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Graph coverings and graph labellings

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Abstract

Regular coverings of a graph or a map admit descriptions in terms of certain group-valued labellings on the graph. Controlling the properties of graphs resulting from such coverings requires to develop a certain type of 'labelling calculus'. We give a brief outline of the related theory and survey the most important results.

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1. Introduction

Graph coverings, and regular coverings in particular, have played an important role in algebraic and topological graph theory. Arguably, the milestone result achieved with the help of regular coverings is the Map Color Theorem [12], providing a solution of the famous problem of Heawood to determine the chromatic number of an arbitrary surface. But constructions by means of regular coverings have also led to a number of other important results in areas such as theory of algebraic invariants (for instance, chromatic polynomials and other parameters derived from matrices associated with graphs), classification of graphs with high level of transitivity, constructions related to the degree-diameter problem and to cages, and theory of maps with emphasis on their symmetries.

It turns out that any regular covering of a 'small' graph by a 'larger' graph can be described by a certain group-valued labelling called voltage assignment. The 'large' graph can then be thought of as a 'blow-up' of the 'small' graph, the blow-up procedure being controlled by the voltage assignment. This way, graphs with appropriate group-valued labellings can be used to construct new 'large' graphs. Properties of the 'large' graphs depend only on the 'small' graph and on the group-valued labelling it carries. Sufficient conditions on the labelling to guarantee specific properties of the 'blow-up' often reduce to conditions involving equations and algebraic expressions in the group, which one can view as a certain type of 'labelling calculus'.

Our goal is to give a brief account on basic facts on representation of regular coverings in terms of voltage assignments and illustrate the way the ‘labelling calculus’ has been used to construct lifts satisfying specific requirements.

2. Group-valued labellings, lifts, and regular coverings

Coverings of graphs and maps (that is, graphs embedded on surfaces) have been an extremely fruitful area of research. In our exposition we confine ourselves to regular coverings, although a similar (but slightly less elegant) theory is available for general coverings. We do not want to go into details regarding abstract definitions of coverings and just mention that the regular coverings introduced below exactly correspond to the regular coverings of topological spaces where graphs are viewed as 1-dimensional complexes [9].

Let G be an undirected graph, possibly with loops, multiple edges, and semi-edges. For auxiliary purpose we think of edges as carrying one of he two possible directions (with exception of semi-edges). An edge with a direction is a *dart*. This way, each edge e which is not a semi-edge gives rise to two distinct darts e^+ and e^-, and we set $e^+ = e^-$ if e is a semi-edge.

Let L be a group. A labelling λ of the dart set D of G by elements of L will be called a *voltage assignment* on G if $\lambda(e^-) = (\lambda(e^+))^{-1}$ for any dart $e \in D(G)$. The *base graph* G together with the labelling λ determine the *lift* G^λ of G as follows. If V is the vertex set of the base graph, then the vertex and the dart sets V^λ and D^λ of the lift are $V^\lambda = V \times L$ and $D^\lambda = D \times L$. In the lift, a dart $(e, a) \in D^\lambda$ points from the vertex (u, a) at the vertex (v, b) if and only if $e \in D$ is a dart from u to v and $b = a\lambda(e)$. We think of the lift G^λ as being undirected, because the pair of darts (e, a) and $(e^{-1}, a\lambda(e))$ are mutually reverse and thus represent an undirected edge of G^λ.

The projection $\pi : G^\lambda \to G$ defined by $\pi(v, a) = v$ and $\pi(e, a) = e$ for $v \in V$, $e \in D$ and $a \in L$ is a *covering* of G by the lift G^λ. Moreover, for any $b \in L$ the mapping I_b given by $(v, a) \mapsto (v, ba)$ and $(e, a) \mapsto (e, ba)$ is and automorphism of the lift such that $\pi I_b = \pi$. Such automorphisms are called *deck transformations*. This way, deck transformations act regularly on the *fibers* $\pi^{-1}(v)$ and $\pi^{-1}(e)$ above vertices and darts of the base graph. A covering having a regular action of deck transformations on fibres is called *regular*. The regular coverings described here are precisely the regular coverings known from algebraic topology in the general context of coverings of topological spaces; see e.g. [3] for details.

As in the case of coverings in topology, all the information about a lift can be extracted in one way or another from lifts of walks in the base graph. Let $W = e_1e_2 \dots e_t$ be a walk of darts in G, that is, for $1 \le i \le t-1$ the terminal vertex of e_i is the initial vertex of e_{i+1}; we say that W is *closed* and *v-based* if v is the initial vertex of e_1 and also the terminal vertex of e_t. The *voltage* of W is $\lambda(W) = \lambda(e_1)\lambda(e_2)\dots\lambda(e_t)$. If L is a finite group and m is the order of the element $\lambda(W)$ in L, it is easy to see that the pre-image $\pi^{-1}(W)$ is a union of $|L|/m$ closed walks, each of length mt. Further details will be explained in the course of discussing particular cases in the next section.

3. Lifts of properties of base graphs

We will present a selection of situations in which one has control over properties of lifts in terms of algebraic conditions on voltage assignments. Throughout we assume that G is a connected graph, λ is a voltage assignment on G in a non-trivial group L, and we denote the lift by G^λ as before. We will introduce step-by-step the various bits of 'labelling calculus' that are necessary to guarantee particular properties of lifts.

3.1. Connectivity

If one just wants the lift G^λ to be connected, there is an easy and almost obvious answer, cf. [3]. Fix a vertex u of G and let L_u be the *local group* consisting of all elements of the form $\lambda(W)$ where W ranges over all closed u-based walks of G. In a connected graph, all local groups are mutually conjugate in L. (Local groups of *connected subgraphs* of G are defined analogously.) Then:

Proposition 1. *The lift G^λ is connected if and only if $L_u = L$ for some (and hence every) vertex u of G.*

The situation gets more complicated if one wants the lift to be 2-connected [2]:

Proposition 2. *The lift G^λ is 2-connected if and only if it is connected and the base graph G has no block H with trivial local group such that H contains a single cut-vertex of G.*

One sees that for (2-)connectivity the 'labelling calculus' reduces to investigation of local groups in the block-structure of the base graph. In particular, if the base graph is 2-connected and the local group is equal to the entire voltage group, then the lift is automatically 2-connected. (Some care is needed when considering 2-connectivity of graphs with loops and semi-edges but details can be worked out easily.) In [2] one can find sufficient conditions for k-connectivity of lifts – however, the conditions involving local groups as well as voltages on walks in the base graph are quite complex to be reproduced here.

3.2. Diameter and girth

The following simple but powerful observation has now become folklore:

Proposition 3. *The diameter of G^λ does not exceed k if and only if for each ordered pair of vertices u, v (allowing $u = v$) in G and for each $b \in L$ there exists a walk W from u to v of length at most k with voltage $\lambda(W) = b$.*

The corresponding dual observation regarding girth uses voltages on cycles as the 'calculus' tool. If C is a cycle of G, then fixing a vertex u on C and an orientation of C one obtains a walk W and can speak about its voltage $\lambda(W)$. It is obvious that fixing a different vertex of C gives just an L-conjugate of $\lambda(W)$ and reversing orientation results in inverting the voltage. We can thus speak about *voltage of a cycle* modulo conjugation and taking inverses. Then we have:

Proposition 4. *The lift G^λ has girth at least k if and only if every cycle of length $\ell < k$ in G has voltage of order at least k/ℓ in the group L.*

Both conditions have been used in the literature. In the degree-diameter problem, Proposition 3 has been applied in two ways. At a theoretical level it was used in the construction of current largest vertex-transitive graphs of a given degree and diameter 2 in [10] (where the condition on voltages on walks of length at most two translated to solvability of a system of linear equations in the product of two additive groups of the same finite field) and later for arbitrary diameter in [7]. At a computational level the condition was used in the process of computational generation of current largest graphs of degree at most 15 and diameter at most 10. Proposition 4 was used in a recent unified treatment [6] of the known constructions of near-cages of girth 6.

3.3. Graph automorphisms

This subsection explains how automorphisms of base graphs can induce automorphisms of lifts. We say that an automorphism $\tilde{A}$ of G^λ) is a *lift* of an automorphism A of the base graph G if $\pi\tilde{A} = A\pi$; alternatively we simply say that the automorphism A *lifts.* A well known consequence of path lifting in coverings of topological spaces translates into the language of graph theory as follows:

Proposition 5. *An automorphism A of G lifts to an automoorphism of G^λ if and only if for each closed walk W based at a fixed vertex of G we have $\lambda(W) = 1 \Leftrightarrow \lambda(AW) = 1$.*

As an immediate consequence we obtain:

Proposition 6. *Let J be a vertex-transitive group of automorphisms of G. If each automorphism in J lifts to an automorphism of G^λ, then the lift G^λ is vertex-transitive.*

Both these principles have been used extensively in the literature. In the degree-diameter problem, Proposition 6 was used in a number of papers presenting constructions of large graphs of given degree and diameter. A prominent example is [10] and others can be extracted from the survey [11]. The same condition is used to guarantee vertex-transitivity of cages in [6]. Proposition 5 was also used to prove non-existence of certain types of automorphisms in lifts [13]. Lifts of graph automorphisms by voltage assignments also play an important role in classification of vertex-transitive graphs satisfying further transitivity conditions (such as s-arc-transitivity for $s \geq 2$); see [5] for examples.

3.4. Map automorphisms

The theory of lifting maps, that is, 2-cell embedded graphs on surfaces, is in its spirit very similar to the theory of lifts of graphs and we just refer to the monograph [3] for details. An automorphism of a map is simply an automorphism of the underlying graph which, in addition, preserved face boundary walks. Surprisingly, Propositions 5 and 6 apply to lifting map automorphisms without alteration. The technique of lifting maps and their automorphisms by voltage assignments was a substantial tool (although in disguise) in the proof of the Map Color Theorem [12]. As it is well known, the Theorem is equivalent to determining genera of complete graphs, which amounts to construct embeddings of complete graphs with as large number of faces as possible. To give an example, in the special cases when $n \equiv 0, 3, 4$ and 7 mod 12, Euler's formula suggests that a triangular embedding of K_n on an orientable surface could exists. To actually construct such a triangulation one begins with a suitable small base graph and endows it with a cleverly chosen labelling in groups that are cyclic or 'close to' cyclic; see [3] for a review of the original approach of [12]. The 'clever choice' refers here to the requirement that, in the lift, all faces must be triangular and the lifted graph must be complete, which is far from easy to satisfy. This is where one has to do 'labelling calculus' to figure out that the two conditions can actually be satisfied. From the point of view of labellings it is worth noting that [12] actually used a dual version of the above by considering a different (but equivalent) type of labelling, assigning *currents* to darts and requiring Kirchhoff's Current Law to be satisfied at vertices.

Lifts of map automorphisms have also been used in constructions of maps with the largest possible 'level of symmetry', called *regular maps*; we refer to [14] for a survey on regular maps and to [4] for a first example of usage of Proposition 5 to explicitly construct regular maps by means of voltage assignments. The same principle can actually be used also for forbidding automorphisms in lifts, as demonstrated by devising an appropriate 'labelling calculus' in [1].

4. Conclusion: Labelling calculus and graph spectra

Most aspects of calculations with voltage labellings which have been mentioned in the above subsections have been explicitly worked out in [15], laying thus foundations of what was called 'walk calculus' which is a part of our 'labelling calculus'. Substantial progress was then achieved in [16] for voltage labellings in cyclic groups. (This looks rather restrictive at a first glance – however, a number of important results have been proved by assignments in cyclic groups.) With an automorphism A of our base graph G one associates a certain square matrix M_A (of dimension equal to the dimension of the cycle space of G) that captures the action of A on G in a natural way. Surprisingly [16], voltage assignments satisfying the lifting condition from Proposition 5 turn out to be *eigenvectors* of M_A. This opens ways to use spectral methods in the theory of voltage assignments. The method has been extended to assignments in elementary abelian groups [8] but further generalizations are completely open.

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