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Exclusive Sum Labeling of Graphs: A Survey

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Abstract

All *sum graphs* are disconnected. In order for a connected graph to bear a sum labeling, the graph is considered in conjunction with a number of isolated vertices, the labels of which complete the sum labeling for the disjoint union. The smallest number of isolated vertices that must be added to a graph H to achieve a sum graph is called the *sum number* of H; it is denoted by $\sigma(H)$. A sum labeling which realizes $H \cup \overline{K_{\sigma(G)}}$ as a sum graph is called an *optimal sum labeling* of H.

In this paper we survey a new type of labeling based on summation, the *exclusive sum labeling.* A sum labeling L is called exclusive sum labeling with respect to a subgraph H of G if L is a sum labeling of G where H contains no working vertex. The *exclusive sum number* $\epsilon(H)$ of a graph H is the smallest number r such that there exists an exclusive sum labeling L which realizes $H \cup \overline{K_r}$ as a sum graph. A labeling L is an *optimal exclusive sum labeling* of a graph H if L is a sum labeling of $H \cup K_{\epsilon(H)}$ and H contains no working vertex.

Keywords: sum graphs, sum number, optimal sum labeling, exclusive sum labeling, exclusive sum number, optimal exclusive sum labeling.

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1. Introduction

All graphs we consider here are finite, simple and undirected. For general terms used in graph theory, please refer to [2].

A *sum labeling* λ of a graph G is a mapping of the vertices of G into distinct positive integers such that for $u, v \in$ V(G), $uv \in E(G)$ if and only if the sum of the labels assigned to u and v equals the label of a vertex w of G. In such a case w is called a *working vertex.* A graph which has a sum labeling is called a *sum graph.* Sum graphs were originally proposed by Harary [6] and later extended to include all integers [7].

Sum graphs cannot be connected graphs since an edge from the vertex with the largest label would necessitate a vertex with a larger label. Graphs which are not sum graphs can be made to support a sum labeling by considering the graph in conjunction with a number of isolated vertices which can bear the labels required by the graph. The fewest number of the additional isolates required by the graph to support a sum labeling is called the *sum number* of the graph; it is denoted by $\sigma(G)$.

Every edge adjacent to the vertex bearing the largest label requires an isolated vertex to witness the edge. Consequently a lower bound for the number of isolates required for a graph to support a sum labeling is $\delta(G)$ - the smallest degree of G. Any graph for which $\sigma(G) = \delta(G)$ is known as a *δ-optimal summable.*

A sum labeling of a graph $G \cup \overline{K_r}$ for some positive integer r is said to be *exclusive* with respect to G if all of its working vertices are in $\overline{K_r}$. Every graph can be made to support an exclusive sum labeling, by adding a required number of isolates. The least possible number of isolates that need to be added to a graph G to obtain an exclusive sum labeling is called the *exclusive sum number* of the graph G, denoted by $\epsilon(G)$.

Observation 1. *Let $\Delta(G)$ be the maximum degree of the vertices of a graph G. Then $\epsilon(G) \geq \Delta(G)$.*

In case $\epsilon(G) = \Delta(G)$, the graph G is said to be a *Δ-optimum summable* graph.

Clearly, every exclusive sum graph is a sum graph but not vice versa and so the exclusive sum number is always greater than or equal to the sum number, that is,

Observation 2. *For any graph G, $\epsilon(G) \geq \sigma(G)$.*

There are several graphs for which $\sigma(G) = \epsilon(G)$ and the optimum sum labeling is exclusive. Examples of such graphs include complete graphs (K_n) [1], cocktail party graphs (regular complete multipartite graphs ($H_{m,n}$)) [12] and odd wheels (W_n, n odd) [9, 11]. Of these examples only the odd wheels are Δ-optimum summable.

An interesting result of Gould and Rödl [5] is that there exist families of sum graphs whose sum number is quadratic in the order of the graph. Despite the work of Nagamochi et al. [13] showing that almost all graphs have superlinear sum number, no families of graphs have been found to exhibit this feature. In light of Observation 2, these results must hold for exclusive sum graphs and while these may prove easier to find, to date none have been discovered.

2. Linear Transformations

It is clear from the nature of sum graphs that if L is a sum labeling of a graph G then so is αL, where α is any positive integer. The case involving addition is not quite so clear. For L a sum labeling, let $L + \beta$ be the labeling formed by adding β to every label. To maintain the sum labeling properties we need to add 2β to working vertices, or more if the working vertex is witnessing an edge between working vertices.

In exclusive sum graphs, adjacent vertices are always non-working, so adding β to every vertex in the graph and 2β to the isolates maintains the exclusive sum labeling properties.

In what follows, when we speak of a linear transformation on a labeling, we mean transform $L(u)$ to $k_1L(u)+k_2$ for u a non-working vertex and $k_1L(v)+2k_2$ for v a working vertex.

Miller et al. showed in [17] that we can use a linear transformation to ensure that the labeling has particular properties such as a specific minimum value, however not all transformations will be suitable to maintain the sum labeling properties as the following example shows.

Table 1 gives a (possibly non-optimal) labeling for $H_{4,4}$. The partite sets, labeled v_1-v_4 etc. are presented in rows in the top part of the table while below are listed (and labeled w_1 etc) the isolates. One method of relabeling this (exclusive) labeling so as to achieve a minimum label 1 would be to add -41 to all vs and -82 to all ws (i.e. set $k_1=1$ and $k_2=-41$). This transformation would label the vertices v_5-v_9 with $2,4,6,8$ thus inducing edges in a partite set.

v_1-v_4	42	44	46	48			
v_5-v_8	43	45	47	49			
v_9-v_{12}	55	57	59	61			
$v_{13}-v_{16}$	56	58	60	62			
w_1-w_6	85	87	89	91	93	95	
w_7-w_{13}	97	99	101	103	105	107	109
$w_{14}-w_{20}$	98	100	102	104	106	108	110
$w_{21}-w_{27}$	111	113	115	117	119	121	123

Table 1: A sum numbering for $H_{4,4}$.

Substituting $k_1=2, k_2=-83$ relabels the first three isolates with $4,8,12$ thus inducing edges among the isolates. $k_1=3, k_2=-125$ assigns labels 5, 17 to w_1 and w_3 respectively while v_8 is assigned 22, again inducing edges between isolates, this time witnessed by a vertex from within the graph.

The following theorem explains the problems from the above example and provides a strategy for choosing a transformation that avoids inducing unwanted edges.

Theorem 1. *If L is an exclusive sum graph labeling of a graph H in $G=H\cup\overline{K_r}$ then so is the labeling $L'(u)=k_1L(u)+k_2$ for $u\in H$ and $L'(u)=k_1L(u)+2k_2$ for $u\in\overline{K_r}$, where k_2 is any integer which results only in positive distinct values in L' and k_1 is any positive integer that does not divide $6k_2$.*

Proof. If w is a vertex witnessing an edge between u and v then $u+v=w$ and in the new labeling

$$u'+v'=k_1u+k_2+k_1v+k_2=k_1(u+v)+2k_2=k_1w+2k_2=w'$$

Then all edges in the original graph will still be present under the new labeling. It remains to show that no new edges are induced. There are four cases to consider.

Case 1. An extra edge may be induced within the graph H whenever $u'+v'=w'$ where $u,v,w\in H$. We need to ensure

$$u+v\neq w-\frac{k_2}{k_1} \tag{1}$$

This can be ensured by forbidding the final term from being an integer, so we choose a k_1 that does not divide k_2.

Case 2. An extra edge may be induced between working vertices by another working vertex whenever $u'+v'=w'$ with $u,v,w\in\overline{K_r}$. Here we need to ensure

$$u+v\neq w-\frac{2k_2}{k_1} \tag{2}$$

Case 3. An extra edge may be induced between working vertices by a non-working vertex whenever $u'+v'=w'$ with $u,v\in\overline{K_r}$ and $w\in H$. To avoid this case we need

$$u+v\neq w-\frac{3k_2}{k_1} \tag{3}$$

Case 4. Extra edges may be induced between H and the working vertices in two different ways. First when $w\in\overline{K_r}$ and secondly when $w\in H$. The former situation is addressed by (1) and the latter by (2).

All three inequalities may be realised by ensuring that k_1 does not divide $6k_2$. □

$v_1 - v_4$	1	11	21	31			
$v_5 - v_8$	6	16	26	36			
$v_9 - v_{12}$	66	76	86	96			
$v_{13} - v_{16}$	71	81	91	101			
$w_1 - w_6$	7	17	27	37	47	57	
$w_7 - w_{13}$	67	77	87	97	107	117	127
$w_{14} - w_{20}$	72	82	92	102	112	122	132
$w_{21} - w_{27}$	137	147	157	167	177	187	197

Table 2: A minimal sum numbering for $H_{4,4}$.

Therefore, for the linear transformation to be suitable we need to answer the following question.

Given positive integers P and Q, can we always find integers s and t where $s > 0$ and does not divide $6t$ such that $P = sQ + t$?

This can be easily answered in the affirmative by choosing an $s \neq 2, 3, 6$ that does not divide P and $t = P - sQ$. In the previous example we can thus choose $k_1 = 5, k_2 = -209$ as shown in Table .

3. Exclusive Sum Labeling of Complete Bipartite Graphs

Hartsfield and Smyth [9] showed that the sum number of a complete bipartite graph $K_{p,q}$ for $q \geq p \geq 2$ is equal to $\left\lceil \frac{3p+q-3}{2} \right\rceil$. In 2001 He et al. [10] showed that this result is only realised for a limited range of p and q. The correct sum number for $K_{m,n}$ was given independently by three sets of authors [18],[10], [14], all published in the same issue of *Discrete Mathematics* in 2001.

However, in the case of exclusive sum number, the following lemma which is modified from the Hartsfield's et al. paper [9] is true in general and provides a lower bound on the exclusive sum number for the graph $K_{p,q}$.

Lemma 1. *For $q \geq 2$ and $p \geq 2, \epsilon(K_{p,q}) \geq p + q - 1$.*

Proof. Let L be any exclusive sum labeling of a complete bipartite graph $K_{p,q}$, $q \geq 2$, $p \geq 2$. Let P and Q be the two partite sets, where $|P| = p \geq 2$, $|Q| = q \geq 2$. Suppose that the labels of $P = \{x_1, x_2, ..., x_p\}$ under L are arranged into an ascending sequence, so that $x_j < x_{j+1}, \quad 1 \leq j \leq p-1$. Similarly, arrange the labels of $Q = \{y_1, y_2, ..., y_q\}$ into an ascending sequence. Observe that each of the following sums is distinct

$$x_1 + y_1 < x_2 + y_1 < ... < x_p + y_1 < x_p + y_2 < ... < x_p + y_q.$$

Since there are exactly $p + q - 1$ distinct sums, it follows that at least $p + q - 1$ isolated vertices are required to label the graph exclusively. □

Next let $k >$ max $\{2p - 2, p + q - 2\}$ and suppose that L labels the vertices of P and Q as follows.

$$\begin{aligned} P &= \{1 + 4i | 0 \leq i \leq p - 1\} \\ Q &= \{1 + 4j | k \leq j \leq k + q - 1\} \end{aligned}$$

Let R be the set of isolated vertices which are labeled by

$$\{(1 + 4i) + (1 + 4k) | 0 \leq i \leq p - 2\} \cup \{(1 + 4(p - 1)) + (1 + 4j) | k \leq j \leq k + q - 1\}.$$

It is clear that $|R| = p + q - 1$. Note that the labels used for P and Q are $1 \pmod 4$ and the labels used for the isolated vertices R are the sums of two numbers of $1 \pmod 4$, that is, $2 \pmod 4$. Therefore, $K_{p,q}$ contains no working vertex.

The sum of any two numbers from P or Q cannot be in R by the choice of k. Moreover, since numbers congruent to 3 (mod 4) and 0 (mod 4) do not occur in this labeling, we conclude that no extra edges are induced between the isolates or between the graph and the isolates. Therefore, we have shown that L is an exclusive sum labeling of $K_{p,q}$ which realises the lower bound of $\epsilon(K_{p,q})$.

Hence we have the following theorem:

Theorem 2. *For $q \geq 2$ and $p \geq 2, \epsilon(K_{p,q}) = p + q - 1$.*

In the next section we present a construction of exclusive sum labeling for paths and cycles and we give their exclusive sum numbers.

4. Paths and Cycles

4.1. Paths

Let $v_1, v_2, ..., v_n$ be the vertices of the path P_n. Label the vertices v_i with $v_i = x+(\frac{i+1}{2}-1)n$, for odd i and $v_i = 2x-((i/2)-1)n$ for even i, where $x > n(n-2)$, then $v_i + v_{i+1} = 3x$ for odd i and $v_i + v_{i+1} = 3x + n$ for even i. Thus P_n has an exclusive labeling with 2 isolated vertices, $v_{n-2} + v_{n-1}$ and $v_{n-1} + v_n$.

We have just proved

Theorem 3. *The exclusive sum number for paths, $\epsilon(P_n) = 2$, for $n \geq 3$.*

4.2. Cycles

We start this subsection with the following lemma.

Lemma 2. *There are at least three distinct edge labels in any exclusive sum labeling of C_n.*

Proof. It is obvious for $n = 3$. Now we assume that $n > 3$. Let w be the largest vertex on C_n. Let u and v be adjacent to w, and (u, w) and (v, w) be their corresponding edges. Without loss of generality, we suppose that $u < v$. Since $n > 3$, it follows that there is a vertex t and a corresponding edge (t, u). It is clear that $t+u \neq u+w$ and $u+w \neq v+w$. Since $v > u$ and $w > t$, we see that $v + w \neq u + t$. Therefore, the sums $t + u, u + w$ and $v + w$ are all different. ☐

It is necessary to deal separately with odd and even cycles.

4.2.1. Odd Cycles

Let v_i, for $1 \leq i \leq n$, be the vertices on the cycle C_n for odd n. Suppose that we label the vertices as follows.

$$v_i = \begin{cases} v_{i-2} + d_i & \text{for odd i} \quad i \geq 3 \\ v_{i-2} - d_i & \text{for even i} \quad i \geq 4 \end{cases}$$

where $v_1 = 1$, $v_2 = v_n - d_2$ and

$$d_i = \begin{cases} 2\left\lceil \frac{n}{4} \right\rceil & \text{if} \quad i \leq 2\left\lceil \frac{n}{4} \right\rceil \\ 2\left\lceil \frac{n}{4} + 1 \right\rceil & \text{otherwise.} \end{cases}$$

Now we sum each pair of adjacent vertices in the two different cases.

(i) For odd i we have,

$$v_i + v_{i+1} = (v_{i-2} + d_i) + (v_{i-1} - d_{i+1}).$$

Note that for odd i, d_i and d_{i+1} have the same value. Therefore,

$$\begin{aligned} v_i + v_{i+1} &= v_{i-2} + v_{i-1} \\ &= v_1 + v_2 \\ &= 1 + v_n - d_2 \\ &= 1 + v_n - 2\left\lceil \frac{n}{4} \right\rceil. \end{aligned}$$

(ii) For even i, we consider the following three cases.

1. for $i < 2\left\lceil \frac{n}{4} \right\rceil$ and $i + 1 < 2\left\lceil \frac{n}{4} \right\rceil$,

$$\begin{aligned} v_i + v_{i+1} &= (v_{i-2} - d_i) + (v_{i-1} + d_{i+1}) \\ &= (v_{i-2} - 2\left\lceil \frac{n}{4} \right\rceil) + (v_{i-1} + 2\left\lceil \frac{n}{4} \right\rceil) \\ &= v_{i-2} + v_{i-1} \\ &= v_2 + v_3 \\ &= (v_n - d_2) + (v_1 + d_3) \\ &= v_n + v_1 \\ &= v_n + 1 \end{aligned}$$

2. for $i = 2\left\lceil \frac{n}{4} \right\rceil$ and $i+1 > 2\left\lceil \frac{n}{4} \right\rceil$,

$$\begin{aligned}
v_i + v_{i+1} &= (v_{i-2} - d_i) + (v_{i-1} + d_{i+1}) \\
&= (v_{i-2} - 2\left\lceil \frac{n}{4} \right\rceil) + (v_{i-1} + 2\left\lceil \frac{n}{4} + 1 \right\rceil) \\
&= (v_{i-2} - 2\left\lceil \frac{n}{4} \right\rceil) + (v_{i-1} + 2\left\lceil \frac{n}{4} \right\rceil + 2) \\
&= v_{i-2} + v_{i-1} + 2 \\
&= v_2 + v_3 + 2 \\
&= (v_n - d_2) + (v_1 + d_3) + 2 \\
&= v_n + v_1 + 2 \\
&= v_n + 3
\end{aligned}$$

3. for $i > 2\left\lceil \frac{n}{4} \right\rceil$ and $i+1 > 2\left\lceil \frac{n}{4} \right\rceil$,

$$\begin{aligned}
v_i + v_{i+1} &= (v_{i-2} - d_i) + (v_{i-1} + d_{i+1}) \\
&= (v_{i-2} - 2\left\lceil \frac{n}{4} + 1 \right\rceil) + (v_{i-1} + 2\left\lceil \frac{n}{4} + 1 \right\rceil) \\
&= v_{i-2} + v_{i-1} \\
&= v_2\left\lceil \frac{n}{4} \right\rceil) + (v_2\left\lceil \frac{n}{4} + 1 \right\rceil
\end{aligned}$$

This is the same as Case 2. Thus

$$v_i + v_{i+1} = v_n + 3$$

We see that there are three distinct edge labels of the cycles C_n for odd n, that is, $1 + v_n - 2\left\lceil \frac{n}{4} \right\rceil$, $v_n + 1$ and $v_n + 3$. Thus, in view of Lemma 2, the construction of an exclusive labeling for odd cycles requires three isolated vertices.

4.2.2. Even Cycles

Let $v_i \in V(C_n)$, $1 \le i \le n$, n even. Suppose that we label the vertices as follows.

$$v_i = \begin{cases} 4i - 3 & \text{if} \quad i \text{ is odd} \\ 4n - 4i + 5 & \text{if} \quad i \text{ is even} \end{cases}$$

Then the sum of each pair of adjacent vertices is: for $1 \le i \le n-1$,

$$v_i + v_{i+1} = \begin{cases} 4n - 2 & \text{if} \quad i \text{ is odd} \\ 4n + 6 & \text{if} \quad i \text{ is even} \end{cases}$$

and

$$v_n + v_1 = 6$$

In view of Lemma 2, an optimum exclusive sum labeling of even cycles requires three isolates and so we have

Theorem 4. *$\epsilon(C_n) = 3$, for $n \geq 3$.*

5. Trees

We refer to [3] for the notions of tree, caterpillar, and shrub. However, it is worth mentioning some terms used in that paper which will be used in this paper.

1. A leaf of a tree is a vertex with degree 1.
2. A near-leaf is a non-leaf that has at most one neighbour which is not a leaf.
3. An inner vertex is a vertex that has at least two neighbours which are not non-leaf.

In [3], M.N. Ellingham proved that if T is a tree of order at least 2, then $\sigma(T) = 1$. Since the introduction of the notions of exclusive sum labeling and exclusive sum number of graphs, it has been being a challenge to find the exclusive sum number of trees. Unlike its counterpart problem in sum number, attempts to solve this problem so far are still unsuccessful. However, some results have come to hand. Some trees are Δ-optimum summable graph, but there exists trees which are not Δ-optimum summable [16]. For example, caterpillars and shrubs (central in Ellingham's proof) along with stars and double stars are Δ-optimum summable graph. The result is cannot be generalised and we conclude with a tree which is not Δ-optimal summable.

Recall that a caterpillar is a graph which has the property that if we remove all the vertices with degree 1 then what remains is a path. A caterpillar can have more than one longest path. Such a path is called the *spine* of the caterpillar. The two end points of a spine are called respectively by *tail* and *head*. Other vertices at the spine are called the *internal vertices*. We shall always consider a spine of a caterpillar as oriented in particular direction from tail to head. The vertices of degree one of a caterpillar other than tail and head will be called the *feet*, which are attached to the internal vertices of its spine by edges called the *legs* of the caterpillar.

Let C be a caterpillar with $\Delta(C) = d$.

Labeling 1 (Exclusive sum labeling of caterpillar)

1. Choose a spine of C and let $P = \{p_1, p_2, ..., p_n\}$ be the the set of vertices of the spine. Let
$$\begin{aligned} f_i &= deg(p_i) - 2, \qquad i = 2, 3, ..., n-1. \\ &= 0, \qquad i = 1, n. \end{aligned}$$
For $2 \leq i \leq n-1$, let $B_i = \{b_{ij} | 1 \leq j \leq f_i\}$ be the set of feet which are attached to the internal vertex p_i. It is clear that $B_i = N(p_i) \backslash (P \cap N(p_i))$ for $2 \leq i \leq n-1$. Let $B = \bigcup_{i=2}^{n-1} B_i$ be the set of all feet of C.

2. Label the spine with a mapping L as follows.

$$\begin{aligned} L(p_i) &= 1+2(i-1)(d-2) \quad \text{for odd} \quad i, \\ &= 1+4(n-i/2)(d-2) \quad \text{for even} \quad i. \end{aligned}$$

It gives:

$$\begin{aligned} L(p_i)+L(p_{i+1}) &= 2+(4n-4)(d-2), \quad \text{for odd} \quad i, \\ &= 2+4n(d-2), \quad \text{for even} \quad i. \end{aligned}$$

3. Let $T_A = \{2+(4n-4)(d-2), 2+4n(d-2)\}$ and choose $a > \max(T_A))$, $a \equiv 1 (\mathrm{mod}\ 4)$.
4. Add two isolates and label with the number from T_A.
5. Add more $d-2$ isolates $T_B = \{t_i^{(b)} | i = 1, 2, ..., (d-2)\}$ and label with $L(t_i^{(b)}) = (a + L(p_2)) + 4(i-1)$.
6. Let $T = T_A \cup T_B$, and for some k=1,2 and l=1,2,...,d-2, let $t_k^{(a)} \in T_A$ and $t_l^{(b)} \in T_B$. Label the vertices of B_i as follows.
$L(b_{ij}) = L(t_j^{(b)}) - L(p_i), j = 1, 2, ..., f_i$.

For convenience, from now on each vertex will be considered as label under L. We will use v for example instead of $L(v)$ for any $v \in V(C) \cup \overline{K_d}$.

Remark 1. *For $i = 1, 2, ..., n$ let $B_i' = \{t_j^{(b)} - p_i | j = 1, 2, ..., d-2\}$. It is obvious that $B_i \subseteq B_i', i = 1, 2, ..., n$. If $B' = \bigcup_{i=1}^{n} B_i'$ then $B \subseteq B'$.*

Before we prove that this labeling is an optimal exclusive sum labeling of C.

We need to consider the following facts:

Observation 3.

1. *$max(P) = p_2, min(P) = p_1$,*
2. *$max(T_A) = p_2 + p_3, min(T_A) = p_1 + p_2$*
3. *$max(T_B) = a + p_2 + 4(d-3), min(T_B) = a + p_2$*
4. *$max(B') = max(T_B) - min(P) = a + p_2 + 4(d-3) - p_1$, and $min(B') = min(T_B) - max(P) = a + p_2 - p_2 = a$.*

Lemma 3. *Let $p \in P, t^{(a)} \in T_A, b \in B$ and $t^{(b)} \in T_B$, where P, T_A and T_B as in Labeling 1, then $p < t^{(a)} < b < t^{(b)}$.*

Proof. There are three parts to prove,

1. We will show that $p < t^{(a)}$ for all $p \in P$ and for all $t^{(a)} \in T_A$. Let $p \in P$, $t^{(a)} \in T_A$,then $p \leq p_2$, and either $t^{(a)} = p_2 + p_1$ or $t^{(a)} = p_2 + p_3$. In both case, we have $p_2 < t^{(a)}$. This gives, $p < t^{(a)}$ for all $p \in P$ and for all $t^{(a)} \in T_A$.

2. We will show that $t^{(a)} < b$, for all $t^{(a)} \in T_A$ and for all $b \in B'$. Let $b \in B'$ and $t^{(a)} \in T_A$ then, $b \geq a > \max(T_A) \geq t^{(a)}$. Therefore, $b > t^{(a)}, \forall t^{(a)} \in T_A$, and $\forall b \in B$.

3. We will show that $t^{(b)} > b$, for all $t^{(b)} \in T_B$ and for all $b \in B$. Let $t^{(b)} \in T_B$ and $b \in B$, then $t^{(b)} \geq \min(T_B) = a+p_2$. $\max(B') = \max(T_B)-\min(P) = (a+p_2)+2(d-3)-p_1$.

$$\begin{aligned} a + p_2 - \max(B') &= (a + p_2) - (a + p_2) + 2(d-3) - p_1 \\ &= p_1 - 2(d-3) \\ &= 2d - 2(d-3) \\ &> 0 \end{aligned}$$

We have, $a + p_2 > \max(B')$, therefore $t^{(b)} > \max(B) \geq b$

From these three facts we have $p < t^{(a)} < b < t^{(b)}$, $\forall p \in P, t^{(a)} \in T_A, b \in B$ and $t^{(b)} \in T_B$. □

Lemma 4. *Let P, B_r' and B_s' as in Labeling 1. If $r \neq s$ then $B_r' \cap B_s' = \emptyset$*

Proof. Suppose on the contrary, $B_r' \cap B_s' \neq \emptyset$. Let $x \in B_r' \cap B_s'$. Then

$x = t_i^{(b)} - p_s$ for some $t_i^{(b)} \in T_B$ and

$x = t_j^{(b)} - p_r$ for some $t_j^{(b)} \in T_B$.

We get $t_i^{(b)} - t_j^{(b)} = p_s - p_r$. But,

$$\begin{aligned} t_i^{(b)} - t_j^{(b)} &\leq \max\{|t_u^{(b)} - t_v^{(b)}||t_u^{(b)}, t_v^{(b)} \in T_B\} \\ &= \max(T_B) - \min(T_B) \\ &= 4(d-3) \end{aligned}$$

On the other hand, $p_s - p_r \geq 4(d-2)$.

Hence, $t_i^{(b)} - t_j^{(b)} \leq 4(d-3) < 4(d-2) \leq p_s - p_r$. This contradicts to the fact that $t_i^{(b)} - t_j^{(b)} = p_s - p_r$.

Therefore we must have $B_r' \cap B_s' = \emptyset$. □

As a consequence,

Lemma 5. *If $r \neq s$ then $B_r \cap B_s = \emptyset$.*

Observation 4. *For all i, $i = 1, 2, ..., n$ $p_i \equiv 1 (mod\ 4)$ and $b_{ij} \equiv 1 (mod\ 4)$ for all $j = 1, 2, ..., f_i$. This gives $t \equiv 2 (mod\ 4)$ $\forall t \in T$.*

We are ready to proof the following theorem.

Theorem 5. *Let C be a caterpillar with vertex maximum degree $\Delta(C)$ then $\epsilon(C) = \Delta(C)$, that is, a caterpillar is Δ-optimum sum graph.*

Proof. We will show that Labeling 1 gives an exclusive sum labeling to the caterpillar C.

By Lemma 3, the labeling is surely a bijection from $V(C \cup T)$ onto distinct positive integer numbers in $P \cup B \cup T_A \cup T_B$. Moreover, it is clear that if $\{u, v\} \in E(C)$ then $u+v \in T$. We need to prove that, there is no extra edge needed,that is, if $\{u, v\} \notin E(C \cup T)$ then $u + v \notin V(C \cup T) = P \cup B \cup T_A \cup T_B$.

1. Let $x \in B$ and $y \in B, \{x, y\} \notin E(C)$. Obviously $x + y \notin B$ and $x + y \notin P$. We will show that $x + y \notin T$.

   - $x + y > t, \forall t \in T_A$ due to Lemma 3.
   - Notice that $a > p_2 + p_3$.

   $$\begin{aligned} x + y &\geq 2a \\ &> (a + p_2) + p_3 \\ &> (a + p_2) + 1 + 4(d - 2) \\ &> (a + p_2) + 4(d - 3) \\ &= \max(T_B). \end{aligned}$$

   Therefore $x + y \notin T_B$.

2. Let $x \in B$, $y \in P$ and $\{x, y\} \notin E(C)$. It is obvious that $x + y \notin P$ and $x + y \notin B$. We will show that $x + y \notin T$. Let $x = b_{ij}$ and $y = p_k$ where $i \neq k$.

   - Due to Lemma 3, $b_{ij} > t, \forall t \in T_A$. So $b_{ij} + p_k \notin T_A$
   - $b_{ij} = t - p_i$ for some $t \in T_B$. If $b_{ij} + p_k \in T_B$, then $b_{ij} + p_k = t'$ for some $t' \in T_B$. We get $b_{ij} = t' - p_k$ This says that $b_{ij} \in B_i' \cap B_k'$, which is contradicts to Lemma 4.

3. Let $\{p_i, p_j\} \notin E(C)$.Clearly $p_i+p_j \notin P$ and $p_i+p_j \notin B$. We will show that $p_i+p_j \notin T$.

   - If $p_i+p_j = t$ for some $t \in T_A$, then $p_i = t-p_j$. This gives $p_i = p_{j-1}$ or $p_i = p_{j+1}$. This of course contradicts to the fact that $p_i, p_j \notin E(C)$.
   - If $p_i + p_j = t$ for some $t \in T_B$, then $p_i = t - p_j \in B_j'$. This contradict to Lemma 1 that $p < b$ $\forall p, \forall b$.

By Observation 2, there is no possibility for the occurrence of any unwanted edge.

The above labeling, is an exclusive labeling of C with the addition of $\Delta(C)$ isolates. □

6. Conclusion

The major unsolved problems in this area include the analog to Gould and Rödl's result (as mentioned in the Introduction) and the classification of Δ-optimal trees. There are, however many avenues for further research in exclusive sum graphs. The conditions for ensuring a linear transformation is an exclusive sum labeling are too strong and something weaker than k_1 not being a factor of $6k_2$ should suffice.

In addition to these, there are many graphs with known sum number whose exclusive sum number has not been investigated. Tuga and Miller [15] have developed a technique for finding optimal exclusive sum labelings for certain graphs with radius 1 including fans (also known as shells), multifans (multishells) and friendship graphs.

Another interesting area might be the study of graphs that are δ-optimal and Δ-optimal. The only members of this class to date are caterpillars (and subsets including paths and stars). In [4] Fernau et al. showed that generalised friendship graphs are δ-optimal. Given the result of Tuga and Miller in [15], the generalised friendship graph might be the first non caterpillar into the ranks of the "double delta optimal" graphs.

References

[1] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski and J. Wiener, The sum number of a complete graph, *Bull. Malaysian Math. Soc.*, **12** (1989), 25-28.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd Edition, Chapman and Hall

[3] M. Ellingham, Sum graphs from trees, *Ars Combin.,* **35**(1993), 335-349.

[4] H. Fernau, J. Ryan and K. A. Sugeng, A sum labeling for the generalised friendship graph, *Discrete Math.,* **308**, (5 6) (2008), 734 740.

[5] R. Gould and V. Rödl, Bounds on the number of isolated vertices in sum graphs, *Graph Theory, Combinatorics and Applications* (edited by Y. Alevi, G. Chartrand, O.R. Oellermann and A.J. Schwenk) John Wiley and Sons (1991), 553-562.

[6] F. Harary, Sum graphs and difference graphs, *Congressus Numerantium*, **72** (1990), 101-108.

[7] F. Harary, Sum graphs over all the integers, *Discrete Math.,* **124** (1994), 99-105.

[8] N. Harstfield and W.F. Smyth, The sum number of complete bipartite graphs *Graphs and Matrices* (edited by Rolf Rees, Marcel Dekker) (1992) 205-211.

[9] N. Hartsfield and W.F. Smyth, A family of sparse graphs of large sum number *Discrete Math.*, **141** (1995), 163-171.

[10] W. He, Y. Shen, L. Wang, Y. Chang, Q. Kang and X. Yu, The integral sum number of complete bipartite graphs $K_{r,s}$, *Discrete Math.*, **239** (2001), 137-146.

[11] M. Miller, J. Ryan, Slamin and W.F. Smyth, Labelling wheels for minimum sum number, *J. Combin. Math. Combin. Comput.,* **28** (1998), 289-297.

[12] M. Miller, J. Ryan and W.F. Smyth, The sum labeling for the cocktail party graph,*Bulletin of ICA*, **22** (1998), 79-90.

[13] H. Nagamochi, M. Miller and Slamin, Bounds on the number of isolates in sum graph labeling, *Discrete Math.*, **240** (2001), 175-185.

[14] A.V. Pyatkin, New formula for the sum number for the complete bipartite graphs, *Discrete Math.,* **239** (2001), 155-160.

[15] M. Tuga and M. Miller, Δ-optimum exclusive sum labeling of certain graphs with radius one, *Proceedings of IJCCGGT 2003*, Bandung, Indonesia, (2003), 216-225.

[16] M. Tuga, M. Miller, J. Ryan and Z. Ryjáček, Sum labelings of trees, *J. Combin. Math. Combin. Comput.,* **55**, (2005), 109-121.

[17] M. Miller, D. Patel, J. Ryan, K. A. Sugeng, Slamin, M. Tuga, Exclusive sum labeling of graphs, *J. Combin. Math. Combin. Comput.,* **55**, (2005), 137 - 148.

[18] Y. Wang and B. Liu, The sum number and integral sum number of complete bipartite graph, *Discrete Math.,* **239** (2001), 69-82.