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On square sum graphs

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Abstract

Let $G = (V, E)$ be a (p, q)-graph and let $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ be a bijection. We define f^* on $E(G)$ by $f^*(uv) = [f(u)]^2 + [f(v)]^2$. If f^* is injective on $E(G)$, then f is called a square sum labeling. If $f^*(E)$ consists of the first q consecutive integers of the form $a^2 + b^2,\ a \leq p-1,\ b \leq p-1,\ a \neq b$, then f is said to be a *strongly square sum labeling* of G. The graph G is said to be a *square sum graph* (strongly square sum graph) if G admits a square sum (strongly square sum) labeling. In this paper we initiae a study of graphs which are square sum and strongly square sum.

Keywords: square sum graph, strongly square sum graph.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [6]. For basic number theoretic results we refer to [2], [3] and [7].

Abundant literature exists concerning the structure of graphs admitting a variety of functions that assign real numbers to their vertices and edges so that certain given conditions are satisfied. For a recent survey on graph labelings we refer to Gallian [4]. Here we are interested in the study of vertex functions $f : V(G) \to \{0, 1, \ldots, p-1\}$ for which the induced edge function $f^* : E(G) \to N$ defined as $f^*(uv) = [f(u)]^2 + [f(v)]^2$, for all $uv \in E(G)$, is injective. This new type of labeling of graphs is closely related to the Diophantine equation $x^2 + y^2 = n$. It is important to note that certain numbers like 3, 6, 7 etc. cannot be written as the sum of two squares. Therefore we study the graphs whose edge labels take the first consecutive numbers that can be expressed as sum of two squares.

We need the following results from number theory.

Theorem 1.1. [7] *The product of any two representable integers is also representable. (The word representable means representable as a sum of two squares).*

Theorem 1.2. [2] *No prime p of the form $4k + 3$ is a sum of two squares.*

Theorem 1.3. [7] *(i) Every square integer is of the form $4q$ or $4q + 1$.*

(ii) Every square integer is of the form $5q, 5q + 1$ or $(5q - 1)$.

Theorem 1.4. [2] *The set $\{3, 4, 5\}$ is the only primitive pythagorian triple involving consecutive integers.*

Theorem 1.5. [7] *The square of an odd integer is of the form $8q + 1$.*

Theorem 1.6. [2] *Let a and b be two integers of opposite parity such that $a > b > 0$ and a and b are relatively prime. Then $(a^2 - b^2, 2ab, a^2 + b^2)$ is a primitive solution of the Pythagorian equation $x^2 + y^2 = z^2$.*

Theorem 1.7. [2] *In any four consecutive integers, at least one is not representable as a sum of two squares.*

2. Square Sum Graphs

Definition 2.1. *Let $G = (V, E)$ be a (p, q)-graph and let $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ be a bijection. We define f^* on $E(G)$ by $f^*(uv) = [f(u)]^2 + [f(v)]^2$. If f^* is injective on $E(G)$, then f is called a square sum labeling. The graph G is said to be a square sum graph if G admits a square sum labeling.*

Following are some simple observations which follow immediately from the definition of a square sum graph.

Observation 2.2. *If G is a square sum graph with a square sum labeling f, then $[f(u)]^2$ occurs $d(u)$ (degree of u) times in the sum $\sum f^*(e)$ and hence we have*

$$\sum_{e \in E(G)} f^*(e) = \sum_{u \in V(G)} [f(u)]^2 d(u).$$

In particular, if G is r-regular then

$$\sum_{e \in E(G)} f^*(e) = \frac{rp(p-1)(2p-1)}{6}.$$

Observation 2.3. *It follows from Theorem 1.1 that* $\prod f^*(e)$ *is a sum of two squares.*

Observation 2.4. *If* $e = uv \in E(G)$ *and* $f(u) = 0$, *then* $f^*(e)$ *is a perfect square and hence is congruent to 0 or* $1(mod\ 4)$.

Theorem 2.5. *Let* G *be a connected square sum graph with a square sum labeling* f. *Then* $f^*(e) \equiv 1(mod\ 2)$ *for at least one edge* $e \in E(G)$. *Further if* $f^*(e) \equiv 1(mod\ 2)$ *for all* $e \in E(G)$, *then* G *is bipartite.*

Proof. Let $X = \{u : f(u) \text{ is even}\}$ and $Y = \{v : f(v) \text{ is odd}\}$. Since G is connected there exists at least one edge $e = uv$ such that $u \in X$ and $v \in Y$. Hence $f^*(e) \equiv 1(mod\ 2)$. If $f^*(e) \equiv 1(mod\ 2)$ for all $e \in E(G)$, it follows that $f(u)$ and $f(v)$ are of opposite parity and X and Y form a bipartition of G. □

In the following theorems we present several families of square sum graphs.

Theorem 2.6. *The graph* $G = K_2 + mK_1$ *is a square sum graph.*

Proof. Let $V(G) = \{u_1, u_2, ..., u_{m+2}\}$ where $V(K_2) = \{u_1, u_2\}$. Define $f : V(G) \to \{0, 1, ..., m+1\}$ by $f(u_i) = i-1$ for $1 \le i \le m+2$. If $f^*(u_1u_i) = f^*(u_2u_j)$, then $[f(u_1)]^2 + [f(u_i)]^2 = [f(u_2)]^2 + [f(u_j)]^2$. Since $f(u_1) = 0$ and $f(u_2) = 1$, we get $[f(u_i)]^2 - [f(u_j)]^2 = 1$, so that either $f(u_i) = 0$ or $f(u_j) = 0$, which is a contradiction. Hence f^* is injective and G is a square sum graph. □

Theorem 2.7. *The complete graph* K_n *is square sum if and only if* $n \le 5$.

Proof. The square sum labelings of K_n for $n \le 5$ are given in Figure 1. Further since $0^2 + 5^2 = 3^2 + 4^2$, it follows that $K_n,\ n \ge 6$, is not a square sum graph. □

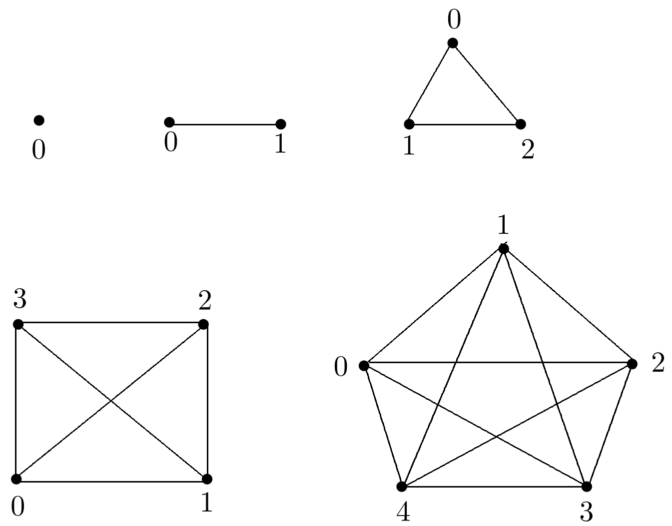

Figure 1. Square Sum Labeling of $K_n, 1 \le n \le 5$

Theorem 2.8. *Cycles are square sum graphs.*

Proof. Let $C_n = (u_1, u_2, \ldots, u_n, u_1)$. Let $e_i = u_i u_{i+1}, 1 \leq i \leq n-1$ and $e_n = u_n u_1$.

Case i. n is odd.

Define $f : V(C_n) \to \{0, 1, ..., n-1\}$ by $f(u_i) = i - 1$ for $1 \leq i \leq n$. Since $f(u_i) < f(u_j)$ if $i < j$, it follows that $f^*(e_i) < f^*(e_j)$ if $1 \leq i < j < n$. Also, since n is odd $f^*(e_n)$ is even and $f^*(e_i)$ is odd if $j < n$. Hence f^* is injective. Hence $f^*(e_i) \neq f^*(e_j), i \neq j$, for every $e_i, e_j \in E(C_n) - \{u_n u_1\}$. When n is odd, $f(u_n)$ is even. Therefore $f^*(u_n u_1)$ is even and hence $f^*(u_n u_1) \neq f^*(e_j)$ for every $e_j \in E(C_n) - \{u_n u_1\}$; hence f^* is injective and f is a square sum labeling on C_n.

Case ii. n is even and $n \neq 6$.

By Theorem 1.4, the equation $a^2 + (a+1)^2 = c^2$ has only one non-zero positive integer solution which are given by $a = 3$ and $c = 5$. Hence for $C_p, p \neq 6$, the labeling given in Case (i) is a square sum labeling.

For C_6 it can be easily verified that the labeling given by $f(u_i) = i - 1, 1 \leq i \leq 4, f(u_5) = 5$ and $f(u_6) = 4$ is a square sum labeling. □

Theorem 2.9. *Trees are square sum graphs.*

Proof. Let T be a tree and let $v_1 \in V(T)$ be a vertex with maximum degree. Starting from the vertex v_1, and applying BFS algorithm [5], label the vertices of T with $0, 1, 2, ..., p-1$ in the order in which they are visited. Since f is increasing on the vertex set of T and $f^*(u_i u_j) = [f(u_i)]^2 + [f(u_j)]^2$, we have f^* is also an increasing function on the edge set of T. Hence f^* is injective and f is a square sum labeling on T. □

Definition 2.10. [1] *A cycle-cactus is a connected separable graph in which every block is a cycle.*

Theorem 2.11. *The cycle-cactus $C_k^{(n)}$, consisting of n copies of C_k, concatenated at exactly one vertex, is a square sum graph.*

Proof. Let $G_1,\ G_2, \ldots, G_n$ be copies of C_k, all concatenated at exactly one vertex, say z. Let $G_i = (z, u_{i1}, u_{i2}, \ldots, u_{i(k-1)}),\ 1 \leq i \leq n$.

Case i. k is odd.

Let $a = \left\lceil \frac{k}{2} \right\rceil$. Define $f : V(G) \to \{0, 1, 2, \ldots, |V(G)| - 1\}$ as follows.

$f(z) = n(k-1)$.

$$f(u_{ij}) = \begin{cases} 2(j-1)n + (i-1) & \text{if } 1 \leq i \leq n,\ 1 \leq j \leq a-1 \\ 2[(j-a)n + (i-1)] + 1 & \text{if } 1 \leq i \leq n,\ a \leq j \leq k. \end{cases}$$

Case ii. k is even.

Let $a = \frac{k}{2}$. Define $f : V(G) \to \{0, 1, 2, \ldots, |V(G)| - 1\}$ as follows.

$f(z) = n(k-1)$.

$$f(u_{ij}) = \begin{cases} i-1 & \text{if } j = a \\ (2(a-j)-1)n + 2i - 2 & \text{if } j < a \\ (2(j-a)-1)n + 2i - 1 & \text{if } j > a. \end{cases}$$

It can be easily verified that the induced edge function f^* is injective on $E(G)$, so that f is a square sum labeling of G. □

Example 2.12. *Square sum labeling of* $C_9^{(6)}$ *and* $C_8^{(5)}$ *are given in Figure 2.*

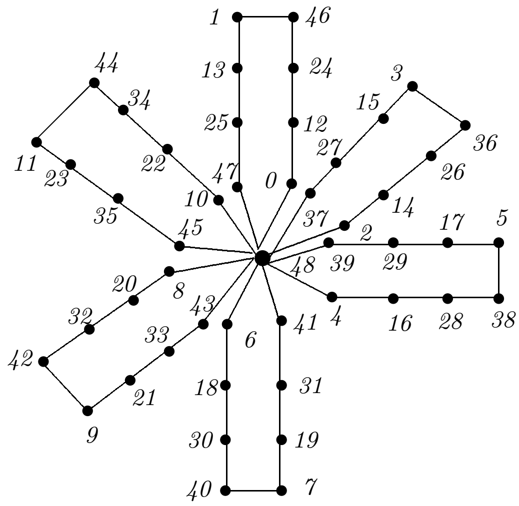

Figure 2 (a). Square Sum Labeling of C_9^6

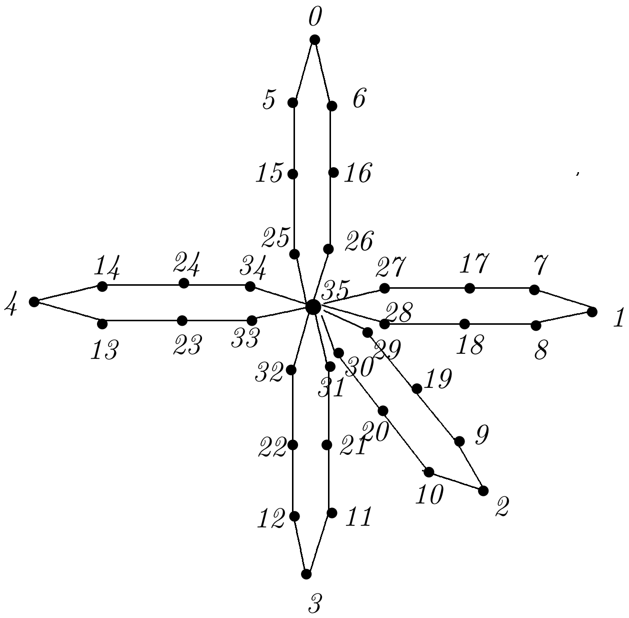

Figure 2(b). Square Sum Labeling of C_8^5

Corollary 2.13. *The friendship graph $C_3^{(n)}$ is a square sum graph.*

Theorem 2.14. *The complete lattice grids $L_{m,n} = P_m \times P_n$ are square sum graphs.*

Proof. Let $V(L_{m,n}) = \{u_{11}, u_{12}, ..., u_{1n}, u_{21}, u_{22}, ..., u_{2n}, ..., u_{m1}, u_{m2}, ..., u_{mn}\}$. We order the set of vertices into different levels as follows: $u_{11}; u_{21}, u_{12}; u_{31}, u_{22}, u_{13};$. Thus the set of vertices at level k is given by $\{u_{(k-1)1}, u_{(k-2)2}, \ldots, u_{1(k-1)}\} = \{v_{ij}; i + j = k\}$. That is $V(L_{m,n}) = \{u_{ij} : i + j = 2, 3, ..., m + n, i \leq m, j \leq n\}$.

Define $f : V(L_{m,n}) \to \{0,\ 1, ...,\ m.n - 1\}$, by $f(u_{11}) = 0,\ f(u_{21}) = 1,\ f(u_{12}) = 2,\ f(u_{31}) = 3,\ f(u_{22}) = 4,\ f(u_{13}) = 5$ and after labeling the i^{th} level vertices we label the $(i+1)^{th}$ level of vertices and so on. Now, let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be any two edges of $L_{m,n}$. If $u_1 = u_2$, then $f(v_1) < f(v_2)$ or $f(v_1) > f(v_2)$ and hence $f^*(e_1) \neq f^*(e_2)$.

Suppose $u_1 \neq u_2$ and $f(u_1) < f(u_2)$. Then $f(v_1) < f(v_2)$ and hence $f^*(e_1) \neq f^*(e_2)$. Thus f^* is injective and hence G is a square sum graph. □

Example 2.15. *Square sum labeling of $L_{4,5}$ is given in Figure 3.*

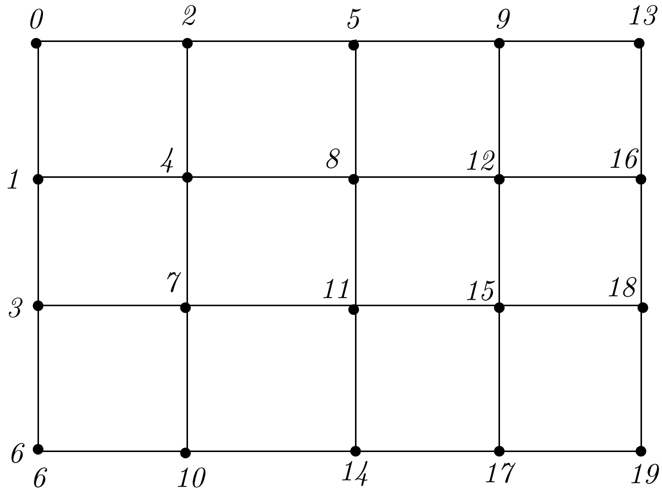

Figure 3. Square Sum Labeling of $L_{4,5}$

Corollary 2.16. *The ladder $L_n = P_2 \times P_n$ is square sum.*

Theorem 2.17. *The complete bipartite graph $K_{m,n}$ is square sum if $m \leq 4$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{m,n}$.

If $m = 1$, define $f(x_1) = 0$ and $f(y_i) = i, 1 \leq i \leq n$.

If $m = 2$, define $f(x_1) = 0, f(x_2) = 1$ and $f(y_i) = i + 1, 1 \leq i \leq n$.

If $m = 3$, define $f(x_1) = 0, f(x_2) = 1, f(x_3) = 2$ and $f(y_i) = i + 2, 1 \leq i \leq n$.

If $m = 4$, define $f(x_1) = 0, f(x_2) = 1, f(x_3) = 2, f(x_4) = 3$ and $f(y_i) = i+3, 1 \leq i \leq n$. It can be easily verified that the induced function f^* is injective on $E(K_{m,n})$. Hence $K_{m,n}$ is square sum for all n and $m \leq 4$. □

The following theorem gives an embedding result for square sum graphs.

Theorem 2.18. *Every (p,q)-graph G can be embedded in a connected square sum graph.*

Proof. Let G be a graph with vertex set $V(G) = \{u_1, u_2, ..., u_p\}$. We shall establish an embedding of G in H, where H is a graph with $|V(H)| = 5^{p-1}+1$ and $|E(H)| = 5^{p-1}+q-p$. Label the vertices of G by $f(u_1) = 0,\ f(u_i) = 5^{i-1},\ 2 \le i \le p$. Let $v_1, v_2, ..., v_n$ be isolated vertices where $n = 5^{p-1} + 1 - p$. Let $X = \{1, 2, \ldots, 5^{p-1} - 1\}$. Label vertices v_i with the numbers from the set $X - \{5, 5^2, \ldots, 5^{p-1}\}$. Let H be the graph obtained from G by joining u_1 to all $v_k,\ 1 \le k \le n$. Clearly f is a bijection from $V(H) \to \{0, 1, \ldots, 5^{p-1}\}$. Now we prove that the induced edge labeling f^* is injective. Assume $f^*(e_1) = f^*(e_2)$ for some $e_1,\ e_2 \in E(G)$. Since f is injective it follows that e_1 and e_2 are nonadjacent. Suppose $e_1 = u_k u_l$ and $e_2 = u_i u_j$ where $1 \le i,\ j,\ k,\ l \le p$. Then $(5^i)^2 + (5^j)^2 = (5^k)^2 + (5^l)^2$.

If $u_i = u_1$, then $(5^j)^2 = (5^k)^2 + (5^l)^2$, which is a contradictions. Otherwise by dividing both sides of the equation by $(5^a)^2$ where $a = min\{i, j, k, l\}$, we get an equation where one side is congrnent to 1 modulo 5 and the other side is congrent to 0 modulo 5, which is a contradiction.

If $e_1 = u_k u_l$ and $e_2 = u_1 v_i$, then $f^*(e_1) = f^*(e_2)$ implies $(5^k)^2 + (5^l)^2 = {x_i}^2$.

Now x_i is even and hence ${x_i}^2 \equiv 0\ (mod\ 4)$ whereas $(5^k)^2 + (5^l)^2 \equiv 2 (mod\ 4)$, which is a contradiction.

Hence f^* is injective and the graph H is a square sum graph which contains G as an induced subgraph. □

3. Strongly Square Sum Graphs

In this section we present a few basic results on strongly square sum Graphs.

Definition 3.1. *Let $G = (V, E)$ be a (p, q)-graph and let $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ be a bijection. We define f^* on $E(G)$ by $f^*(uv) = [f(u)]^2 + [f(v)]^2$. If $f^*(E)$ consists of the first q consecutive integers of the form $a^2 + b^2,\ a \le p-1,\ b \le p-1,\ a \ne b$, then f is said to be a* strongly square sum labeling *of G. The graph G is called strongly square sum graph if G admits a strongly square sum labeling.*

Observation 3.2. *The complete graph K_n is strongly square sum if and only if $n \le 5$.*

Theorem 3.3. *Every strongly square sum graph except K_1, K_2 and $K_{1,2}$ contains a triangle.*

Proof. Clearly K_1, K_2 and $K_{1,2}$ are triangle-free strongly square sum graphs. For a strongly square sum graph with at least three edges. The numbers 1, 4 and 5 appear as edge labels. To obtain the edge value 5, the vertices labelled 1 and 2 should be adjacent. Hence the vertices u, v and w with $f(u) = 0, f(v) = 1$ and $f(w) = 2$, form a triangle. □

Corollary 3.4. *The cycles $C_n, n \geq 4$ are not strongly square sum.*

Corollary 3.5. *The Complete bipartite graph $K_{a,b}$ is strongly square sum if and only if $a = 1$ and $b \leq 2$.*

Theorem 3.6. *The cycles C_4 and C_5 can be embedded as an induced subgraph of a strongly square sum graph.*

Proof. We first embed C_4 into a (p, q)-graph G which is strongly square sum. Let $V(G) = \{1, 2, \ldots, 10\}$ with the edges $\{0, 3\}, \{3, 4\}, \{4, 10\}$ and $\{10, 3\}$ forming an induced C_4. Join other pairs of vertices in G by edges in such a way that the set of consecutive numbers of the form $a^2 + b^2, 0 \leq a \leq 10, 0 \leq b \leq 10$ and $a \neq b$ namely $1, 4, 5, \ldots, 181$ appear as edge labels. Observe that the integers 25 and 100 can be expressed as $a^2 + b^2$ in two different ways, namely $25 = 0^2 + 5^2 = 3^2 + 4^2$ and $100 = 0^2 + 10^2 = 8^2 + 6^2$. Since $\{3, 4\}$ and $\{0, 10\}$ are edges of the induced C_4 we shall not add the edges $\{3, 4\}$ and $\{6, 8\}$ in G. Clearly G is a strongly square sum graph having C_4 as an induced subgraph. The proof for C_5 is similar. □

Conjecture 3.7. *The cycles $C_n, n \geq 4$ can be embedded as an induced subgraph of a strongly square sum graph.*

Theorem 3.8. *A unicyclic graph is square sum if and only if it is either C_3 or C_3 with one pendant edge.*

Proof. Strongly square sum labelings for C_3 and C_3 with one pendant edge are given in Figure 4 Conversely, suppose G is a unicyclic graph that is strongly square sum. Then $f^*(E(G)) = \{1, 4, 5, 9, 10, ...\}$. If $f^*(E(G)) = \{1, 4, 5\}$, then $G \cong C_3$. If $f^*(E(G)) = \{1, 4, 5, 9\}$, then G is isomorphic to C_3 along with one pendant edge. If $f^*(E(G)) = \{1, 4, 5, 9, 10\}$, then to obtain the edge value 10, the vertices with labels 3 and 1 must be adjacent. In this case one more triangle is formed. Therefore, the unicyclic graphs that admits a strongly square sum labeling are either C_3 or C_3 with one pendant edge. □

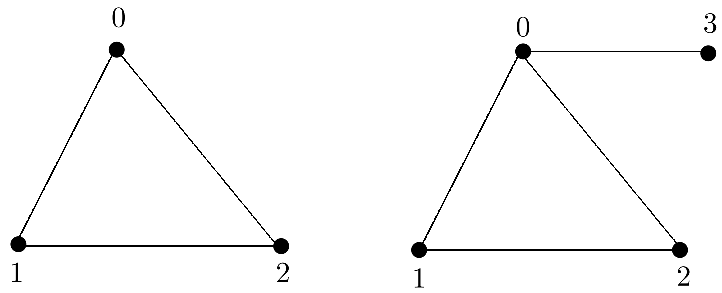

Figure 4. Strongly Square Sum Unicyclic Graphs

Observation 3.9. *There exists a unique strongly square sum graph with q edges when $6 \leq q \leq 9$. When $q = 6$, the set of edge labels is $\{1, 4, 5, 9, 10, 13\}$ and the corresponding strongly square sum graph is isomorphic to K_4. The unique strongly square sum graphs with $q = 7, 8$ and 9 are given in Figure 5*

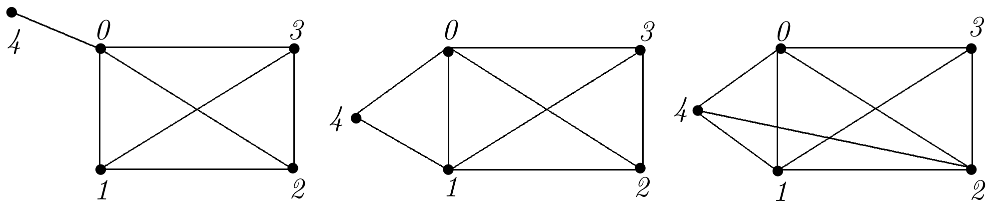

Figure 5. Strongly Square Sum Graphs with $q = 7, 8, 9$

Observation 3.10. *The exist two strongly square sum graphs with* $q = 10$. *The set of edge labels is* $\{1, 4, 5, 9, 10, 13, 16, 17, 20, 25\}$. *Since* $25 = 0^2 + 5^2 = 3^2 + 4^2$, *we get two strongly square sum graphs of size 10, which are given in Figure 6.*

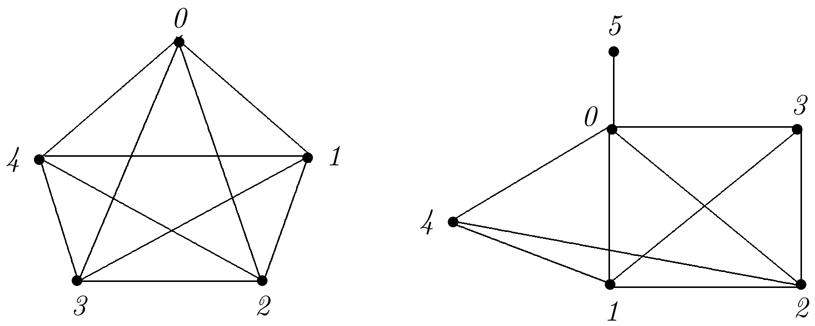

Figure 6. Strongly Square Sum graphs with $q = 10$

Problem 3.11. *Find the maximum and minimum number of edges of a strongly square sum graph with* p *vertices.*

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