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On Super mean graphs

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Abstract

Let G be a (p, q) graph and let $f : V(G) \to \{1, 2, 3, \cdots, p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = (f(u)+f(v))/2$ if $f(u)+f(v)$ is even and $f^*(e) = (f(u)+f(v)+1)/2$ if $f(u) + f(v)$ is odd. Then f is called a super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, p+q\}$. A graph that admits a super mean labeling is called a super mean graph. In this paper we present several infinite families of super mean graphs.

Keywords: super mean labeling, super mean graph, armed crowns.

2000 Mathematics Subject Classification: 05C78

1. Introduction

By a graph we mean a finite, simple and undirected one. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The disjoint union of m copies of a graph G is denoted by mG. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is obtained by taking one copy of G_1(with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. Armed crown $C_n \ominus P_m$ is the graph obtained from a cycle C_n by identifying the pendent vertex of a path P_m at each vertex of the cycle. Bi-armed crown $C_n \ominus 2P_m$ is a graph obtained from a cycle C_n by identifying the pendent vertices of two vertex disjoint paths of same length $m-1$ at each vertex of the cycle. The graph $\langle B_{n,n} : w\rangle$ is the graph obtained by subdividing the central edge of the bistar $B_{n,n}$ with the vertex w. The graph $S_{m,n}$ is obtained from a star with n spokes in which each spoke is a path of length m. Terms and notations not defined here are used in the sense of Harary[1].

2. Preliminary Results

The concept of super mean labeling was introduced in [3]. Let G be a (p,q)-graph and $f : V(G) \to \{1,2,3,\cdots,p+q\}$ be an injection. For each edge $e = uv$, let $f^*(e) = (f(u)+f(v))/2$ if $f(u)+f(v)$ is even and $f^*(e) = (f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd. Then f is called a super mean labeling if $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1,2,3,\cdots,p+q\}$. A graph that admits a super mean labeling is called a super mean graph. The following results have been proved in [3]:

- Path, Comb and any cycle of odd length are super mean graphs.
- If $n > 3$, $K_{1,n}$ is not a super mean graph.
- The bistar $B_{m,n}$ is a super mean graph for $m = n$ or $n+1$.
- The graph $L_n \odot K_1$ is a super mean graph.
- If $n > 3$, K_n is not a super mean graph.
- The graph P_n^2 is a super mean graph.
- $C_m \cup P_n$ is a super mean graph for all $m \geq 3$ and $n \geq 2$.

In [2], super mean labeling of graphs of order 5 are discussed. We use the following results in the subsequent theorems.

Theorem 2.1. [3] *Union of two super mean graphs is a super mean graph.*

Theorem 2.2. [3] *Let G_1 be a (p_1,q_1)-graph and G_2 be a (p_2,q_2)-graph with super mean labelings f and g respectively. Let $f(u) = p_1 + q_1$ and $g(v) = 1$. Then the graph $(G_1)_f * (G_2)_g$ obtained from G_1 and G_2 by identifying the vertices u and v is also a super mean graph.*

3. Super Mean Graphs

Theorem 3.3. *The armed crown $C_n \ominus P_m$ is a super mean graph for all $m \geq 2$ and $n \geq 3$.*

Proof. Let $u_1, u_2, u_3, \cdots, u_n$ be the vertices of the cycle. let $v_i^1, v_i^2, v_i^3, \cdots, v_i^m$ be the vertices of the path of length $m-1$ attached with u_i $(1 \leq i \leq n)$ with identification of u_i and v_i^m.

Case (i) n is odd.

Then $n = 2k+1$ for some k. Define $f : V(G) \to \{1, 2, \cdots, p+q = 2mn\}$ by

$$\begin{aligned}
f(v_1^i) &= 2i-1 && \text{for } 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i-1+2jm && \text{for } j \text{ } odd, 1 \le j \le k, 1 \le i \le m\\
f(v_{1+j}^i) &= 2i-1+2jm && \text{for } j \text{ } even, 1 \le j \le k, 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i+2jm && \text{for } j \text{ } odd, k+1 \le j \le 2k, 1 \le i \le m\\
f(v_{1+j}^i) &= 2i+2jm && \text{for } j \text{ } even, k+1 \le j \le 2k, 1 \le i \le m.
\end{aligned}$$

Then f is a super mean labeling.

Case (ii) n is even.

Then $n = 2k$ for some k.

Subcase (i) $k \equiv 0 \pmod 2$.
Define $f : V(G) \to \{1, 2, \cdots, p+q = 2mn\}$ by

$$\begin{aligned}
f(v_1^i) &= 2i-1 && \text{for } 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i-1+2jm && \text{for } j \text{ } odd, 1 \le j \le k-1, 1 \le i \le m\\
f(v_{1+j}^i) &= 2i-1+2jm && \text{for } j \text{ } even, 1 \le j \le k-1, 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i+2(j+1)m && \text{for } j \text{ } odd, k \le j \le 2k-3, 1 \le i \le m\\
f(v_{1+j}^i) &= 2i+2(j+1)m && \text{for } j \text{ } even, k \le j \le 2k-3, 1 \le i \le m\\
f(v_{2k-1}^{m+1-i}) &= (4k-2)m+2i && \text{for } 1 \le i \le m\\
f(v_{2k}^1) &= (4k-4)m+1 && \text{and}\\
f(v_{2k}^{1+i}) &= (4k-4)m+2+2i && \text{for } 1 \le i \le m-1.
\end{aligned}$$

Then f is a super mean labeling.

Subcase (ii) $k \equiv 1 \pmod 2$.

Define $f : V(G) \to \{1, 2, \cdots, p+q = 2mn\}$ by

$$\begin{aligned}
f(v_1^i) &= 2i-1 && \text{for } 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i-1+2jm && \text{for } j \text{ } odd, 1 \le j \le k-1, 1 \le i \le m\\
f(v_{1+j}^i) &= 2i-1+2jm && \text{for } j \text{ } even, 1 \le j \le k-1, 1 \le i \le m\\
f(v_{1+j}^{m+1-i}) &= 2i+2jm && \text{for } j \text{ } odd, k \le j \le 2k-3, 1 \le i \le m
\end{aligned}$$

$$f(v_{1+j}^{i}) = 2i + 2jm \quad \text{for } j \text{ even}, k \le j \le 2k-3, 1 \le i \le m$$
$$f(v_{2k-1}^{m+1-i}) = (4k-2)m + 2i \quad \text{for } 1 \le i \le m$$
$$f(v_{2k}^{1}) = (4k-4)m + 1 \quad \text{and}$$
$$f(v_{2k}^{1+i}) = (4k-4)m + 2 + 2i \quad \text{for } 1 \le i \le m-1.$$

Then f is a super mean labeling. Hence $C_n \ominus P_m$ is a super mean graph for all $m \ge 2$ and $n \ge 3$. □

Example 3.4. *A super mean labeling of $C_{12} \ominus P_5$ is given in Figure 1.*

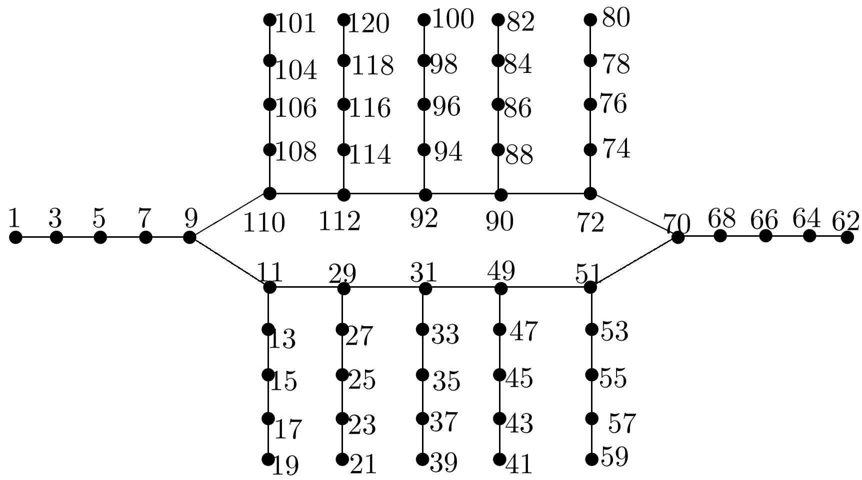

Figure 1.

In [3] we have shown that $K_{1,4}$ is not a super mean graph. The following theorem shows that $nK_{1,4}, n > 1$ is a super mean graph.

Theorem 3.5. *The disconnected graph $nK_{1,4}, n > 1$ is a super mean graph.*

Proof. The super mean labeling of $2K_{1,4}$ and $3K_{1,4}$ are given in Figures 2 and 3.

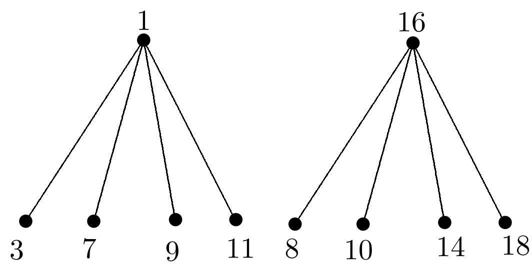

Figure 2.

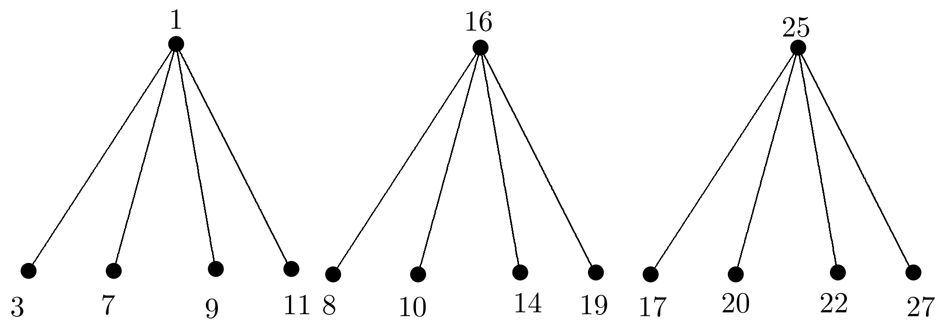

Figure 3.

Let $G_1 = 2K_{1,4}$ and $G_2 = 3K_{1,4}$ Now $G_1 \cup G_1 = 4K_{1,4}$ and $G_1 \cup G_2 = 5K_{1,4}$. By Theorem 2.1, $4K_{1,4}$ and $5K_{1,4}$ are super mean graphs. Again $G_1 \cup G_1 \cup G_1 = G_1 \cup 4K_{1,4} = 6K_{1,4}$ and $G_1 \cup G_1 \cup G_2 = G_1 \cup 5K_{1,4} = 7K_{1,4}$. By Theorem 2.1, $6K_{1,4}$ and $7K_{1,4}$ are super mean graphs. Now for any integer $m \geq 2$, $mG_1 = 2mK_{1,4}$ and $(m-1)G_1 \cup G_2 = (2m+1)K_4$ are super mean graphs. Hence $nK_{1,4}$ is a super mean graph for all $n > 1$. □

Theorem 3.6. *Let $\wp$ be the collection of paths $P_n^i = (u_1^i, u_2^i, u_3^i, \cdots, u_n^i), 1 \leq i \leq m$. Let $1 \leq t \leq n$. Let G be the graph with $V(G) = \bigcup_{i=1}^{m} V(P_n^i)$ and $E(G) = \left(\bigcup_{i=1}^{m} E(P_n^i)\right) \cup \left\{u_t^i u_t^{i+1} : 1 \leq i \leq m-1\right\}$. Then G is a super mean graph.*

Proof. Define $f : V(G) \to \{1, 2, \cdots, p+q = 2mn-1\}$ by $f(u_i^1) = 2i-1$ for $1 \leq i \leq n$; $f(u_{n+1-j}^{2k}) = f(u_n^{2k-1}) + 2j$ for $1 \leq j \leq n$and $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$, $f(u_j^{2k+1}) = f(u_1^{2k}) + 2j$ for $1 \leq j \leq n$ and $1 \leq k \leq r$ where $r = \left\lfloor \frac{m}{2} \right\rfloor$ if m is odd and $r = \left\lfloor \frac{m-1}{2} \right\rfloor$ if m is even.
Then f is a super mean labeling of G. Hence G is a super mean graph. □

Example 3.7. *An illustration Theorem 3.6 with $m = 3$ and $n = 4$ is given in Figure 4.*

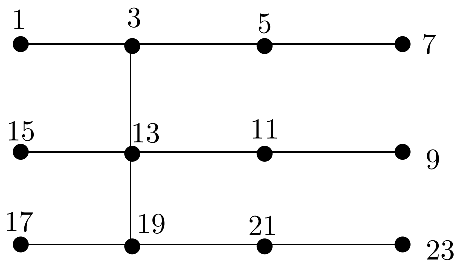

Figure 4.

Theorem 3.8. *The graph $< B_{n,n} : w >$ obtained by the subdivision of the central edge of $B_{n,n}$ with a vertex w, is a super mean graph.*

Proof. Let $V(< B_{n,n} : w >) = \{u, v, w, u_i, v_i : 1 \leq i \leq n\}$ and $E(< B_{n,n} : w >) = \{uw, vw, uu_i, vv_i : 1 \leq i \leq n\}$.

Case (i) n is even.

Then $n = 2k$ for some k. Define $f : V(< B_{n,n} : w >) \to \{1, 2, 3, \cdots, p+q = 4n+5\}$ by $f(u) = 1$; $f(w) = 4n+5$; $f(v) = 4n+3$; $f(u_i) = 4i-1$ if $1 \leq i \leq 2k$ and if $i \neq k+1$; $f(u_{k+1}) = 4k+2$ and $f(v_i) = 4i+1$ if $1 \leq i \leq 2k$. Then f is a super mean labeling.

Case (ii) n is odd.

Then $n = 2k+1$ for some k. Define $f : V(< B_{n,n} : w >) \to \{1, 2, 3, \cdots, p+q = 4n+5\}$ by $f(u) = 1$; $f(w) = 4n+5$; $f(v) = 4n+3$; $f(u_i) = 4i-1$ if $1 \leq i \leq 2k+1$; $f(v_i) = 4i+1$ if $1 \leq i \leq 2k+1$ and if $i \neq k+1$ and $f(v_{k+1}) = 4k+4$. Then f is a super mean labeling. Hence $< B_{n,n} : w >$ is a super mean graph. □

Example 3.9. *A super mean labeling of $< B_{6,6} : w >$ is given in Figure 5.*

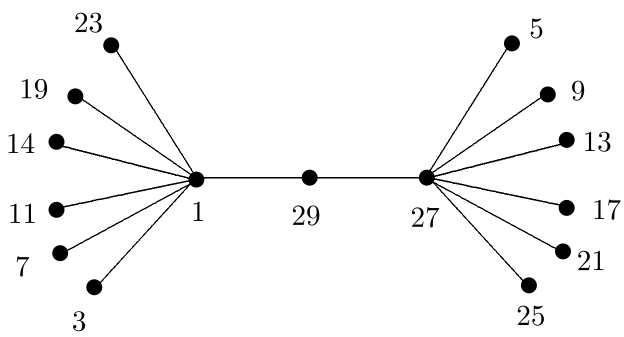

Figure 5.

Theorem 3.10. *The graph $< B_{n,n} : w > @P_m$ obtained by attaching a path P_m with the central vertex w of $< B_{n,n} : w >$ is a super mean graph.*

Proof. Let $V(< B_{n,n} : w >) = \{u, v, w, u_i, v_i : 1 \leq i \leq n\}$. Let $P_m = (w_1, w_2, \cdots, w_m)$. The path P_m is a super mean graph [3]. Now it follows from Theorem 3.8 and Theorem 2.2 $< B_{n,n} : w > @P_m$ is a super mean graph. □

Theorem 3.11. *The bi-armed crown $C_n \ominus 2P_m$ is a super mean graph for all odd $n \geq 3$ and $m \geq 2$.*

Proof. Let $C_n = (u_1, u_2, u_3, \cdots, u_n, u_1)$. Let $(v_{i1}^1, v_{i1}^2, v_{i1}^3, \cdots, v_{i1}^m)$ and $(v_{i2}^1, v_{i2}^2, v_{i2}^3, \cdots, v_{i2}^m)$ be the paths of length $m-1$ with $V_{i1}^m = v_{i2}^m$ and attached with u_i and $v_{i1}^m = v_{i2}^m (1 \leq i \leq n)$ with identification of u_i and $v_{i1}^m = (v_{i2}^m), 1 \leq i \leq n$.

Let $n = 2k+1$ for some k. Define $f : V(G) \to \{1, 2, \cdots, p+q = 4mn - 2n\}$ by

$$\begin{aligned}
f(v_{j1}^{i}) &= 4(j-1)m - (2j-1) + 2i && \text{for } 1 \le j \le k+1,\ 1 \le i \le m \\
f(v_{j2}^{m+1-i}) &= 2(2j-1)m - (2j+1) + 2i && \text{for } 1 \le j \le k,\ 2 \le i \le m \\
f(v_{k+12}^{m-1}) &= 2(2k+1)m - (2k+2) && \\
f(v_{k+12}^{m-1-i}) &= 2(2k+1)m - 2k + 2 + 2i && \text{for } 1 \le i \le m-2 \\
f(v_{k+1+j1}^{i}) &= 4(j+k)m - 2(k+j) + 2i && \text{for } 1 \le j \le k,\ 1 \le i \le m \\
f(v_{k+1+j2}^{m+1-i}) &= (4j+4k+2)m - 2(k+1+j+2i) && \text{for } 1 \le j \le k,\ 2 \le i \le m.
\end{aligned}$$

Then f is a super mean labeling. Hence $C_n \ominus 2P_m$ is a super mean graph for all odd $n \ge 3$ and $m \ge 2$. □

Example 3.12. *A super mean labeling of $C_5 \ominus 2P_4$ is given in Figure 6.*

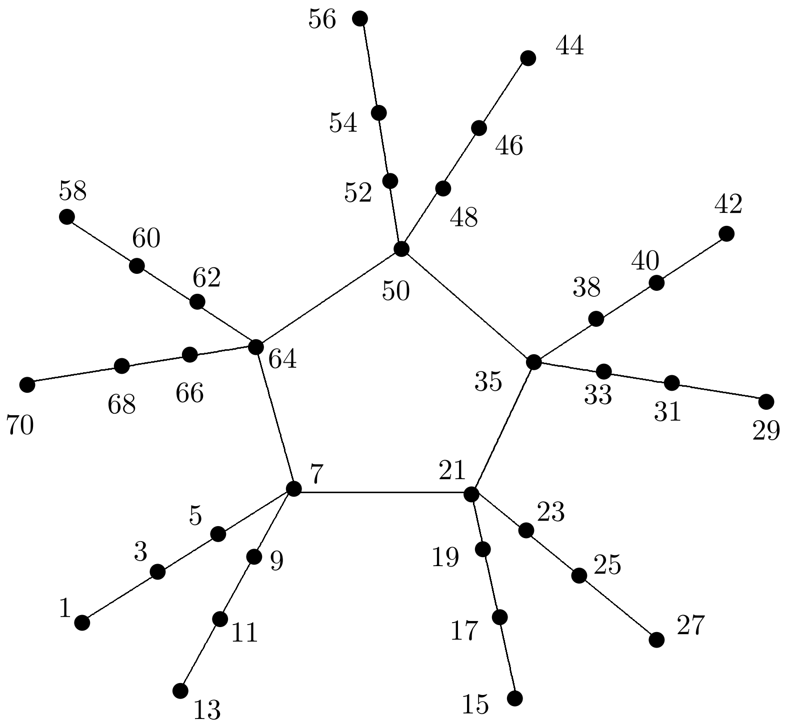

Figure 6.

The graph $S_{1,3}$ is the star $K_{1,3}$ which is a super mean graph [3]. Next we prove that $S_{m,3}$ is a super mean graph for $m > 1$.

Theorem 3.13. *The graph $S_{m,3}, m > 1$ is a super mean graph.*

Proof. Let v_0 be the centre of the star and v_i^j, $i = 1, 2, 3, \cdots, m$ and $j = 1, 2, 3$ be the vertices of $S_{m,3}$. The graph $S_{m,3}$ has $3m+1$ vertices and $3m$ edges. Define $f : V(S_{m,3}) \to \{1, 2, \cdots, p+q = 6m+1\}$ by

$$\begin{aligned} f(v_i^1) &= 2i-1 \text{ for } 1 \le i \le m; & f(v_0) &= 2m+1; \\ f(v_{m+1-i}^2) &= 2m-1+2i \text{ for } 1 \le i \le m-1; & f(v_1^2) &= 4m+2; \\ f(v_1^3) &= 4m+3; & f(v_i^3) &= 4m+1+2i \text{ for } 2 \le i \le m-2; \\ f(v_{m-1}^3 &= 6m+1; & f(v_m^3) &= 6m-2. \end{aligned}$$

Then f is a super mean labeling. Hence $S_{m,3}, m > 1$ is a super mean graph. □

Example 3.14. *A super mean labeling of $S_{5,3}$ is given in Figure 7.*

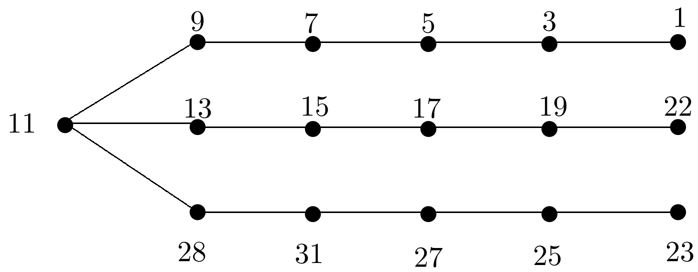

Figure 7.

The graph $S_{1,4}$ is the star $K_{1,4}$ which is not a super mean graph [3]. We now prove that $S_{m,4}$ is a supermean graph for all $m \ge 2$.

Theorem 3.15. *The graph $S_{m,4}, m \ge 2$ is a super mean graph.*

Proof. A super mean labeling of $S_{2,4}$ is given in Figure 8.

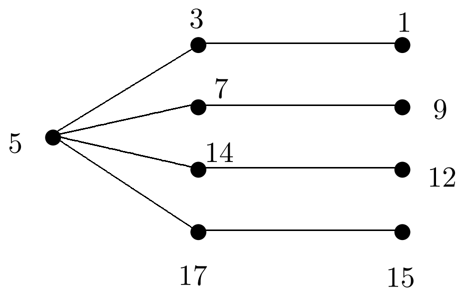

Figure 8.

Now $m \geq 3$. Let v_o be the centre of the star and v_i^j , $i = 1, 2, 3, \cdots, m$ and $j = 1, 2, 3, 4$ be the vertices of $S_{m,4}$. The graph $S_{m,4}$ has $4m + 1$ vertices and $4m$ edges.

Case (i) m is odd.

Define $f : V(S_{m,4}) \to \{1, 2, \cdots, p + q = 8m + 1\}$ by

$$f(v_i^1) = 2i - 1 \text{ for } 1 \leq i \leq m; \quad f(v_0) = 2m + 1;$$
$$f(v_{m+1-i}^2) = 2m + 1 + 2i \text{ for } 1 \leq i \leq m; \quad f(v_1^3) = 4m + 4;$$
$$f(v_i^3) = 4m + 2 + 2i \text{ for } 2 \leq i \leq m; \quad f(v_i^4) = 6m + 4i - 3 \text{ for } 1 \leq i \leq \frac{m-1}{2};$$
$$f(v_{\frac{m+1}{2}}^4) = 8m + 1; \quad f(v_{m+1-i}^4) = 6m + 4i - 2 \text{ for } 1 \leq i \leq \frac{m-1}{2}.$$

Then f is a super mean labeling.

Case (ii) m is even.

Define $f : V(S_{m,4}) \to \{1, 2, \cdots, p + q = 8m + 1\}$ by

$$f(v_i^1) = 2i - 1 \text{ for } 1 \leq i \leq m; \quad f(v_0) = 2m + 1;$$
$$f(v_{m+1-i}^2) = 2m + 1 + 2i \text{ for } 1 \leq i \leq m; \quad f(v_1^3) = 4m + 4;$$
$$f(v_i^3) = 4m + 2 + 2i \text{ for } 2 \leq i \leq m; \quad f(v_i^4) = 6m + 4i - 3 \text{ for } 1 \leq i \leq \frac{m-1}{2};$$
$$f(v_{\frac{m}{2}}^4) = 8m - 4; \quad f(v_{\frac{m+2}{2}}^4) = 8m + 1;$$
$$f(v_{m+1-i}^4) = 6m + 4i - 2 \text{ for } 1 \leq i \leq \frac{m-1}{2}.$$

Then f is a super mean labeling. Hence $S_{m,4}, m \geq 3$ is a super mean graph. □

Example 3.16. *A super mean labeling of $S_{6,4}$ is given in Figure 9.*

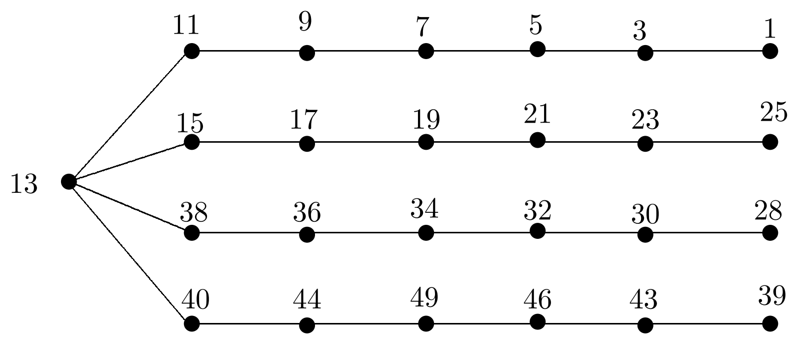

Figure 9.

Acknowledgement

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