*AKCE J. Graphs. Combin., 6, No. 1 (2009), pp. 155-160*

# Bimagic Labelings

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**Abstract**

An *edge-magic* (total) *labeling* $\lambda$ of a graph $G$ is a one-to-one mapping from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, \dots n\}$ for which there exists a constant $k$ such that $\lambda(x) + \lambda(xy) + \lambda(y) = k$ whenever $x$ and $y$ are adjacent vertices. In a *bimagic labeling*, there are two constants $k_1$ and $k_2$ such that all sums of the specified type equal one or other of those two sums. We discuss edge-bimagic labelings of graphs for which no edge-magic labeling exists. In particular, two cases are of special interest: when the number of edges with one sum is (approximately) the same as the number with the other; or when all edges but one have the common sum.

**Keywords:** graph labeling, magic labeling.

**2000 Mathematics Subject Classification:** 05C78

## 1. Definitions

A (total) *labeling* $\lambda$ of a graph $G$ is a one-to-one mapping from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, \dots n\}$. Various arithmetical properties of graph and digraph labelings have been studied (a good survey is [3]). In particular, *magic* denotes the requirement that all sums of labels of a certain kind have the same value. For example, a total labeling $\lambda$ is *edge-magic* if there exists a constant $k$ (the "magic constant") such that $\lambda(x)+\lambda(xy)+\lambda(y) = k$ whenever $x$ and $y$ are adjacent vertices. Babujee [1, 2] introduces the idea of a *bimagic labeling*, in which there are two constants $k_1$ and $k_2$ such that all sums of the specified type equal one or other of those two sums. For example, we would define an *edge-bimagic total labeling* of $G$ to be a total labeling $\lambda$ such that, when $x$ and $y$ are adjacent vertices of $G$, $\lambda(x) + \lambda(xy) + \lambda(y)$ equals either $k_1$ or $k_2$. (Babujee uses a more complicated terminology, but we have standardized to be consistent with [6].)As one would expect, it is very easy to find bimagic labelings. However, it is of interest to consider bimagic labelings of graphs for which no magic labeling exists. In particular, two cases are of special interest. If the number of edges with one sum is the same as the number with the other, or (when $|E(G)|$ is odd) differs by 1, we refer to the labeling as *equitable*. When all edges but one have the common sum, the labeling is *almost magic*. Our aim here is to give some elementary results, in the hope that further work will be forthcoming, particularly on the families we discuss.

## 2. Classes of Graphs

### 2.1. Wheels

The wheel $W_n$ consists of an $n$-cycle $C_n$ together with a central vertex $c$ joined to the other $n$ vertices. It is known (see [5]) that $W_n$ has no edge-magic total labeling when $n \equiv 3 \pmod 4$. But an edge-bimagic total labeling is easy to construct. Say $n = 4t+3$. Call the center vertex of the wheel $c$ and the other vertices (in clockwise order) $x_1, x_2, \ldots, x_n$. We define a labeling $\lambda$ as follows.Rim vertex labels:

$$\lambda(x_n) = 1, \lambda(x_1) = 2t+1, \ldots, \lambda(x_j) = 1 + j(2t+1), \ldots, \lambda(x_{n-1}) = 2t+3,$$

with the labels reduced modulo $n$ to lie in the range 1, 2, ..., $n$. Rim edge labels:

$$\lambda(x_j, x_{j+1}) = 8t + 6 + j.$$

(In particular, $\lambda(x_n, x_1) = 12t + 9 = 3n$.)Spoke labels:

$$\lambda(x_j, c) = 8t + 7 - \lambda(x_j).$$

The rim edges have sum $\lambda(x_j) + \lambda(x_j, x_{j+1}) + \lambda(x_{j+1}) = 14t + 12$ and the spokes have sum $\lambda(x_j) + \lambda(x_j, c) + \lambda(c) = 8t + 7 + \lambda(c)$. If we put $\lambda(c) = 6t + 5$ the resulting mapping is edge-magic, but it is not a labeling as label $12t + 10$ has not been used and there is a repeated value:

$$\lambda(c) = 6t + 5 = \lambda(c, x_1).$$

If we set $\lambda(c, x_1) = 12t + 10$ the resulting labeling is almost magic (the magic constant is $14t + 12$, but one edge has sum $20t + 17$). If we set $\lambda(c) = 12t + 10$ the resulting labeling is equitable (the rim edges have sum $14t + 12$, while the spokes have sum $20t + 17$). So:

**Theorem 2.1.** *When $n \equiv 3(mod\ 4)$, the wheel $W_n$ has both an equitable bimagic labeling and an almost magic labeling.*

Figure 1 shows the two labelings of $W_7$.

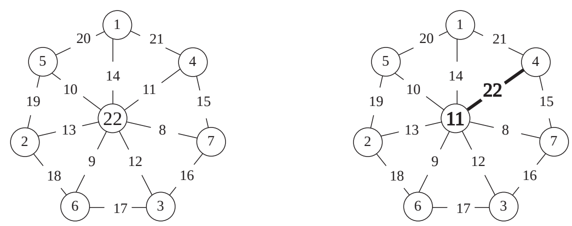


Figure 1: Equitable and almost magic labelings of $W_7$

### 2.2. One-Factors

The one-factor $nK_2$, consisting of $n$ independent edges, has an edge-magic total labeling if and only if $n$ is odd. For $n$ even, a bimagic total labeling is easily constructed: take an edge-magic total labeling of $(n-1)K_2$, and add a copy of $K_2$ with its three elements labeled $\{3n-2, 3n-1, 3n\}$.

### 2.3. Complete Graphs

The only complete graphs with edge-magic total labelings are $K_2$, $K_3$, $K_5$ and $K_6$. We know that $K_4$ has an edge-bimagic total labeling because $K_4 = W_3$. We now exhibit edge-bimagic total labelings of $K_7$ and $K_8$. The left-hand array below represents a suitable labeling of $K_7$ with magic constants 22 and 31; the diagonal entries are the vertex labels and the $(i, j)$ entry is the label for the edge joining vertices $i$ and $j$. Similarly, the right-hand array represents a labeling of $K_8$ with magic constants 27 and 41.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 28 | 27 | 26 | 25 | 23 | 21 |
| | 2 | 17 | 16 | 24 | 22 | 20 |
| | | 3 | 15 | 14 | 12 | 19 |
| | | | 4 | 13 | 11 | 18 |
| | | | | 5 | 10 | 8 |
| | | | | | 7 | 6 |
| | | | | | | 9 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 24 | 23 | 36 | 35 | 33 | 31 | 28 |
| | 2 | 22 | 21 | 34 | 32 | 30 | 27 |
| | | 3 | 20 | 19 | 17 | 29 | 26 |
| | | | 4 | 18 | 16 | 14 | 25 |
| | | | | 5 | 15 | 13 | 10 |
| | | | | | 7 | 11 | 8 |
| | | | | | | 9 | 6 |
| | | | | | | | 12 |

We conjecture that there exists a constant $N$ such that $K_n$ has an EBTL if and only if $n \leq N$. If this is true, we now know $N \geq 8$. So far the search for an EBTL of $K_9$ has been unsuccessful.

### 2.4. Duplicated Graphs

Suppose $G$ is an edge-magic graph with EMTL $\lambda$. Then it is easy to construct an EBTL of $2G$. One copy of $G$ is labeled with $\lambda$. In the other copy, the label is increased by $|V(G)|+|E(G)|$. If $k$ was the original magic constant, then the new labeling has constants $k$ and $k+|V(G)|+|E(G)|$. As an example, the graph $2K_3$, the union of two disjoint triangles, is not edge-magic. The problem is so small that an exhaustive computation is very short (and hand computation is feasible). Moreover a theoretical proof is now available [4]. But $K_3$ is edge-magic, so $2K_3$ is edge-bimagic. Similarly $2K_5$ and $2K_6$ are edge-bimagic. However, all these labelings are equitable; the more interesting question of almost magic labelings is much harder.

## 3. Strong Labelings

An edge-magic total labeling is called *strong* or *super-magic* if the smallest labels are affixed to the vertices. In other words, if there are $v$ vertices, the vertex labels are $\{1, 2, \ldots, v\}$. Even cycles are known to have no strong edge-magic total labeling. But a strong edge-bimagic total labeling is available for every cycle. If there are $2t$ vertices, then the vertices are labeled $1$, $t+1$, $2$, $t+2$, $\ldots$, $t$, $2t$; the edge from 1 to $t+1$ is labeled $4t$, and subsequent edges are $4t-1$, $4t-2$, $\ldots$, $2t+1$.

## 4. Computations for Duplicated Complete Graphs

Our first computational results on $2K_n$ concerned the question: is $2K_n$ edge-magic? In general, $2K_n$ has $2n$ vertices and $n(n-1)$ edges, a total of $n(n+1)$ elements. Suppose it has an edge-magic total labeling with constant $k$, and suppose the sum of the vertex labels is $V$ and the sum of the edge-labels is $E$. $V+E$ equals the sum of all the labels:

$$V + E = 1 + 2 + \ldots + n(n+1) = n(n+1)(n^2+n+1)/2,$$

which is odd when $n \equiv 2(mod\ 4)$. Each vertex lies in $n-1$ edges, so if we sum $\lambda(x) + \lambda(xy) + \lambda(y)$ over all edges $xy$, the result will be $(n-1)V + E = (n-2)V + (V+E)$. So we have

$$(n-2)V + (V+E) = k|E(G)| = kn(n-1).$$

When $n \equiv 2(mod\ 4)$, the left hand term is odd, but the right-hand is even. We have investigated $2K_n$ for the small values $n = 4$ and 5, and found that neither $2K_4$ nor $2K_5$ is edge magic. The computation for $2K_7$ is considerably larger.When looking for almost magic labelings of $2K_n$, our technique was very simple. We first chose a value $k$ and then attempted to apply the labels 1, 2, $\ldots, n(n+1)$ to the vertices and edges so that every edge-sum equalled $k$, using a backtrack search. However, we did not backtrack when this proved impossible. Rather, in that case, we recorded the fact that a problem had occurred, and went on to label the next element. We only backtracked when a second problem occurred. Surprisingly, this rather primitive approach seemed, in the case of $2K4$, to be just as efficient as any sophisticated algorithm we concocted.We found that there are 140 almost magic labelings of $2K_3$, 64 of $2K_4$ and 4 of $2K_5$. There are no examples for $2K_6$.Examples for $2K_3$ (with $k = 18$, one edge sum is 10), $2K_4$ (with $k = 24$, one edge sum is 30) and $2K_5$ (with $k = 48$, one edge sum is 33) are shown in Figures 3, 4 and 5. In order to avoid cluttering the diagram, edge labels are omitted, except for the one edge label that gives rise to an anomalous sum; all other edge labels can be computed by subtracting the labels on the endpoints from $k$.

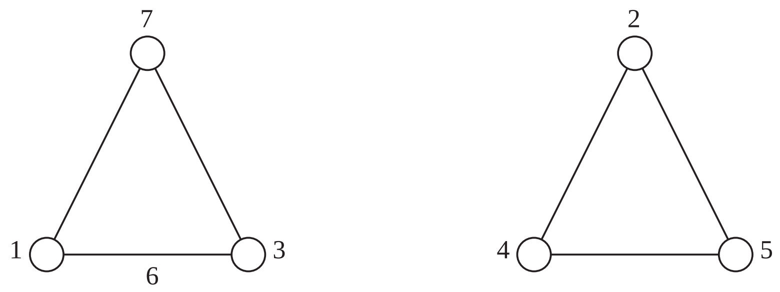


Figure 2: $2K_3$ with main edge sum $k = 18$, one edge sum 10

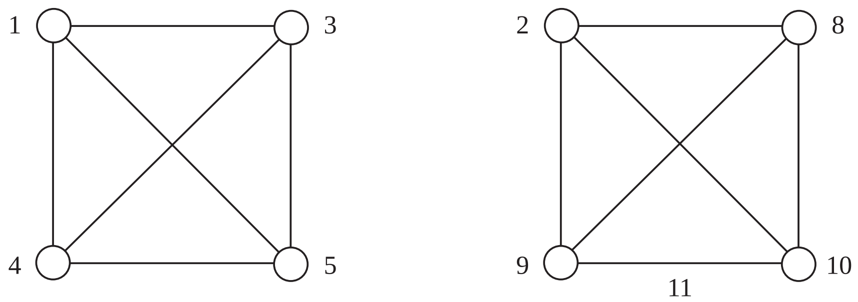


Figure 3: $2K_4$ with main edge sum $k = 24$, one edge sum 30

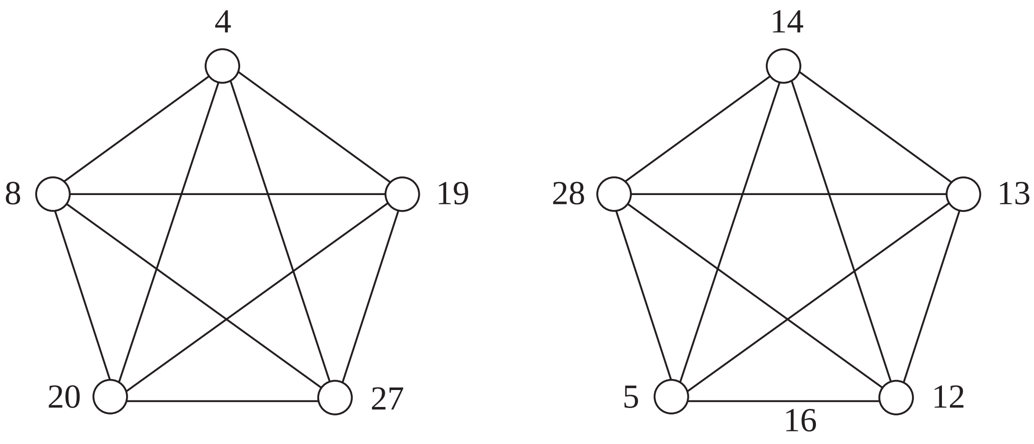


Figure 4: $2K_5$ with main edge sum $k = 48$, one edge sum 33

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