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Prime Labeling of Grids

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Abstract

A labeling f on a graph G on n vertices is called a prime labeling if f is a bijection from the vertex set $V(G)$ to $\{1, 2, \cdots, n\}$ such that $f(x)$ and $f(y)$ are coprime if x and y are adjacent. It was shown by Sundaram et al. [1] that the planar grid $P_m \times P_n$ has a prime labeling if $m \leq n$ and n is a prime. In this paper it is proved that the following grids have a prime labeling:

(i) $P_{n+1} \times P_{n+1}$ has a prime labeling where n is an odd prime, $n = 5$ or $n \equiv 3$ or $9 \pmod{10}$ and $(n+1)^2 + 1$ is also a prime.

(ii) $P_n \times P_{n+2}$ has a prime labeling where n is an odd prime and $n \not\equiv 2 \pmod{7}$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. Prime labelings were defined by Sundaram et al. [1] who also gave constructions of prime labelings for certain classes of grids.

Definition 1.1. *A simple graph $G = (V, E)$ is said to have a prime labeling if its vertices are labeled with distinct integers from $\{1, 2, \cdots, |V|\}$ such that for each edge xy, the labels assigned to x and y are relatively prime. A graph with a prime labeling defined on it is called a prime graph or simply prime.*

Definition 1.2. *For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ their cartesian product $G_1 \times G_2$ is defined as the graph whose vertex set is $V_1 \times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) in $G_1 \times G_2$ are adjacent if $u_1 = u_2$ and v_1 is adjacent to v_2 or u_1 is adjacent to u_2 and $v_1 = v_2$. A path of order n is denoted by P_n. The planar grid $P_m \times P_n$ is the cartesian product of paths P_m and P_n.*

Sundaram, Ponraj and Somasundaram [1] have given two constructions to label grids. The first construction gives a prime labeling for the planar grid $P_m \times P_n$ where n is an odd prime and $m < n$, whereas in the second construction they have proved that the grid $P_n \times P_n$ has a prime labeling where n is an odd prime. They have also conjectured that $P_m \times P_n$ is prime for m, n natural numbers and $m \leq n$. The following is the labeling due to Sundaram, Ponraj and Somasundaram [1] given to the planar grid $P_n \times P_n$ where n is an odd prime.

1.1. Labeling of $P_n \times P_n$

Let $V(P_n \times P_n) = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$ and $E(P_n \times P_n) = \{u_{ij}u_{i,j+1} : 1 \leq i \leq n, 1 \leq j \leq n-1\} \cup \{u_{ij}u_{i+1,j} : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. f is defined by :

$$f(u_{ij}) = (j-1)n + i \text{ for } (1 \leq i \leq n-1, 2 \leq j \leq n-1).$$

For $1 \leq i \leq n-1$,

$$f(u_{i1}) = \begin{cases} ni & \text{if } i = 1, 3, 5, \ldots, n-2 \\ i & \text{otherwise.} \end{cases}$$

$f(u_{nj}) = j$ or nj as j is odd or even $1 \leq j \leq n-1$.

$f(u_{in}) = (n-1)n + i, 1 \leq i \leq n, i \neq n-2, i \neq n.$

$f(u_{n1}) = 1, f(u_{nn}) = n^2 - 2,\ f(u_{n-2,n}) = n^2.$

The object of this paper is to establish prime labelings for the following grids by using the labeling described above as a base and building on it.

1. Grid $P_{n+1} \times P_{n+1}$, n an odd prime, $n = 3$ or $n \geq 13$ and $(n+1)^2 + 1$ is a prime. When $n \geq 13$, n odd and $(n+1)^2 + 1$ is a prime it can be checked that n cannot be congruent to 1, 5 or 7 (mod 10). We also observe that $n = 5$ is a special case for which the grid $P_{n+1} \times P_{n+1}$ is prime.

2. Grid $P_n \times P_{n+2}$, n an odd prime and $n \not\equiv 2(mod\ 7)$.

2. Main Results

Theorem 2.1. *Let n be an odd prime, $n = 5$ or $n \equiv 3$ or 9 (mod 10) and $(n+1)^2 + 1$ also a prime, then the grid $P_{n+1} \times P_{n+1}$ has a prime labeling.*

Proof. Let n be an odd prime, $n = 5$ or $n \equiv 3$ or 9 (mod 10) and $(n+1)^2 + 1$ is also a prime. For odd prime $n = 5$ the prime labeling for $P_6 \times P_6$ is given in Table 1:

36	5	6	11	16	21
35	2	7	12	17	22
34	3	8	13	18	25
33	4	9	14	19	24
32	1	10	3	20	23
31	26	27	28	29	30

Table 1: Prime labeling for the grid $P_6 \times P_6$

Suppose $n \neq 5$. First give a prime labeling to the grid $P_n \times P_n$ using the method described above. Construct the grid $P_{n+1} \times P_{n+1}$ by adding a new column 0 at the left of the grid $P_n \times P_n$ and adding a new row $n+1$ at the bottom of the existing grid $P_n \times P_n$. Our rows are thus numbered $1, 2, \cdots, n+1$ while columns are numbered $0, 1, \cdots, n$. Denote the vertices of the newly added column 0 by u_{i0} where $i = 1, 2, \cdots, n+1$ and the vertices of the row $n+1$ by $u_{n+1,j}$ where $j = 1, 2, \cdots, n$. From the labeling described above, the vertices of the grid $P_n \times P_n$ are given the labels $1, 2, \cdots, n^2$. The remaining labels are $n^2+1, n^2+2, \cdots, (n+1)^2$. These are assigned to the newly added column 0 and row $n+1$ as follows:

$f(u_{i0}) = (n+1)^2 + 1 - i$ where $i = 1, 2, \cdots, n+1$.

$f(u_{n+1,j}) = n^2 + j$ where $j = 1, 2, \cdots, n$.

We now show that the labeling given above is prime for the grid $P_{n+1} \times P_{n+1}$ for all primes n satisfying the following conditions:

i) $n \equiv 3$ or $9 \pmod{10}$

ii) $(n+1)^2 + 1$ is a prime

The labels $f(u_{i,0})$, $i = 1, 2, \cdots, n+1$ are from $(n+1)^2, (n+1)^2 - 1, \cdots, (n+1)^2 - n$. Therefore $gcd(f(u_{i,0}), f(u_{i+1,0})) = 1$ for all $i = 1, 2, \cdots, n$. Also since the labels $f(u_{n+1,j})$, $j = 1, 2, \cdots, n$ are from $n^2+1, n^2+2, \cdots, n^2+n$, $gcd(f(u_{n+1,j}), f(u_{n+1,j+1})) = 1$ for all $j = 1, 2, \cdots, n-1$. We now show that:

Case a. The labels $f(u_{i,0})$ and $f(u_{i,1})$ are co-prime for $i = 1, 2, 3 \cdots, n+1$.

Case b. The labels $f(u_{n+1,j})$ and $f(u_{n,j})$ are co-prime for $j = 0, 1, 2, \cdots, n$.

Case a.

$f(u_{i,1}) = ni$ where $i = 1, 3, \cdots, n-2$

$f(u_{i,1}) = i$ where $i = 2, 4, \cdots, n-1$

$f(u_{n,1}) = 1$ where $i = 1, 3, \cdots, n-2$,

$$\begin{aligned}
gcd(f(u_{i,0}), f((u_{i,1})) &= gcd((n+1)^2 + 1 - i, ni) \\
&= gcd(n^2 + 2n + 1 + 1 - i, ni) \\
&= gcd(n^2 + 2n + 2 - i, ni) \\
&= gcd(n^2 + 2n + 2 - i, i) \textit{ as } n \textit{ does not divide } ((n+1)^2 + 1 - j) \\
&= gcd(n^2 + 2n + 2, i) \\
&= gcd((n+1)^2 + 1, i) \\
&= 1 \textit{ as } (n+1)^2 + 1 \textit{ is prime.}
\end{aligned}$$

When $i = 2, 4, \cdots, n$,

$$\begin{aligned}
gcd(f(u_{i,0}), f(u_{i,1})) &= gcd((n+1)^2 + 1 - i, i), i = 2, 4, \cdots, n-1 \\
&= gcd((n+1)^2 + 1 - i + i, i) \\
&= gcd((n+1)^2 + 1, i) \\
&= 1 \textit{ as } (n+1)^2 + 1 \textit{ is prime.}
\end{aligned}$$

Hence $gcd(f(u_{n,0}), f(u_{n,1})) = 1$. Also

$f(u_{n,0}) = (n+1)^2 - (n-1)$

$f(u_{n,1}) = 1$.

Thus $gcd(f(u_{i,0}), f(u_{i,1})) = 1$ for $i = 1, 2, \cdots, n$.

Case b.

$$\begin{aligned}
gcd(f(u_{n+1,0}), f(u_{n+1,1}) &= gcd((n+1)^2 - n, n^2 + 1) \\
&= gcd(n^2 + 2n + 1 - n, n^2 + 1) \\
&= gcd(n^2 + 2n + 1 - n - n^2 - 1, n^2 + 1) \\
&= gcd(n, n^2 + 1) \\
&= gcd(n^2 + 1 - n^2, n) \\
&= gcd(1, n) \\
&= 1.
\end{aligned}$$

For $j = 1, 3, \ldots, n-2$,

$$\begin{aligned} gcd(f(u_{n+1,j}), f(u_{n,j}) &= gcd(n^2 + j, j) \\ &= gcd(n^2 + j - j, j) \\ &= gcd(n^2, j) \\ &= 1. \end{aligned}$$

For $j = 2, 4, \ldots, n-1$,

$$\begin{aligned} gcd(f(u_{n+1,j}), f(u_{n,j}) &= gcd(n^2 + j, nj) \\ &= gcd(n^2 + j, j) \textit{ as } n \textit{ does not divide } (n^2 + j) \\ &= gcd(n^2 + j - j, j) \\ &= gcd(n^2, j) \\ &= 1 \textit{ as } n \textit{ is an odd prime.} \end{aligned}$$

Also

$$\begin{aligned} gcd(f(u_{n+1,n}), f(u_{n,n})) &= gcd(n^2 + n, n^2 - 2) \\ &= gcd(n^2 + n - n^2 + 2, n^2 - 2) \\ &= gcd(n + 2, n^2 - 2) \\ &= gcd(n^2 - 2 - n^2 - 2n, n + 2) \\ &= gcd(2n + 2, n + 2) \\ &= gcd(2n + 2 - 2n - 4, n + 2) \\ &= gcd(2, n + 2) \\ &= gcd(n, 2) \\ &= 1 \textit{ as } n \textit{ is an odd prime,} \end{aligned}$$

Therefore for n an odd prime, $n = 5$ or $n \equiv 3$ or $9 \pmod{10}$ and $(n+1)^2 + 1$ also a prime the grid $P_{n+1} \times P_{n+1}$ has a prime labeling. □

Theorem 2.2. *Let n be an odd prime. Then the grid $P_n \times P_{n+2}$ has a prime labeling for $n \not\equiv 2(mod\ 7)$.*

Proof. Let n be an odd prime such that $n \not\equiv 2(mod\ 7)$. The grid $P_n \times P_n$ is first constructed and labels are assigned using the method outlined earlier.

Construct the grid $P_n \times P_{n+2}$ by adding rows R_{n+1} and R_{n+2} after the last row of the grid $P_n \times P_n$. Denote the vertices corresponding to the rows R_{n+1} and R_{n+2} by $u_{n+1,j}$

and $u_{(n+2),j}$ respectively, where $j = 1, 2, \cdots, n$. Since the labels $1, 2, \cdots, n^2$ are already assigned to the vertices of the grid $P_n \times P_n$ the remaining labels $n^2+1, n^2+2, \cdots, (n+1)^2-1$ are to be assigned to the vertices of the rows R_{n+1} and R_{n+2} which is done as follows:

$f(u_{n+1,j}) = n^2 + j$ where $j = 1, 2, \cdots, n$.

$f(u_{n+2,j}) = n^2 + n + j$ where $j = 2, \cdots, n-1$.

$f(u_{n+2,1}) = n$

$f(u_{n+2,n}) = n^2 + n + 1$

$f(u_{11}) = n^2 + 2n$

Note: The labeling given to u_{11} is n in the labeling for the grid $P_n \times P_n$. We assign the label n to the vertex $u_{(n+2),1}$ and the label $n^2 + 2n$ is assigned to u_{11}.

We now show that the labeling given above is prime for the grid $P_n \times P_{n+2}$ for all primes n, $n \not\equiv 2(mod\ 7)$. The labels $f(u_{n+1,j})$, $j = 1, 2, \cdots, n$ are from $n^2+1, n^2+2, \cdots, n^2+n$. Therefore $gcd(f(u_{n+1,j}), f(u_{n+1,j+1})) = 1$ for all $j = 1, 2, \cdots, n$.

Clearly $gcd(f(u_{n+2,j}), f(u_{n+2,j+1})) = 1$ for all $j = 2, \cdots, n-1$ since the labels $f(u_{n+2,j})$, for $j = 2, \cdots, n-1$ are from $n^2+n+1, n^2+n+2, \cdots, (n+1)^2-1$. We now show that the labeling is prime for the remaining cases:

$$\begin{aligned} gcd(f(u_{1,1}), f(u_{1,2})) &= gcd(n^2+2n, n+1) \\ &= gcd(n^2+2n-n^2-n, n+1) \\ &= gcd(n, n+1) \\ &= 1. \end{aligned}$$

$$\begin{aligned} gcd(f(u_{1,1}), f(u_{2,1})) &= gcd(n^2+2n, 2) \\ &= gcd(n^2+2n-2n, 2) \\ &= gcd(n^2, 2) \\ &= 1. \end{aligned}$$

Now, for $j = 2, 3, \ldots, n-1$.

$$\begin{aligned} gcd(f(u_{n+1,j}), f(u_{n+2,j})) &= gcd(n^2+j, n^2+n+j) \\ &= gcd(n^2+n+j-n^2-j, n^2+j) \\ &= gcd(n, n^2+j) \\ &= gcd(n^2+j-n^2, n) \\ &= gcd(j, n) \\ &= 1. \end{aligned}$$

Also

$$\begin{aligned} gcd(f(u_{n+1,1}), f(u_{n+2,1})) &= gcd(n^2+1, n) \\ &= gcd(n^2+1-n^2, n) \\ &= gcd(1, n) \\ &= 1. \end{aligned}$$

$$\begin{aligned} gcd(f(u_{n+1,1}), f(u_{n+1,2})) &= gcd(n^2+1, n^2+2) \\ &= 1.gcd(f(u_{n+2,1}), f(u_{n+2,2}) \qquad = gcd(n, n^2+n+2) \\ &= gcd(n^2+n+2-n, n) \\ &= gcd(n^2+2, n) \\ &= gcd(n^2+2-n^2, n) \\ &= gcd(2, n) \\ &= 1. \end{aligned}$$

$$\begin{aligned} gcd(f(u_{n+1,n}), f(u_{n+2,n}) &= gcd(n^2+n, n^2+n+1) \\ &= 1. \\ gcd(f(u_{n+2,n}), f(u_{n+2,n-1}) &= gcd(n^2+n+1, n^2+2n-1) \\ &= gcd(n^2+2n-1-n^2-n-1, n^2+n+1) \\ &= gcd(n-2, n^2+n+1) \\ &= gcd(n^2+n+1-n^2+2n, n-2) \\ &= gcd(3n+1, n-2) \\ &= gcd(3n+1-3n+6, n-2) \\ &= gcd(7, n-2) \\ &= 1 \ as \ n \ 2(mod \ 7) \end{aligned}$$

Thus for n an odd prime and $n \not\equiv 2(mod\ 7)$ the grid $P_n \times P_{n+2}$ has a prime labeling asdescribed above. □

Example 2.3.

Table 2 gives the prime labeling for n = 13.
Table 3 gives the prime labeling for n = 7.

196	13	14	27	40	53	66	79	92	105	118	131	144	157
195	2	15	28	41	54	67	80	93	106	119	132	145	158
194	39	16	29	42	55	68	81	94	107	120	133	146	159
193	4	17	30	43	56	69	82	95	108	121	134	147	160
192	65	18	31	44	57	70	83	96	109	122	135	148	161
191	6	19	32	45	58	71	84	97	110	123	136	149	162
190	91	20	33	46	59	72	85	98	111	124	137	150	163
189	8	21	34	47	60	73	86	99	112	125	138	151	164
188	117	22	35	48	61	74	87	100	113	126	139	152	165
187	10	23	36	49	62	75	88	101	114	127	140	153	166
186	143	24	37	50	63	76	89	102	115	128	141	154	169
185	12	25	38	51	64	77	90	103	116	129	142	155	168
184	1	26	3	52	5	78	7	104	9	130	11	156	167
183	170	171	172	173	174	175	176	177	178	179	180	181	182

Table 2: Prime labeling for the grid $P_{14} \times P_{14}$

63	8	15	22	29	36	43
2	9	16	23	30	37	44
21	10	17	24	31	38	45
4	11	18	25	32	39	46
35	12	19	26	33	40	49
6	13	20	27	34	41	48
1	14	3	28	5	42	47
50	51	52	53	54	55	56
7	58	59	60	61	62	57

Table 3: Prime labeling for the grid $P_7 \times P_9$

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References

[1] M. Sundaram, R. Ponraj and S. Somasundaram On a Prime Labeling Conjecture, *Ars Combin.*, **LXXIX** 2006, 205-209.