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On Harmoniousness of Hypercubes

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Abstract

In the introductory paper on harmonious labelling, in 1980, Graham and Sloane have stated that by using a computer it has been verified that the 3-dimensional hypercube Q_3 is not harmonious. Theoretical proof is not available till today. In this paper we give a theoretical proof that Q_3 is not harmonious. Moreover, whether Q_n, $n \geq 4$, is harmonious or not is an open problem. Here we give harmonious labellings to Q_4 and Q_5. We also conjecture that Q_n is harmonious for every n, $n \geq 6$.

Keywords: Harmonious labelling, Harmonious graph, n-dimensional Hypercube, Even and Odd edges.

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1. Introduction

The concept of harmonious graph was introduced by Graham and Sloane [1] in 1980. It arose in their study of modular versions of additive bases problems stemming from error correcting codes. Let $G = (V, E)$ be a simple connected graph different from a tree with q edges. An injective mapping $f : V(G) \to \mathbb{Z}_q$ (the set of integers modulo q) is said to be a harmonious labelling of G if the induced mapping $f^* : E(G) \to \mathbb{Z}_q$ defined by $f^*(uv) = f(u) + f(v) \pmod q$, for each edge $uv \in E(G)$, is bijective. In case of a tree exactly one vertex label is allowed to repeat. A graph G with a harmonious labelling is called a harmonious graph. Existing results on harmonious graphs can be found in the survey by Gallian [2].

An n-dimensional hypercube Q_n is obtained by taking two copies of of Q_{n-1} and making their corresponding vertices adjacent, where Q_1 is the path P_2 on two vertices. The hypercube Q_n is n-regular bipartite graph with 2^n vertices and $n2^{n-1}$ edges. Trivially Q_1 is harmonious. One can easily verify that Q_2 is not harmonious. By using a computer Graham and Sloane [[1],Theorem 22] have checked that the 3-dimentional hypercube Q_3 is not harmonious. In this paper we give a theoretical proof of the result Q_3 is not harmonious. Although the proof is lengthy and bit involved, the idea used here may be extended to get harmonious labelling of higher dimensional hypercubes. Here we also give harmonious labellings to Q_4 and Q_5 and conjecture that Q_n is harmonious for every n, $n \geq 6$.

Next we state a result in [1] which is very important for this paper.

Theorem 1.1. [1] *If f is a harmonious labelling of a graph G with q edges, then so is $rf+s$, where r is an invertible element of $\mathbb{Z}_q$ and s is any element of $\mathbb{Z}_q$.*

Corollary 1.2. *Any vertex in a harmonious graph can be assigned the label* 0.

The following terminologies are used in this paper.

Definition 1.3. *Let G be a harmonious graph with a harmonious labelling f. By the notation $f(G)$, we mean the labelled graph G by the labelling f. An edge e (respectively a vertex x) of $f(G)$, is called an odd or even edge (respectively vertex) if $f^*(e)$ (respectively $f(x)$) is odd or even.*

2. Results

Motivated by a result in [3] (Theorem 2), we give the following analogous result for a harmonious graph with an even number of edges.

Theorem 2.1. *Let G be a harmonious graph with an even number of edges and a harmonious labelling f. Then every cycle C of $f(G)$ contains an even number of odd edges.*

Proof. Let q be the number of edges of G. Then $\sum_{e\in E(C)} f^*(e) = 2\sum_{x\in V(C)} f(x) (\mathrm{mod}\, q)$ for any cycle C of G. Since q as well as $2\sum_{x\in V(C)} f(x)$ are even numbers, $\sum_{e\in E(C)} f^*(e)$ is also an even integer. If C contains an odd number of odd edges then $\sum_{e\in E(C)} f^*(e)$ cannot be even. □

Next, in the following observation we list some of the easy properties of the hypercube Q_3 which we use in the sequel.

Observation 2.2.

(*i*) *Any two faces (or 4-cycles) in Q_3 are either disjoint or share exactly one edge. Every edge in Q_3 is contained in exactly two faces. Any two non-adjacent vertices in Q_3 have either zero or two common neighbors in it.*

(*ii*) *Using first part of this observation one gets that in Q_3 every 6-cycle occurs from two intersecting faces by removing their common edge. Similarly every 8-cycle occurs from the union of three faces C_1, C_2, C_3, by removing the two common edges, where C_1 and C_3 are disjoint but C_2 shares edges with each of them.*

Theorem 2.3. *If Q_3 is harmonious with a harmonious labelling f then the following are all possible combinations of odd and even edges of $f(Q_3)$. Here e stands for an even edge and o stands for an odd edge.*

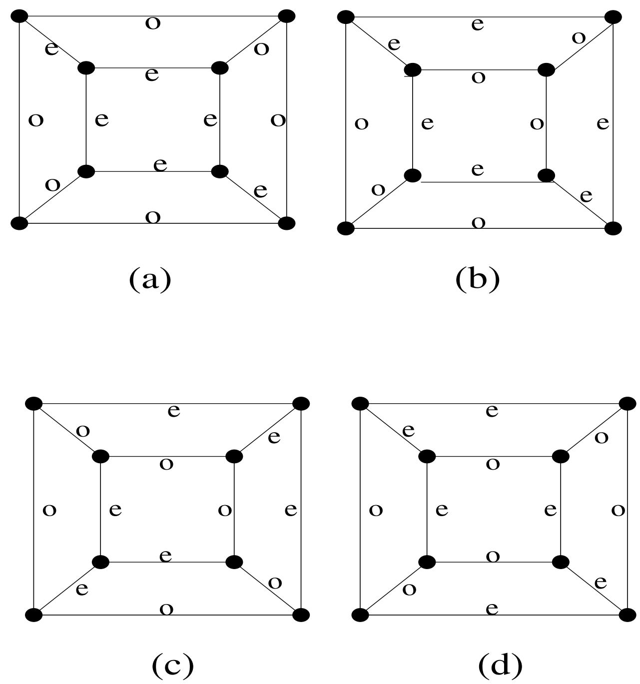


Figure 1: Possible combinations of odd and even edges of $f(Q_3)$

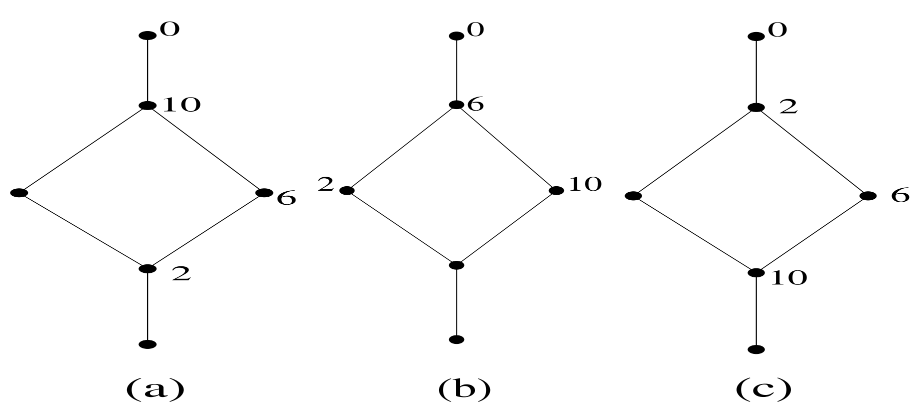
Proof. In view of Theorem 2.1, every cycle in $f(Q_3)$ contains an even number of odd edges and hence an even number of even edges also as Q_3 has only even cycles. In $f(Q_3)$ either there is a 4-cycle for which every edge is an even edge or there is no such cycle at all. If $f(Q_3)$ contains such a cycle then by Theorem 2.1 and since there are six even edges only, the combination of odd and even edges of $f(Q_3)$ is as in Figure 1(a).

Next suppose that $f(Q_3)$ contains no 4-cycle where every edge is an even edge. If $f(Q_3)$ contains a 4-cycle for which every edge is an odd edge then we get the same combination as in Figure 1(a). Then it remains to see the case where every 4-cycle of $f(Q_3)$ contains exactly two even edges. In this case, $f(Q_3)$ may contain a 4-cycle for which two opposite edges are even edges or may not contain such a 4-cycle. If $f(Q_3)$ contains such a cycle then by Theorem 2.1 (and since there are only six possible even edges) we get the combination of $f(Q_3)$ as in Figure 1(d). Finally, let for every 4-cycle of $f(Q_3)$ exactly two edges are even and these two edges be adjacent. Let C be a 4-cycle in $f(Q_3)$ with only two (adjacent) even edges e_1 and e_2. Then the even and odd edges of the opposite 4-cycle C' (the 4-cycle with no common edge with C) of C will be determined. Next, assignment of either odd or even label to an edge that is not in $C \cup C'$ leads to the combination as in Figure 1(b) or 1(c). The combinations in Figure 1(b) and 1(c) are different because if we go for their vertex labels then we find that in one of them four even and in the other two or six even labels are there.

In the above we have seen that in any combination of Figure 1, every 4-cycle has an even number of odd edges. But this should also hold for all other cycles according as Theorem 2.1, otherwise we get the result here itself that Q_3 is not harmonious. Note that Q_3 is bipartite and therefore other possible cycles in it are 6-cycles and 8-cycles only. Let S be a 6-cycle in any of the combinations of Figure 1. Then S can be obtained from two intersecting 4-cycles C_1 and C_2 by removing their common edge l (in view of observation 2.2 (ii)). Let e_i be the number of odd edges in C_i, $i = 1, 2$. Now the number of odd edges in S is $e_1 + e_2 - 2$ or $e_1 + e_2$ according as l is odd or even. Similarly, we check that every 8-cycle in all the combinations of Figure 1 contains an even number of odd edges. □

Theorem 2.4. *If f is a harmonious labelling of Q_3 then the combination of odd and even edges of $f(Q_3)$ cannot be as in Figure 1(a), 1(b) or 1(c).*

Proof. If the combination of $f(Q_3)$ is as in Figure 1(a), then the six even edges in $f(Q_3)$ induce a subgraph S consisting of a 4-cycle and two pendant edges incident on (any) two opposite vertices of the 4-cycle. Then the vertex labels of S are either all odd or all even. Because of Theorem 1.1, we may assume that all the vertex labels of S are even and by the Corollary 1.2, we may assign the label zero to any vertex of S. So let the label of a pendant vertex, say x_1, of S be zero. Then the neighbor of x_1, say x_2, in S may get the label 2,4,6,8 or 10. The edge label 4 can be obtained from the pairs $\{0, 4\}$ or $\{6, 10\}$ only and the edge label 8 can be obtained from the pairs $\{0, 8\}$ or $\{2, 6\}$ only. So until now the vertex label of some of the vertices in S will have the following form.



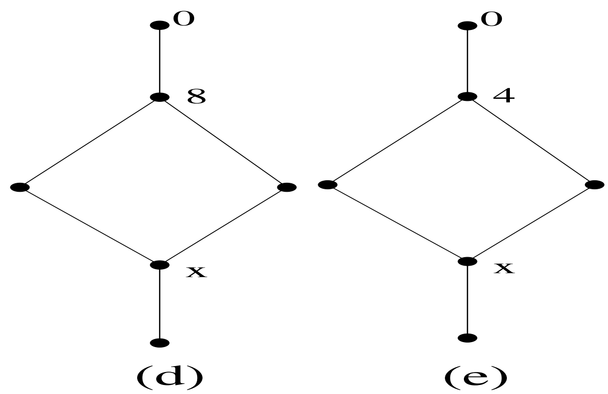


Figure 2: Possible labelling of vertices in S

It is not possible to proceed further for Figures 2(a),(b) and (c) because assigning any new label to any unlabelled vertex give a contradiction. In Figures 2(d) and 2(e) let x be the degree three vertex other than x_2. In Figure 2(d), to generate edge label 4, vertex labels 6 and 10 have to be adjacent. So $x = 6$ or 10. If $x = 6$ then vertex label 6 is adjacent to the vertex labels 2, 4 and 10, which gives a contradiction as 6+2 = 0+8. If $x = 10$ then the vertex label 10 is adjacent to the vertex labels 2,4 and 6, and the vertex label 8 is at least adjacent to 4 or 6 or both. This leads to a contradiction that either the edge label zero or the edge label 2 appears twice. In the similar way Figure $2(e)$ can also be ruled out.

If the combination of $f(Q_3)$ is as in Figure 1(b), then the six even edges in $f(Q_3)$ form a cycle C on six vertices, say a_i, $1 \le i \le 6$. The labels of the vertices of C are either all even or all odd. By Theorem 1.2 we may assume that the label of each a_i, $1 \le i \le 6$, is even. Let $l = \sum_{i=1}^{6} f(a_i)$. Then we have $2l = 0+2+4+6+8+10\ (=30) = 6\ (\text{mod } 12)$ or $l = 3\ (mod\ 6)$. So $l-3$ is even as is divisible by 6. This is a contradiction because l itself is even as each $f(a_i)$ is even.

If the combination of $f(Q_3)$ is as in Figure 1(c), then the six odd edges in $f(Q_3)$ form a cycle C_1 on six vertices, say b_i, $1 \le i \le 6$. The labels of the vertices of C_1 are alternately even and odd. So three vertex labels of C_1 are even and three are odd. Let $l_1 = \sum_{i=1}^{6} f(b_i)$. Then l_1 is an odd integer. Now we have $2l_1 = 36\ (\text{mod } 12)$. Then $l_1 = 0\ (\text{mod } 6)$ and we get l_1 is an even integer, which is a contradiction. □

Next we show that the combination of odd and even edges of $f(Q_3)$ cannot be also as in Figure 1(d), if f is a harmonious labelling of Q_3. But, before this we make one observation and prove a lemma below.

Observation 2.5. *If f is a harmonious labelling of Q_3 and if the combination of $f(Q_3)$ is as in Figure 1(d) then we observe the following.*

(*i*) *All the even edges in* $f(Q_3)$ *are partitioned in to two disjoint paths* P *and* P' *of length three each. There are two odd edges in* $f(Q_3)$ *together with which* P *and* P' *form a* C_8 *in* $f(Q_3)$*. Because of these two odd edges the vertex labels of one path are all even and the vertex labels of the other path are all odd.*

(*ii*) *Let* P *be the path* $v_1v_2v_3v_4$, P' *be the path* $v'_1v'_2v'_3v'_4$, *and the pair of odd edges together with which* P *and* P' *form a* C_8 *in* $f(Q_3)$ *be* $v_1v'_1$ *and* $v_4v'_4$*. Then the remaining four odd edges in* $f(Q_3)$ *are* $v_2v'_4$, $v_1v'_3$, $v_3v'_1$ *and* $v_4v'_2$.

(*iii*) *By picking of any even edge in* $f(Q_3)$ *and assigning either even or odd labels to both the end vertices of it, the labels of other vertices whether even or odd are determined. So we find that there are four even and four odd vertices in* $f(Q_3)$*. Consider the partition* $\{A_i : A_i = \{x \in \mathbb{Z}_{12} : x = i \pmod 4\},\ 0 \le i \le 3\}$ *of* $\mathbb{Z}_{12}$*. Then the set of vertex labels of* $f(Q_3)$ *intersects each* A_i, $0 \le i \le 3$.

(*iv*) *Here we make one number theoretic observation that for any* $x \in A_i$ *and* $y \in A_j$, $0 \le i,j \le 3$, $x + y \in A_{i+j(\text{mod } 4)}$.

Definition 2.6. *Partition the odd edges (here we do not distinguish between an edge and its label) of* $f(Q_3)$ *into three pairs of independent edges* $R_1 = \{v_1v'_1, v_4v'_4\}$, $R_2 = \{v_1v'_3, v_2v'_4\}$ *and* $R_3 = \{v_3v'_1, v_4v'_2\}$*. We shall say that edges in any* R_i *are of the same parity if they are from the same* A_j *(*$j =$ *1 or 3), otherwise they are of different parity.*

Lemma 2.7. *If* f *is a harmonious labelling of* Q_3 *and the combination of* $f(Q_3)$ *is as in Figure 1(d) then the following hold.*

(*i*) *The edges in any* R_i *are of the same parity if either the even ends of both the edges are from the same* A_k *and the odd ends of both the edges are also from the same* A_j *or all the four labels of the end vertices of the edges in* R_i *are from the different* A_l*. The edges in any* R_i *are of different parity if either the even ends are from the same set and the odd ends are from the different sets or viceversa.*

(*ii*) *We will have* $R_1 \cup R_2 \cup R_3 = A_1 \cup A_3$ *if and only if either for every* R_i, $1 \le i \le 3$, *the edges are of different parity or for only one* R_i, *the edges are of different parity and for the remaining two if the edges in one are from* A_1 *then the edges in the other are from* A_3.

Proof. (*i*) For any t, $t = 1, 2$ or 3, let the two edges of R_t, be x_1y_1 and x_2y_2. If $x_1, x_2 \in A_i$ and $y_1, y_2 \in A_j$ then by Observation 2.5 (*iv*), both x_1y_1 and x_2y_2 are in $A_{i+j(\text{mod } 4)}$. If $x_1 \in A_i$, $y_1 \in A_j$, $x_2 \in A_k$ and $y_2 \in A_l$, where i, j, k, l are all distinct, $\{i,k\} = \{0,2\}$ and $\{j,l\} = \{1,3\}$ then we get $i = k + 2 \pmod 4$, $j = l + 2 \pmod 4$, and $i + j = k + 2 + l + 2 \pmod 4 = k + l \pmod 4$. So both x_1y_1 and x_2y_2 are in $A_{i+j(\text{mod } 4)}$. Next, let $x_1, x_2 \in A_i$, $y_1 \in A_j$ and $y_2 \in A_k$, where $j \ne k$. Then $i + j \ne i + k \pmod 4$

and the edges of R_t are of different parity.
(ii) This is true because the three elements of A_1 (or A_3) can be distributed to R_1, R_2 and R_3 one each or two to one, only one to one and nothing to the remaining one. □

Theorem 2.8. *If f is a harmonious labelling of Q_3 then the combination of $f(Q_3)$ cannot be as in Figure 1(d).*

Proof. From Observation 2.5 (iii) since there are four even and four odd vertices in $f(Q_3)$ and the set of vertex labels of $f(Q_3)$ intersects each $A_i,\ 0 \le i \le 3,$ the vertex labels of $f(Q_3)$ have the following possibilities (1) two from each $A_i,\ 0 \le i \le 3,$ (2) three from any of A_0 or A_2 and three from any of A_1 or A_3 and one each from the rest and (3) three from any of $A_i,\ 0 \le i \le 3,$ one from $A_{i+2(\text{mod } 4)}$ and two each from the rest.

Here onwards we use notations a and b to represent elements of $A_0 \cup A_2$ such that whenever $a \in A_0,\ b \in A_2$ and viceversa. Similarly we use notations c and d to represent elements of $A_1 \cup A_3$. We make clarification that whenever we use any of the above notations more than once for vertex labelling of $f(Q_3)$ it does not mean that they are equal. However, it means that they are all different but from the same set $A_i,$ for some i.

By Observation 2.5 (i), the vertex labels of the path P are either all even or all odd. Because of Theorem 1.1, without loss of generality, let the vertex labels of P be even.

Now in the possibility (1) the vertex labels of the path P are consist of two a and two b and the vertex labels of the path P' are consist of two c and two d. If a and b appear alternately in the vertices of P then c and d also appear alternately in the vertices of P' because in the former all the edge labels are from A_2 and in the latter all the edge labels are from A_0 and we have that the set of even edge labels in $f(Q_3)$ is $A_0 \cup A_2$. Here we find that the labels of the end vertices of the pair of edges in every R_i are from the different A_i. So we get contradiction due to Lemma 2.7 (ii). Next we may have either $f(v_1) = a,\ f(v_2) = b,\ f(v_3) = b,\ f(v_4) = a,\ f(v'_1) = c,\ f(v'_2) = d,\ f(v'_3) = d,\ f(v'_4) = c$ or $f(v_1) = a,\ f(v_2) = a,\ f(v_3) = b,\ f(v_4) = b,\ f(v'_1) = c,\ f(v'_2) = c,\ f(v'_3) = d,\ f(v'_4) = d$. Here also we get that the edges in each $R_i,\ 1 \le i \le 3,$ are of the same parity and is a contradiction due to Lemma 2.7 (ii).

In the possibility (2) the vertex labels of the path P consist of three a and one b and the vertex labels of the path P' are also consist of three c and one d. Since the set of edge labels of $P \cup P'$ is $A_0 \cup A_2,$ we have either the three a are consecutive in P and the three c are consecutive in P' or the three a are not consecutive in P and the three c are not consecutive in P'. One can check that in all these cases we get either the edges in all R_i are of the same parity or the edges in only one R_i are of the same parity, which lead to contradiction because of Lemma 2.7 (ii).

In the possibility (3) we have that either the vertex labels of P consist of three a and one b and that the vertex labels of P' consist of two c and two d or the vertex labels of P consist of two a and two b and that the vertex labels of P' consist of three c and one d. The latter possibility can be converted to the former by Theorem 1.1. So we shall look at the former only. In the former one we have the following six possibilities given in Figure 3.

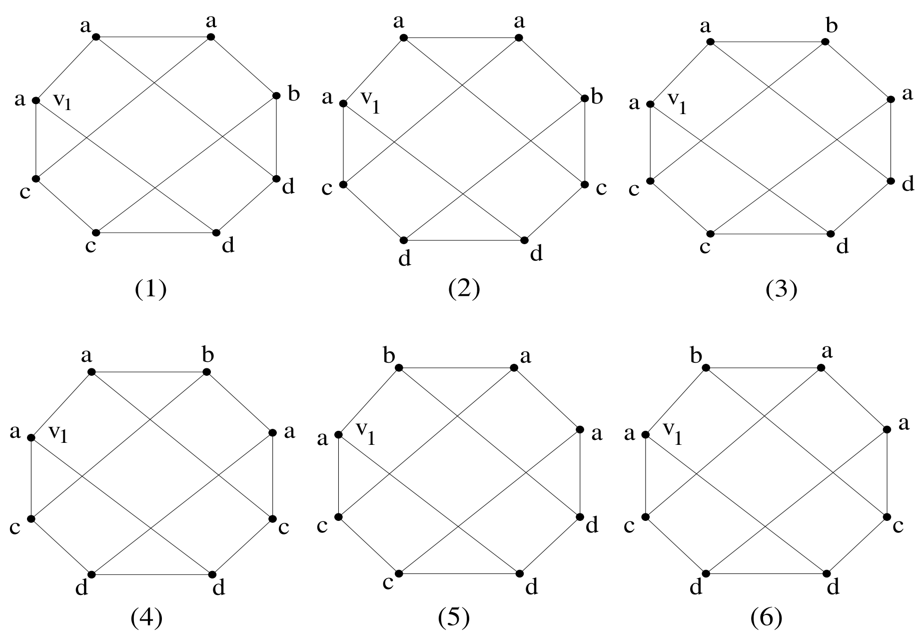


Figure 3:

In each of the Figures 3.2, 3.3 and 3.5, there are exactly two R_i for which the edges are of different parity and so we get contradiction by Lemma 2.7 (ii).

In Figure 3.1, without loss of generality (by corollary 1.2) let $f(v_1) = 0$. So $f(v_1), f(v_2)$, $f(v_3) \in A_0$, $f(v_4) \in A_2$. Recall that all equations and equalities of vertex or edge labels here onwards are up to modulo 12. Now by Observation 2.5 (iv), $f^*(v_1v_3')$, $f^*(v_2v_4')$ and $f^*(v_4v_2')$ are all in the same set A_k ($k =$ 1 or 3), where $f(v_3')$ and $f(v_4')$ are also there. Since $f^*(v_1v_3') = f(v_3')$ and $f^*(v_2v_4')$ cannot be equal to $f(v_4')$, we get the following.

$$f(v_4) + f(v_2') = f^*(v_4v_2') = f(v_4') \qquad (2.1)$$

Since $f^*(v_1v_2)$, $f^*(v_2v_3)$, $f^*(v_2'v_3')$ and $f(v_3)$ are all in A_0 and first two edge labels are not equal to $f(v_3)$, we have the following.

$$f(v_2') + f(v_3') = f^*(v_2'v_3') = f(v_3) \qquad (2.2)$$

Now $A_2 = \{f^*(v_3v_4), f^*(v_1'v_2'), f^*(v_3'v_4')\}$ and $f(v_4)$ is also in A_2. So we get either $f^*(v_1'v_2') = f(v_4)$ or $f^*(v_3'v_4') = f(v_4)$. If $f^*(v_1'v_2') = f(v_4)$ then putting this value of $f(v_4)$ in equation 2.1 we get $f(v_2') + f(v_1') + f(v_2') = f(v_4')$. Adding both sides $f(v_3')$ and using equation 2.2 we get $f(v_3) + f(v_4) = f(v_3') + f(v_4')$, which is a contradiction because all edge labels have to be distinct. Similarly if $f^*(v_3'v_4') = f(v_4)$ then together with equation 2.1 we get $f(v_2') + f(v_3') + f(v_4') = f(v_4')$. So $f(v_2') + f(v_3') = 0$, that is from equation 2.2 $f(v_3) = 0$, which is a contradiction because $f(v_1) = 0$.

Next consider Figures 3.4 and 3.6 simultaneously. Without loss of generality let $f(v_1) = 0$. In both these cases we have $f^*(v_1'v_2')$ and $f^*(v_3'v_4')$ are in the same A_i, so $f^*(v_3'v_4') = f^*(v_1'v_2') \pm 4$. Since $f(v_2')$ and $f(v_3')$ are also both in the same A_j, we have $f(v_3') = f(v_2') \pm 4$. From the previous two statements we get

$$f(v_2') + f(v_4') \pm 4 = f(v_1') + f(v_2') \pm 4 \tag{2.3}$$

If on both sides of equation (2.3) either +4 or -4 is there then we get $f(v_4') = f(v_1')$, which is a contradiction because all vertex labels are distinct. If on one side of equation (2.3), +4 appears and on the other side -4 appears then we get $f(v_4') + f(v_2') \pm 4 = f(v_1') + f(v_2')$, because +8= -4 and -8 = +4. But we have $f(v_2') \pm 4 = f(v_3')$ or $f^*(v_2'v_4)$, because $\{f(v_2'), f(v_3'), f^*(v_2'v_4)\} = A_1$ or A_3. If $f(v_2') \pm 4 = f(v_3')$ then we get $f(v_4') + f(v_3') = f(v_1') + f(v_2')$, which is a contradiction as two distinct edges $v_4'v_3'$ and $v_1'v_2'$ have the same label. If $f(v_2') \pm 4 = f^*(v_2'v_4)$ then $f(v_4') + f(v_4) = f(v_1')$, which is a contradiction because $f^*(v_1v_1') = f(v_1')$ and two distinct edges $v_4'v_4$ and v_1v_1' get the same label. □

The following result is clear from Theorems 2.3, 2.4 and 2.8.

Theorem 2.9. *The three dimensional hypercube Q_3 is not harmonious.*

Theorem 2.10. *The 4-dimensional and 5-dimensional hypercubes are harmonious.*

Proof. The following labellings of Q_4 and Q_5 proves the theorem.

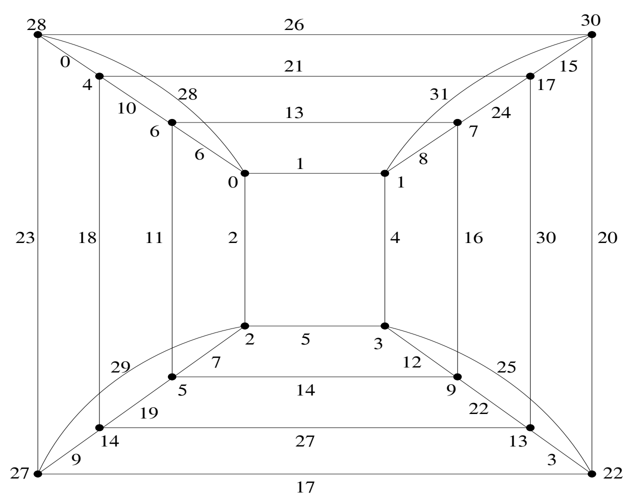


Figure 4: Harmonious labelling of Q_4

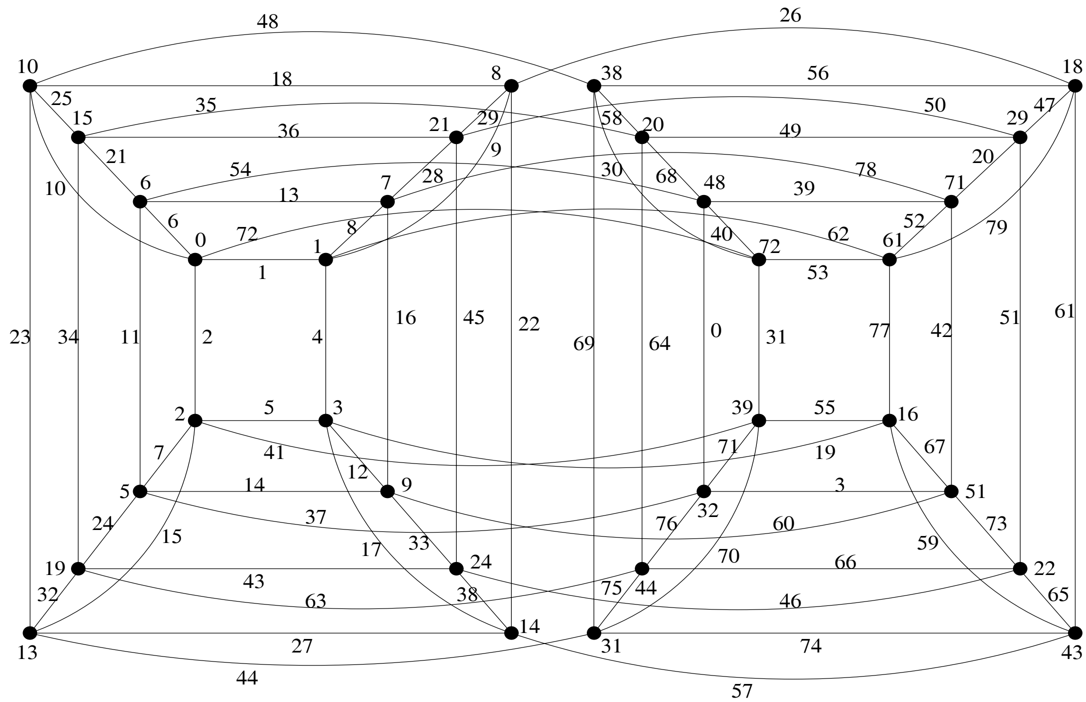


Figure 5: Harmonious labelling of Q_5

While proving that Q_3 is not harmonious we have found several labellings which were very close to a harmonious labelling of Q_3. We realize that the drawback found in Q_3 can be overcome in higher dimensional hypercubes. As we have also obtained harmonious labelling of Q_4 and Q_5 we make the following conjecture. □

Conjecture: The hypercube Q_n, $n \geq 6$, is harmonious.

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