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A CONSTRUCTION OF SUPERMAGIC GRAPHS

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Abstract

A graph is called supermagic if it admits a labelling of the edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we will introduce a new construction of supermagic graphs using the special structure of their subgraphs. The supermagic complements of some bipartite graphs and joins of some graphs are presented.

Keywords: supermagic graphs, bipartite graphs, join of graphs.

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. For $X, Y \subseteq V(G)$ the subgraph of a graph G induced by $\{uv \in E(G) : u \in X, v \in Y\}$ is denoted by $G(X, Y)$. If $X = Y$ then we put $G(X) = G(X, X)$. For two disjoint graphs G_1 and G_2 $G_1 \oplus G_2$ and $G_1 \cup G_2$ denote their join and their union, respectively. The union of m disjoint copies of a graph G is denoted by mG. If $f : A \longrightarrow B$ is a mapping and $M \subseteq A$, then $f(M)$ stands for the set $\{f(x) \in B : x \in M\}$.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v, e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling f of G is called a *supermagic labelling* if the set $f(E(G))$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of a graph G.

The concept of magic graphs was introduced by Sedláček [16]. The regular magic graphs are characterized in [4]. Two different characterizations of all magic graphs are given in [14] and [13]. Supermagic graphs were introduced by Stewart [19]. In [1], [7], [8], [10], [17], [18] and [20] there are described some classes of regular supermagic graphs. Some constructions of supermagic labellings of various classes of graphs are described in [5], [7], [8], [9], [11] and [12]. More comprehensive information on magic and supermagic graphs can be found in [6].

A graph G is called *bipartite* if its vertex set can be partitioned into disjoint parts $V_1(G)$, $V_2(G)$ such that every edge of G joins vertices of different parts. A bipartite graph G is called *balanced*, if $|V_1(G)| = |V_2(G)|$. In the paper we will introduce a new method to construct supermagic graphs and apply it to complements of some balanced bipartite graphs and joins of some graphs.

2. Construction

Bodendiek and Walther [2] introduced the special labelling of graphs. For a graph G and positive integers a, d, a bijection $f : E(G) \longrightarrow \{1, 2, \ldots, |E(G)|\}$ is called an (a, d)-*antimagic labelling* of G if

$$f^*(V(G)) = \{a, a + d, \ldots, a + (|V(G)| - 1)d\}.$$

Now we define a similar notion. Let U_1, U_2 be subsets of the vertex set of G such that $|U_1| = |U_2|$, $U_1 \cup U_2 = V(G)$, $U_1 \cap U_2 = \emptyset$. An injective mapping f from $E(G)$ into positive integers is called a *double-consecutive labelling* (*DC-labelling*) with respect to (U_1, U_2) if its index-mapping f^* satisfies

$$f^*(U_1) = f^*(U_2) = \{a, a + 1, \ldots, a + |U_1| - 1\} \quad \text{for some integer } a.$$

Let f_i, $i \in \{1, 2\}$, be a DC-labelling of a graph G_i. The labellings f_1 and f_2 are called *complementary* if $f_1(E(G_1)) \cap f_2(E(G_2)) = \emptyset$ and $f_1(E(G_1)) \cup f_2(E(G_2))$ consists of consecutive integers.

Now we are able to prove

Proposition 2.1. *Let G_1, G_2 be disjoint graphs each of order n and let G_3 be a balanced bipartite graph of order $2n$ with parts U_1 and U_2. Let f be a DC-labelling of $G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ and let g be a DC-labelling of G_3 with respect to (U_1, U_2). If f and g are complementary, then there exists a supermagic graph G such that its vertex set is $V(G) = U_1 \cup U_2$, $G(U_1)$ is isomorphic to G_1, $G(U_2)$ is isomorphic to G_2 and $G(U_1, U_2)$ is isomorphic to G_3.*

Proof. As g is a DC-labeling of G_3 with respect to (U_1, U_2), we can denote the vertices of U_i, $i \in \{1, 2\}$, by $u_{i,1}$, $u_{i,2}$, …, $u_{i,n}$ in such a way that $g^*(u_{i,j}) + 1 = g^*(u_{i,j+1})$ for all $j \in \{1, \ldots, n-1\}$. Similarly, we can denote the vertices of G_i, $i \in \{1, 2\}$, by $v_{i,1}$, $v_{i,2}$, …, $v_{i,n}$ in such a way that $f^*(v_{i,j}) = 1 + f^*(v_{i,j+1})$ for all $j \in \{1, \ldots, n-1\}$. Clearly, $f^*(v_{i,j}) + g^*(u_{i,j}) = f^*(v_{i,1}) + g^*(u_{i,1}) = f^*(v_{1,1}) + g^*(u_{1,1})$.

Let G be a graph with the vertex set $V(G) = U_1 \cup U_2$ and the edge set $E(G) = E(G_3) \cup \{u_{1,i}u_{1,j} : v_{1,i}v_{1,j} \in E(G_1)\} \cup \{u_{2,i}u_{2,j} : v_{2,i}v_{2,j} \in E(G_2)\}$. Now consider a mapping φ from $E(G)$ to positive integers given by

$$\varphi(u_{i,j}u_{k,l}) = \begin{cases} g(u_{i,j}u_{k,l}) & \text{if } i \neq k, \\ f(v_{i,j}v_{k,l}) & \text{if } i = k. \end{cases}$$

Evidently, $\varphi^*(u_{i,j}) = f^*(v_{i,j}) + g^*(u_{i,j}) = f^*(v_{1,1}) + g^*(u_{1,1})$. Moreover, $\varphi(V(G)) = f(V(G_1 \cup G_2)) \cup g(V(G_3))$ and labellings f and g are complementary, thus φ is a supermagic labelling of G. Therefore, G is a desired graph. □

The proposition describes a method to construct supermagic graphs. However, there are needed appropriate DC-labellings of some graphs.

Lemma 2.2. *Let G be a 1-regular bipartite graph of order $2n$ with parts U_1 and U_2 and let k be a positive integer. Then there is a DC-labelling f of G with respect to (U_1, U_2) such that $f(E(G)) = \{k, k+1, \ldots, k+n-1\}$.*

Proof. Clearly, any bijection $f : E(G) \longrightarrow \{k, k+1, \ldots, k+n-1\}$ has a desired properties. □

Lemma 2.3. *Let G be a connected 2-regular bipartite graph of order $2n$ with parts U_1 and U_2 and let k, t, $t \geq k+n$, be positive integers. If n is an odd integer, then there is a DC-labelling f of G with respect to (U_1, U_2) such that*

$$f(E(G)) = \{k, k+1, \ldots, k+n-1, t, t+1, \ldots, t+n-1\}.$$

Proof. As G is a connected 2-regular graph it is a cycle. Let us denote the vertices of G by u_1, u_2, …, u_{2n} in such a way that its edges are u_iu_{i+1} for $i = 1, 2, \ldots, 2n$, the subscripts being taken modulo $2n$. Obviously, $U_1 = \{u_1, u_3, \ldots, u_{2n-1}\}$ and $U_2 = \{u_2, u_4, \ldots, u_{2n}\}$ are parts of G. Consider a mapping f from $E(G)$ to positive integers given by

$$f(e) = \begin{cases} i+k-1 & \text{if } e = u_{2i-1}u_{2i},\ 1 \leq i \leq n, \\ i+t-1 & \text{if } e = u_{n+2i-1}u_{n+2i},\ 1 \leq i \leq n. \end{cases}$$

It is not difficult to check that $f^*(U_1) = f^*(U_2) = \{a, a+1, \ldots, a+n-1\}$, where $a = k + t + \frac{1}{2}(n-1)$. Thus, f is a desired DC-labelling. □

As every $2k$-regular graph is decomposable into k edge-disjoint 2-factors, the following assertion will be very useful.

Lemma 2.4. *Let H be a balanced bipartite graph of order $2n$ with parts U_1 and U_2. Let F be a 2-factor of H and let g be a DC-labelling of $G = H - F$ with respect to (U_1, U_2). If k, t, $t \geq k + n$, are positive integers such that*

$$g(E(G)) \cap \{k, k+1, \dots, k+n-1, t, t+1, \dots, t+n-1\} = \emptyset,$$

then there exists a DC-labeling f of H with respect to (U_1, U_2) such that

$$f(E(H)) = g(E(G)) \cup \{k, k+1, \dots, k+n-1, t, t+1, \dots, t+n-1\}.$$

Proof. As g is a DC-labeling of G with respect to (U_1, U_2), we can denote the vertices of U_i, $i \in \{1, 2\}$, by $u_{i,1}$, $u_{i,2}$, $\dots$, $u_{i,n}$ in such a way that $g^*(u_{i,j}) = 1 + g^*(u_{i,j+1})$ for all $j \in \{1, \dots, n-1\}$. As F is a 2-factor of a bipartite graph there are disjoint 1-factors F_1 and F_2 such that $F = F_1 \cup F_2$. Denote the edges of the factor F_i, $i \in \{1, 2\}$, by $e_{i,1}$, $e_{i,2}$, $\dots$, $e_{i,n}$ in such a way that $e_{i,j}$ is incident to $u_{i,j}$. Consider a mapping f from $E(H)$ to positive integers given by

$$f(e) = \begin{cases} g(e) & \text{if } e \in E(G), \\ j + k - 1 & \text{if } e = e_{1,j}, \\ j + t - 1 & \text{if } e = e_{2,j}. \end{cases}$$

Since $g^*(u_{i,j}) + f(e_{i,j}) = g^*(u_{i,1}) + f(e_{i,1})$ for all j, we get

$$f^*(U_1) = f^*(U_2) = \{g^*(u_{1,1}) + k + t, \dots, g^*(u_{1,1}) + k + t + n - 1\}.$$

Therefore, f is a required DC-labelling of H. □

Lemma 2.5. *Let G_1 and G_2 be disjoint 3-regular Hamiltonian bipartite graphs each of order $n = 4k$, $k \geq 2$. Then there exists a DC-labelling f of $G = G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ such that $f(E(G)) = \{1, 2, \dots, 3n\}$.*

Proof. Graphs G_1 and G_2 are Hamiltonian, so there are their Hamilton cycles C_1 and C_2. Denote the vertices of G_i, $i \in \{1, 2\}$, by $u_{i,1}$, $u_{i,2}$, $\dots$, $u_{i,n}$ in such a way that the edges of C_i are $u_{i,j}u_{i,j+1}$ for $j = 1, 2, \dots, n$, the subscripts being taken modulo n. Obviously, $U_{i,1} = \{u_{i,1}, u_{i,3}, \dots, u_{i,2n-1}\}$ and $U_{i,2} = \{u_{i,2}, u_{i,4}, \dots, u_{i,2n}\}$ are parts of G_i. Consider an auxiliary mapping g from $E(C_1 \cup C_2)$ to $\{1, 2, \dots, 2n\}$ given by

$$g(e) = \begin{cases} j & \text{if } e = u_{1,2j}u_{1,2j+1}, \\ j + 2k & \text{if } e = u_{2,2j}u_{2,2j+1}, \\ 1 + 6k - j & \text{if } e = u_{2,2j-1}u_{2,2j}, \\ 1 + 2n - j & \text{if } e = u_{1,2j-1}u_{1,2j}. \end{cases}$$

It is not difficult to check that $g^*(u_{i,1}) = 10k$, $g^*(u_{i,3}) = \cdots = g^*(u_{i,n-1}) = 8k$ and $g^*(u_{i,2}) = \cdots = g^*(u_{i,n}) = 1 + 8k$.

Finally, let $f : E(G_1 \cup G_2) \longrightarrow \{1, 2, \dots, 3n\}$ be an arbitrary bijection satisfying

$f(e) = g(e)$ whenever $e \in E(C_1 \cup C_2)$,

$f(e) \in \{8k+1, 8k+3, \dots, 10k-1, 10k+2, 10k+4, \dots, 12k\}$ when $e \in E(G_1) - E(C_1)$,

$f(e) \in \{8k+2, 8k+4, \dots, 10k, 10k+1, 10k+3, \dots, 12k-1\}$ when $e \in E(G_2) - E(C_2)$,

$f(e) = 8k + 1$ when $e \in E(G_1) - E(C_1)$ is incident to $u_{1,1}$,

$f(e) = 10k + 1$ when $e \in E(G_2) - E(C_2)$ is incident to $u_{2,1}$.

It is easy to see that $f^*(V(G_1)) = f^*(V(G_2)) = \{16k + 2, 16k + 3, \dots, 20k + 1\}$. Thus, f is a required DC-labeling of $G_1 \cup G_2$. □

Lemma 2.6. *Let G_1 and G_2 be disjoint 4-regular Hamiltonian graphs each of odd order n. Then there exists a DC-labelling f of $G = G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ such that $f(E(G)) = \{1, 2, \dots, 4n\}$.*

Proof. Graphs G_1 and G_2 are Hamiltonian, so there are their Hamilton cycles C_1 and C_2. Denote the vertices of G_i, $i \in \{1, 2\}$, by $u_{i,1}$, $u_{i,2}$, $\dots$, $u_{i,n}$ in such a way that the edges of C_i are $u_{i,j}u_{i,j+1}$ for $j = 1, 2, \dots, n$, the subscripts being taken modulo n. The subgraph F_i of a graph G_i induced by $E(G_i) - E(C_i)$ is a 2-factor. So there is a digraph $\vec{F_i}$ which we get from F_i by an orientation of its edges such that the outdegree of every vertex of $\vec{F_i}$ is equal to 1. Let $\vec{e}$ denote an arc of $\vec{F_i}$ corresponding to $e \in E(F)$. Therefore, we can denote the edges of F_i by $e_{i,j}$ in such a way that the initial vertex of $\vec{e}_{i,j}$ is $u_{i,j}$ for all $i \in \{1, 2\}$, $j \in \{1, 2, \dots, n\}$. Let $f : E(G_1 \cup G_2) \longrightarrow \{1, 2, \dots, 4n\}$ be a mapping given by

$$f(e) = \begin{cases} j & \text{if } \ e = u_{1,2j-1}u_{1,2j}, \\ j + n & \text{if } \ e = u_{2,2j-1}u_{2,2j}, \\ 1 + 3n - j & \text{if } \ e = e_{2,j}, \\ 1 + 4n - j & \text{if } \ e = e_{1,j}. \end{cases}$$

As $f(u_{1,j-1}u_{1,j}) + f(u_{1,j}u_{1,j+1}) = a + j$, where $a = \frac{1}{2}(n + 1)$, we get

$$f^*(u_{1,j}) = (a + j) + (1 + 4n - j) + (1 + 4n - t) = a + 2 + 8n - t$$

when the terminal vertex of $\vec{e}_{1,t}$ is $u_{1,j}$, $j \neq t$. Therefore

$$f^*(V(G_1)) = \{a + 7n + 2, a + 7n + 3, \dots, a + 8n + 1\}.$$

Similarly, $f(u_{2,j-1}u_{2,j}) + f(u_{2,j}u_{2,j+1}) = 2n + a + j$, and

$$f^*(u_{2,j}) = (2n + a + j) + (1 + 3n - j) + (1 + 3n - t) = a + 2 + 8n - t.$$

Hence $f^*(V(G_2)) = f^*(V(G_1))$ and thus f is a desired DC-labelling. □

Lemma 2.7. *Let G_1 be a 4-regular Hamiltonian graph of odd order n and let G_2 be a cycle of order n. Then there is a DC-labelling f of $G = G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ such that $f(E(G)) = \{\frac{1}{2}(n+1), \frac{1}{2}(n+3), \dots, \frac{1}{2}(7n-1)\}$.*

Proof. Graphs G_1 and G_2 are Hamiltonian, so there are their Hamilton cycles C_1 and C_2. Denote the vertices of G_i, $i \in \{1,2\}$, by $u_{i,1}$, $u_{i,2}$, $\dots$, $u_{i,n}$ in such a way that the edges of C_i are $u_{i,j}u_{i,j+1}$ for $j = 1, 2, \dots, n$, the subscripts being taken modulo n. The subgraph F of a graph G_1 induced by $E(G_1) - E(C_1)$ is a 2-factor. So there is a digraph $\vec{F}$ which we get from F by an orientation of its edges such that the outdegree of every vertex of $\vec{F}$ is equal to 1. Therefore, we can denote the edges of F by e_j in such a way that the initial vertex of $\vec{e}_j$ is $u_{1,j}$ for all $j \in \{1, 2, \dots, n\}$. Let f be a mapping from $E(G_1 \cup G_2)$ to positive integers given by

$$f(e) = \begin{cases} \frac{1}{2}(n-1) + j & \text{if } e = u_{1,2j-1}u_{1,2j}, \\ \frac{1}{2}(n-1) + 1 + 2n - j & \text{if } e = e_j, \\ \frac{1}{2}(n-1) + 2n + j & \text{if } e = u_{2,2j-1}u_{2,2j}. \end{cases}$$

As $f(u_{1,j-1}u_{1,j}) + f(u_{1,j}u_{1,j+1}) = a + j$, where $a = \frac{1}{2}(3n-1)$, we get

$$f^*(u_{1,j}) = (a+j) + (\frac{1}{2}(n-1) + 1 + 2n - j) + (\frac{1}{2}(n-1) + 1 + 2n - t) = a + 1 + 5n - t$$

when the terminal vertex of $\vec{e}_t$ is $u_{1,j}$. Therefore

$$f^*(V(G_1)) = \{a + 4n + 1, a + 4n + 2, \dots, a + 5n\}.$$

Similarly, $f^*(u_{2,j}) = f(u_{2,j-1}u_{2,j}) + f(u_{2,j}u_{2,j+1}) = 4n + a + j$. Hence we get $f^*(V(G_2)) = f^*(V(G_1))$. Thus f is a desired DC-labelling. □

Lemma 2.8. *Let G_1 and G_2 be disjoint 6-regular Hamiltonian graphs each of odd order n. Then there exists a DC-labelling f of $G = G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ such that $f(E(G)) = \{1, 2, \dots, 5n\} \cup \{6n+1, 6n+2, \dots, 7n\}$.*

Proof. Let C_i be a Hamilton cycle of G_i, $i \in \{1,2\}$. Let us denote the vertices of G_i by $u_{i,1}$, $u_{i,2}$, $\dots$, $u_{i,n}$ in such a way that the edges of C_i are $u_{i,j}u_{i,j+1}$ for $j = 1, 2, \dots, n$, the subscripts being taken modulo n. The subgraph F_i of G_i induced by $E(G_i) - E(C_i)$ is 4-regular. Evidently, $F = F_1 \cup F_2$ is a 4-factor of $G_1 \cup G_2$. As every 4-regular graph is decomposable into two edge-disjoint 2-factors, there are edge-disjoint 2-factors X_i and Z_i such that C_i, X_i and Z_i form a decomposition of G_i. Let $\vec{X}_i$ and $\vec{Z}_i$ denote digraphs which we get from X_i and Z_i by an orientation of their edges such that the outdegree of every their vertex is equal to 1. Denote the edges of X_i by $x_{i,j}$ in such a way that the initial vertex of $\vec{x}_{i,j}$ is $u_{i,j}$ for all $i \in \{1,2\}$, $j \in \{1, 2, \dots, n\}$. Similarly, denote the edges of Z_i by $z_{i,j}$ in such a way that the initial vertex of $\vec{z}_{i,j}$ is the terminal vertex of $\vec{x}_{i,j}$ for all $i \in \{1,2\}$, $j \in \{1, 2, \dots, n\}$.

Let $f : E(G_1 \cup G_2) \longrightarrow \{1, 2, \ldots, 5n\} \cup \{6n+1, 6n+2, \ldots, 7n\}$ be a bijection given by

$$f(e) = \begin{cases} j & \text{if } e = u_{1,2j-1}u_{1,2j}, \\ j+n & \text{if } e = u_{2,2j-1}u_{2,2j}, \\ 1+3n-j & \text{if } e = x_{1,j}, \\ 1+4n-j & \text{if } e = x_{2,j}, \\ j+4n & \text{if } e = z_{2,j}, \\ j+6n & \text{if } e = z_{1,j}. \end{cases}$$

As $f(x_{1,p})+f(z_{1,p}) = 1+9n$ and $f(u_{1,j-1}u_{1,j})+f(u_{1,j}u_{1,j+1}) = a+j$, where $a = \frac{1}{2}(n+1)$, we get

$$f^*(u_{1,j}) = (a+j) + (1+3n-j) + (1+9n) + (t+6n) = a+2+18n+t$$

when the terminal vertex of $\vec{z}_{1,t}$ is $u_{1,j}$. Therefore

$$f^*(V(G_1)) = \{a+18n+3, a+18n+4, \ldots, a+19n+2\}.$$

Similarly, $f(x_{2,p}) + f(z_{2,p}) = 1+8n$, $f(u_{2,j-1}u_{2,j}) + f(u_{2,j}u_{2,j+1}) = 2n+a+j$,

$$f^*(u_{2,j}) = (2n+a+j) + (1+4n-j) + (1+8n) + (t+4n) = a+2+18n+t.$$

Hence $f^*(V(G_2)) = f^*(V(G_1))$ and thus f is a required DC-labelling. $\square$

Lemma 2.9. *Let H_1 and H_2 be disjoint graphs each of order n. Let F be a 4-factor of $H = H_1 \cup H_2$ and let g be a DC-labelling of $G = H - F$ with respect to $(V(H_1), V(H_2))$. If k is a positive integer such that $k \geq g(e)$ for each $e \in E(G)$, then there exists a DC-labeling f of H with respect to $(V(H_1), V(H_2))$ such that*

$$f(E(H)) = g(E(G)) \cup \{k+1, k+2, \ldots, k+4n\}.$$

Proof. Since g is a DC-labeling of G with respect to $(V(H_1), V(H_2))$, we can denote the vertices of H_i, $i \in \{1, 2\}$, by $u_{i,1}$, $u_{i,2}$, $\ldots$, $u_{i,n}$ in such a way that $g^*(u_{i,j}) + 1 = g^*(u_{i,j+1})$ for all $j \in \{1, \ldots, n-1\}$. As in the proof of previous lemma denote the edges of F by $x_{i,j}$ and $z_{i,j}$, $i \in \{1, 2\}$, $j \in \{1, 2, \ldots, n\}$.

Let f be a mapping from $E(H)$ to positive integer given by

$$f(e) = \begin{cases} g(e) & \text{if } e \in E(G), \\ k+1+n-j & \text{if } e = x_{1,j}, \\ k+1+2n-j & \text{if } e = x_{2,j}, \\ k+2n+j & \text{if } e = z_{2,j}, \\ k+3n+j & \text{if } e = z_{1,j}. \end{cases}$$

As $f(x_{1,p}) + f(z_{1,p}) = 2k + 4n + 1$ and $g^*(u_{1,j}) + f(x_{1,j}) = a + k + n$, where $a = g^*(u_{1,1})$, we get

$$f^*(u_{1,j}) = (a + k + n) + (2k + 4n + 1) + (k + 3n + t) = a + 4k + 8n + 1 + t$$

when the terminal vertex of $\vec{z}_{1,t}$ is $u_{1,j}$. Therefore

$$f^*(V(H_1)) = \{a + 4k + 8n + 2, a + 4k + 8n + 3, \ldots, a + 4k + 9n + 1\}.$$

Similarly, $f(x_{2,p}) + f(z_{2,p}) = 2k + 4n + 1$, $g^*(u_{2,j}) + f(x_{2,j}) = a + k + 2n$ and

$$f^*(u_{2,j}) = (a + k + 2n) + (2k + 4n + 1) + (k + 2n + t) = a + 4k + 8n + 1 + t.$$

Hence $f^*(V(G_2)) = f^*(V(G_1))$ and thus f is a required DC-labelling. □

3. Complements of bipartite graphs

Suppose that G is a balanced bipartite graph of order $2n$ with parts U_1 and U_2. For the complement $\overline{G}$ of G it holds

$\overline{G}(U_1)$ and $\overline{G}(U_2)$ are isomorphic to a complete graph K_n,

$\overline{G}(U_1, U_2)$ is a balanced bipartite graph.

Therefore we can construct some supermagic complements of bipartite graphs using Proposition 2.1.

Theorem 3.10. *Let G be a d-regular bipartite graph of order $8k$. The complement of G is a supermagic graph if and only if d is odd.*

Proof. In [8] it is proved that there is no supermagic r-regular graph of order n for $r \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 4$. Since $\overline{G}$ is a regular graph of degree $8k - d - 1$, the condition $d \equiv 1 \pmod 2$ is necessary for $\overline{G}$ to be supermagic.

On the other hand, the graph $\overline{G}(U_1, U_2)$ is regular bipartite of odd degree $8k - d$, when U_1, U_2 are parts of G. According to Lemma 2.2 and Lemma 2.4 there is a DC-labelling g of $\overline{G}(U_1, U_2)$ with respect to (U_1, U_2) such that

$$g(E(\overline{G}(U_1, U_2)) = \{4k(4k-1) + 1, 4k(4k-1) + 2, \ldots, 4k(4k-1) + 8k(8k-d)\}.$$

Similarly, combining Lemma 2.5 and Lemma 2.9 there is a DC-labelling f of the graph $\overline{G}(U_1) \cup \overline{G}(U_2) = 2K_{4k}$, for $k \geq 2$, with respect to (U_1, U_2) such that

$$f(E(2K_{4k})) = \{1, 2, \ldots, 4k(4k-1)\}.$$

The labellings f and g are complementary and so, by Proposition 2.1, the graph $\overline{G}$ is supermagic for $k \geq 2$.

If G is a 1-regular graph of order 8, then the complement $\overline{G}$ is supermagic because it is a regular complete 4-partite graph (see [8]). Finally, the complement of a 3-regular bipartite graph of order 8 is isomorphic to the Cartesian product $K_4 \times K_2$. Some supermagic labelling of $K_4 \times K_2$ is described below in the matrix.

	u_1	u_2	u_3	u_4		v_1	v_2	v_3	v_4	
u_1	–	1	3	16	14	–	6	2	12	v_1
u_2	1	–	11	9	13	6	–	7	8	v_2
u_3	3	11	–	5	15	2	7	–	10	v_3
u_4	16	9	5	–	4	12	8	10	–	v_4

Therefore, $\overline{G}$ is a supermagic graph. □

Theorem 3.11. *Let G be a d-regular bipartite graph of order $2n$, where n is odd and d is even. The complement of G is supermagic if and only if $(n, d) \neq (3, 2)$.*

Proof. The graph $\overline{G}(U_1, U_2)$ is regular bipartite of an odd degree $n - d$, when U_1, U_2 are parts of G. If $d = 0$, then the complement of G is isomorphic to a complete graph K_{2n}. In [20] it is proved that K_{2n} is a supermagic graph.

The complement of a 2-regular bipartite graph of order 6 is a graph of 3-side prism which is not magic.

Suppose that $5 \leq n \equiv 1 \pmod 4$. According to Lemma 2.2 and Lemma 2.4 there is a DC-labelling g of $\overline{G}(U_1, U_2)$ with respect to (U_1, U_2) such that

$$g(E(\overline{G}(U_1, U_2)) = \{n(n-1)+1, n(n-1)+2, \dots, n(n-1)+n(n-d)\}.$$

Similarly, using Lemma 2.6 and Lemma 2.9 there exists a DC-labelling f of the graph $\overline{G}(U_1) \cup \overline{G}(U_2) = 2K_n$ with respect to (U_1, U_2) such that

$$f(E(2K_n)) = \{1, 2, \dots, n(n-1)\}.$$

The labellings f and g are complementary and so, by Proposition 2.1, the graph $\overline{G}$ is supermagic.

Assume that $7 \leq n \equiv 3 \pmod 4$. By Lemma 2.2 and Lemma 2.4 there is a DC-labelling g of $\overline{G}(U_1, U_2)$ with respect to (U_1, U_2) such that

$$g(E(\overline{G}(U_1, U_2)) = \{5n+1, 5n+2, \dots, 6n\} \cup \{n^2+1, n^2+2, \dots, n^2+n(n-d-1)\}.$$

According to Lemma 2.8 and Lemma 2.9 there exists a DC-labelling f of the graph $\overline{G}(U_1) \cup \overline{G}(U_2) = 2K_n$ with respect to (U_1, U_2) such that

$$f(E(2K_n)) = \{1, 2, \dots, 5n\} \cup \{6n+1, 6n+2, \dots, n^2\}.$$

By Proposition 2.1, the graph $\overline{G}$ is supermagic because the labellings f and g are complementary. □

Theorem 3.12. *Let G be a d-regular bipartite graph of order $2n$ with parts U_1 and U_2. If $n \geq 5$ and d are odd and $\overline{G}(U_1, U_2)$ is a Hamiltonian graph, then the complement of G is a supermagic graph.*

Proof. As in the proof of Theorem 3.11, but using Lemma 2.3 instead of Lemma 2.2, there are complementary DC-labellings of $\overline{G}(U_1, U_2)$ and $\overline{G}(U_1) \cup \overline{G}(U_2)$. Thus, by Proposition 2.1, the graph $\overline{G}$ is supermagic. □

In [15] there is proved that any balanced bipartite graph of order $2n$ with minimum degree greater than $n/2$ is Hamiltonian. So, we get immediately

Corollary 3.13. *Let G be a d-regular bipartite graph of order $2n$. If $2d < n$ and $5 \leq n \equiv d \equiv 1 \pmod 2$, then the complement of G is a supermagic graph.*

4. Joins of graphs

The join $G = G_1 \oplus G_2$ of graphs G_1 and G_2 satisfies

$G(V(G_i)$ is isomorphic to G_i, $i \in \{1, 2\}$,

$G(V(G_1), V(G_2))$ is a complete bipartite graph.

Therefore we can construct some supermagic joins of graphs by Proposition 2.1.

Theorem 4.14. *Let G_1, G_2 be disjoint d-regular Hamiltonian graphs of order n. If $d \geq 4$ is even and n is odd, then the join $G_1 \oplus G_2$ is a supermagic graph.*

Proof. As in the proof of Theorem 3.11, there are complementary DC-labellings of $K_{n,n}$ and $G_1 \cup G_2$. Thus, by Proposition 2.1, the graph $G_1 \oplus G_2$ is supermagic. □

As any graph of order $n \geq 3$ with minimum degree at least $n/2$ is Hamiltonian (see [3]), we have immediately

Corollary 4.15. *Let G_1, G_2 be disjoint d-regular graphs of order n. If $2d \geq n$, $5 \leq n \equiv 1 \pmod 2$ and $4 \leq d \equiv 0 \pmod 2$, then the join $G_1 \oplus G_2$ is supermagic.*

Theorem 4.16. *Let G_i, $i \in \{1, 2\}$, be a d_i-regular Hamiltonian graph of order n. If $4 \leq d_1 \equiv 0 \pmod 4$, $d_1 = d_2 + 2$ and n is odd, then the join $G_1 \oplus G_2$ is a supermagic graph.*

Proof. As in the proof of Theorem 3.11, but using Lemma 2.7 instead of Lemma 2.6, there are complementary DC-labellings of $K_{n,n}$ and $G_1 \cup G_2$. Therefore, by Proposition 2.1, the graph $G_1 \oplus G_2$ is supermagic. □

For dense graphs we get immediately

Corollary 4.17. *Let G_i, $i \in \{1, 2\}$, be a d_i-regular graph of odd order n. If $n \leq 2d_2$, $d_1 = d_2 + 2$ and $4 \leq d_1 \equiv 0 \pmod 4$, then the join $G_1 \oplus G_2$ is a supermagic graph.*

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References

[1] M. Bača, I. Holländer and Ko-Wei Lih, Two classes of super-magic graphs, *J. Combin. Math. Combin. Comput.,* **23** (1997) 113–120.

[2] R. Bodendiek and G. Walther, *Aritmethisch antimagische graphen*, in: K. Wagner, R. Bodendiek, eds., Graphentheorie III (BI-Wiss. Verl., Mannheim, 1993).

[3] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc., Ser. 3*, **2** (1952), 69–81.

[4] M. Doob, Characterizations of regular magic graphs, *J. Combin. Theory, Ser. B,* **25** (1978), 94–104.

[5] S. Drajnová, J. Ivančo and A. Semaničová, Numbers of edges in supermagic graphs, *J. Graph Theory,* **52** (2006), 15–26.

[6] J.A. Gallian, A dynamic survey of graph labeling, *Electronic J. Combinatory,*#**DS6** 36 (2003).

[7] N. Hartsfield, G. Ringel, *Pearls in Graph Theory* (Academic Press, Inc., San Diego, 1990).

[8] J. Ivančo, On supermagic regular graphs, *Mathematica Bohemica,* **125** (2000), 99–114.

[9] J. Ivančo, Magic and supermagic dense bipartite graphs, *Discuss. Math. Graph Theory,* **27** (2007), 583–591.

[10] J. Ivančo, Z. Lastivková, A. Semaničová, On magic and supermagic line graphs, *Mathematica Bohemica,* **129** (2004), 33–42.

[11] J. Ivančo and A. Semaničová, Some constructions of supermagic graphs using antimagic graphs, *SUT Journal of Mathematics,* **42** (2006), 177–186.

[12] J. Ivančo and A. Semaničová, Some constructions of supermagic non-regular graphs, *Australasian J. Combin.,* **38** (2007), 127–139.

[13] R.H. Jeurissen, Magic graphs, a characterization, *Europ. J. Combin.,* **9** (1988), 363–368.

[14] S. Jezný and M. Trenkler, Characterization of magic graphs, *Czechoslovak Math. J.,* **33** (1983) 435–438.

[15] J. Moon, L. Moser, On Hamiltonian bipartite graphs, *Isr. J. Math.,* **1** (1963), 163–165.

[16] J. Sedláček, *Problem 27. Theory of Graphs and Its Applications*, Proc. Symp. Smolenice, Praha (1963) 163–164.

[17] J. Sedláček, On magic graphs, *Math. Slovaca,* **26** (1976), 329–335.

[18] A. Semaničová, On magic and supermagic circulant graphs, *Discrete Math.,* **306** (2006), 2263–2269.

[19] B.M. Stewart, Magic graphs, *Canad. J. Math.,* **18** (1966), 1031–1059.

[20] B.M. Stewart, Supermagic complete graphs, *Canad. J. Math.,* **19** (1967), 427–438.