{G}^4|} \geqslant 2,$$

for each $i \in \{7, 9, 11, 13\}$, the fact that $\{0, a_i, a_{i+1}, a_{i+2}\}$ is a subgroup implies that $H \cap \{a_i, a_{i+1}, a_{i+2}\} \neq \emptyset$; from this, we find that $a_9, a_{13} \in H$. Thus $\{0, a_1, a_2, a_3, a_4, a_5, a_9, a_{13}\}$ is a subgroup; by symmetry, $\{0, a_{15}, a_{14}, a_{13}, a_{12}, a_{11}, a_7, a_3\}$ is also a subgroup. The cardinality of the intersection of these two subgroups is 3—a contradiction. ∎

01-000001	*16-010000*	17-010001	*35-100011*	50-110010	*15-001111*	61-111101
						33-100001
44-101100	*04-000100*	40-101000	*38-100110*	14-001110	*18-010010*	28-011100
63-111111						
19-010011	*49-110001*	34-100010	*46-101110*	12-001100	*21-010101*	25-011001
						27-011011
30-011110	*23-010111*	09-001001	*36-100100*	45-101101	*47-101111*	02-000010
43-101011						
53-110101	*13-001101*	56-111000	*39-100111*	31-011111	*11-001011*	20-010100
						32-100000
48-110000	*10-001010*	58-111010	*57-111001*	03-000011	*55-110111*	52-110100
06-000110						
54-110110	*08-001000*	62-111110	*59-111011*	05-000101	*24-011000*	29-011101
						07-000111
42-101010	*60-111100*	22-010110	*37-100101*	51-110011	*41-101001*	26-011010

Figure 2

Remark 4. *In Figure* 2*, a sequence of 63 terms is displayed; each term has an element of* $\mathbb{F}^6$ *and its value in decimal form. It can be verified that the strings of this sequence form a ternary permutation of* $\mathbb{F}^6$.

Now combining the information we have from Propositions 1 and 3 and Remarks 2 and 4, we get the following.

Theorem. *For any integer* $n \geqslant 2$, $\mathbb{G}^n$ *is sequentially ternary if and only if* n *is neither 3 nor 4.*

References

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