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Amicability of Forests

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Abstract

This article concerns a labeling-property which distinguishes forests: we prove that if G is a forest and X is a set of integers such that the size of G equals the cardinality of X, then there exists a bijection from the vertex set of G to a set of integers such that the set of all numbers, each of which is the sum of the integers assigned to the ends of some edge, is X itself; if a graph G is not acyclic, we exhibit a set X of integers for which $|X| = |E(G)|$ such that the conclusion of the preceding statement does not hold.

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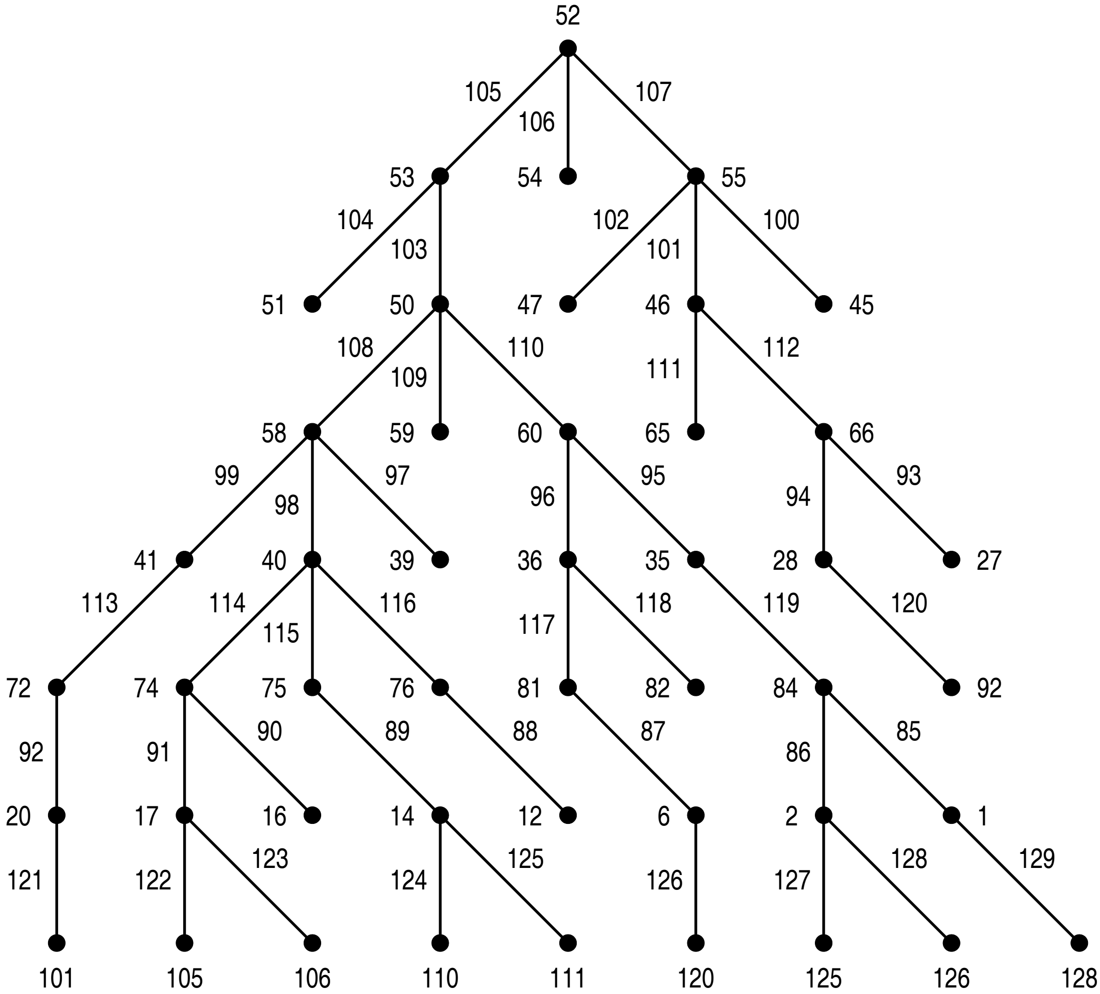
All graphs considered in this note are finite and simple. The set of all positive integers and the set of all integers are denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. For basic information about graph theory and order relations of sets, we rely on [3] and [2], respectively. Let $G = (V, E)$ be a graph. For any map $f : V \mapsto \mathbb{Z}$, we associate a map f^+ from E to $\mathbb{Z}$: for all $xy \in E$, $f^+(xy) = f(x) + f(y)$. If for each set X of integers such that $|X| = |E|$, there exists an injective function $f : V \mapsto \mathbb{Z}$ such that $f^+(E) = X$, then G is called *amicable*. This article is the result of an attempt to derive Proposition 3; its objective is to show that *a graph is amicable if and only if it is acyclic.* The main results of this article are the following one and Theorem 4.

Theorem 1. *Every tree is amicable.*



An arithmetic labeling.

Proof. Let $T = (V, E)$ be a rooted tree and ρ be its root. If a vertex α is the parent of a vertex β—i.e., if $\alpha\beta$ is an edge such that $d(\rho, \alpha) < d(\rho, \beta)$—call α as the *head* of $\alpha\beta$ and β as the *tail* of this edge. Let '$\prec$' be any liner order of V such that the following hold. (Henceforth, by the phrase 'x precedes y', we mean the relation '$x \prec y$'.)

(1) If u, v are two vertices such that $d(\rho, u) < d(\rho, v)$, then $u \prec v$.

(2) If the head of an edge e_1 precedes that of another edge e_2, then the tail of e_1 also precedes that of e_2.

The order '$\prec$' of V induces a linear order on E: An edge e_1 precedes another edge e_2, if the tail of e_1 precedes that of e_2. (The induced order is also denoted by '$\prec$' because any ambiguity is unlikely.)

Let $E_o = \{e \in E : d(\rho, e) \text{ is odd}\}$ and $V_o = \{v \in V : d(\rho, v) \text{ is odd}\}$; let $E_\varepsilon = E - E_o$ and $V_\varepsilon = V - V_o$. Let X be any subset of $\mathbb{Z}$ such that $|X| = |E|$. Let g be the bijection from E to X such that the following hold.

(1) For each $e_1 \in E_o$ and $e_2 \in E_\varepsilon$, $g(e_1) < g(e_2)$.

(2) If e_1, $e_2 \in E_\varepsilon$ and $e_1 \prec e_2$, then $g(e_1) < g(e_2)$.

(3) If e_1, $e_2 \in E_o$ and $e_1 \prec e_2$, then $g(e_1) > g(e_2)$.

(For the illustration of the edge-labeling just described, see the figure: the uppermost vertex is the root and $X = \{85, 86, 87, \ldots, 127, 128, 129\}$.) Now we define a map $f : V \mapsto \mathbb{Z}$ inductively:

(4) Let $f(\rho)$ be any integer such that $2f(\rho) < \min\{g(e) : e \in E_\varepsilon\}$.

If a vertex α is the parent of a vertex β and $f(\alpha)$ is known, let $f(\beta) = g(\alpha\beta) - f(\alpha)$.

Let $\rho, x_1, y_1, x_2, y_2, \ldots$ be a path; by (4) and by repeatedly using (1), we get
$2f(\rho) < f(\rho) + f(x_1) > f(x_1) + f(y_1) < f(y_1) + f(x_2) > f(x_2) + f(y_2) < \cdots$.
Therefore, $\cdots < f(y_2) < f(y_1) < f(\rho) < f(x_1) < f(x_2) < \cdots$.

From the above relation, we get the following.

(5) For each $x \in V_o$, $f(x) > f(\rho)$; for each $x \in V_\varepsilon - \{\rho\}$, $f(x) < f(\rho)$.

Let us show the following.

(∗∗) If $x, x' \in V$ and $x \prec x'$, then $\begin{cases} x, x' \in V_o \Rightarrow f(x) < f(x') \text{ and} \\ x, x' \in V_\varepsilon \Rightarrow f(x) > f(x'). \end{cases}$

Let $\beta, \beta' \in V$ such that $\beta \prec \beta'$. If $\beta = \rho$, then by (5), the conclusion of (∗∗) holds for $x = \beta$ and $x' = \beta'$. So, let $\beta \neq \rho$; we can assume that

(6) for each vertex x which precedes β, (∗∗) holds.

Let the parents of β and β' be α and α', respectively. Suppose that $\beta, \beta' \in V_o$. Then $\alpha\beta$, $\alpha'\beta' \in E_\varepsilon$. Since $\alpha\beta \prec \alpha'\beta'$, by (2), $g(\alpha\beta) < g(\alpha'\beta')$; i.e., $f(\alpha) + f(\beta) < f(\alpha') + f(\beta')$. Since $\alpha, \alpha' \in V_\varepsilon$ and $\alpha \preccurlyeq \alpha'$, by (6), $f(\alpha) \geqslant f(\alpha')$. Therefore, from the earlier numeric inequality, $f(\beta) < f(\beta')$. Similarly, we can show that $\beta, \beta' \in V_\varepsilon \Rightarrow f(\beta) > f(\beta')$ [by using (3) and (6)]. Thus (∗∗) holds; from this and (5), it follows that f is injective. □

Corollary 2. *Every forest is amicable.*

Proof. Let G be an amicable graph and T be a tree such that $V(G) \cap V(T) = \emptyset$; since trees are amicable, it is enough to show that so is $G \cup T$. Let X and Y be arbitrary disjoint subsets of $\mathbb{Z}$ such that $|X| = |E(G)|$ and $|Y| = |E(T)|$. Then there are injective functions $g : V(G) \mapsto \mathbb{Z}$ and $h : V(T) \mapsto \mathbb{Z}$ such that $g^+(E(G)) = X$ and $h^+(E(T)) = Y$. Since T is bipartite, its vertex set can be partitioned into two sets P and Q such that every edge of T has one end in P and the other in Q. Let μ be any integer; define a map $f : V(G) \cup V(T) \mapsto \mathbb{Z}$ as follows.

$$f(x) = \begin{cases} g(x) & \text{when } x \in V(G); \\ h(x) + \mu & \text{when } x \in P; \\ h(x) - \mu & \text{when } x \in Q. \end{cases}$$

Obviously $f^+(E(G) \cup E(T)) = X \cup Y$. It is easy to see that μ can be chosen so that f is injective. Therefore, the amicability of $G \cup T$ follows. □

Let $G = (V, E)$ be a graph; if f is an injective function from V to $\mathbb{N}$ such that the elements of $f^+(E)$ form an arithmetic progression of $|E|$ terms—i.e., if this set can be written as $\{k + d,\ k + 2d,\ \ldots,\ k + |E|d\}$ for some $k \in \mathbb{Z}$ and $d \in \mathbb{N}$—then f is called an *arithmetic labeling* of G. This labeling was introduced in [1]. If a graph admits an arithmetic labeling, then it is called *arithmetic*. Let $G = (V, E)$ be an acyclic graph; by Corollary 2, there is an injective function $f : V \mapsto \mathbb{Z}$ such that $f^+(E) = \{1, 2, \ldots, |E|\}$. Now, choose an integer μ such that for all $x \in V$, $\mu + f(x) > 0$. It is easy to verify that the function from V to $\mathbb{N}$ which assigns to each $x \in V$, $\mu + f(x)$ is an arithmetic labeling of G. Thus we have the following result.

Proposition 3. *Every forest is arithmetic.*

Next, we settle the second part of our objective:

Theorem 4. *Every amicable graph is a forest.*

Proof. The set $\mathcal{X} := \{3^n : n \in \mathbb{N}\}$ has the following property.

(∗∗) If $\mu_1, \mu_2, \ldots, \mu_k$ are distinct numbers in $\mathcal{X}$ and $s_1, s_2, \ldots, s_k \in \{-1, 1\}$, then $s_1\mu_1 + s_2\mu_2 + \cdots + s_k\mu_k$ is non-zero. (Reasoning: this expression is not divisible by $3 \times \min\{\mu_1, \mu_2, \ldots, \mu_k\}$.)

Let $G = (V, E)$ be a graph containing a cycle $(w_1, w_2, \ldots, w_m, w_{m+1} = w_1)$; suppose the existence of a map $f : V \mapsto \mathbb{N}$ such that $f^+(E) \subset \mathcal{X}$. Then

$$f^+(w_1w_2) + f^+(w_2w_3) + \cdots + f^+(w_mw_{m+1}) = 2[f(w_1) + f(w_2) + \cdots + f(w_m)].$$

Since each term on the left side is odd, m is even; therefore the right side is $2[f^+(w_1w_2) + f^+(w_3w_4) + \cdots + f^+(w_{m-1}w_m)]$ whence by simplification we obtain

$$f^+(w_1w_2) - f^+(w_2w_3) + \cdots + f^+(w_{m-1}w_m) - f^+(w_mw_{m+1}) = 0.$$

Now by (∗∗), f^+ is not injective. Therefore G is not amicable. □

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