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Cordial labelings of a Class of Planar Graphs *

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Abstract

Let $G = (V, E)$ be a graph and let $f : V \to \{0, 1\}$ be a mapping from the set of vertices to $\{0, 1\}$ and for each edge $(u, v) \in E$ assign the label $|f(u) - f(v)|$. If the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labled with 0 and the number of edges labeled with 1 differ by at most 1, then f is called a cordial labeling. In this paper, we present two families of planar graphs, Pl_n and $Pl_{m,n}$ defined shortly, that admit a cordial labeling. We also show that the class $Pl_{m,n}$ admits a total product cordial labeling under certain conditions.

Keywords: cordial labeling, Pl_n, $Pl_{m,n}$, total product cordial labeling.

2000 Mathematics Subject Classification: 057C8

1. Introduction

All graphs considered here are finite, simple, and undirected. The origin of graph labelings can be attributed to Rosa [11, 8] or Graham and Sloane [9, 8]. Several types of graph labelings have been investigated both from a purely combinatorial perspective as well as from an application point of view. A class of labelings called magic labelings motivated by magic squares was introduced by Sedlacek [14, 10].

In [5], Cahit defines cordial labeling, a variation of both graceful and harmonious labelings, as follows.

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Definition 1.1. *Let G be a graph and f be a function from the set of vertices V to $\{0,1\}$ and for each edge uv assign the label $\mid f(u) - f(v) \mid$. The graph G is said to be a cordial graph if (i) the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and (ii) the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.*

Cahit [6] showed that all trees, all complete bipartite graphs, and all complete graphs of order at most 3 are cordial. It is also shown in [6] that a k-angular cactus graph, a graph whose blocks are cycles with k vertices, with kt vertices is cordial if and only if $kt \not\equiv 2 \bmod 4$. Du [7] provides a bound on the maximum number of edges in an n–vertex cordial graph apart from giving necessary conditions for a regular graph to be cordial. Further progress in this direction is presented in works such as [16, 1, 2, 3]. In [16], it is shown that if G and H are cordial then $G \cup H$ is cordial if at least one of G or H has an even size and $G + H$ is cordial if both have even size.

Another type of labeling we consider in this report is product cordial labeling and total product cordial labeling. This is a simple extension of cordial labeling where the edge labels are defined as the product of the vertex labels. Sundaram and Somasundaram [13, 15] introduced the notion of product cordial labelings and total product cordial labelings.

Definition 1.2. [13] *A product cordial labeling of a graph G with vertex set V is a function f from V to $\{0,1\}$ such that if each edge uv is assigned the label $f(u)f(v)$ then (i) the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and (ii) the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph.*

Definition 1.3. [15] *A total product cordial labeling of a graph G with vertex set V is a function f from V to $\{0,1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a total product cordial graph.*

In this paper, we show that the class of planar graphs Pl_n, $Pl_{m,n}$ are cordial graphs and $Pl_{m,n}$ is a total product cordial graph under certain conditions. All the results are based on an embedding of the planar graphs in question.

2. Cordial Labelings of Planar Graphs Pl_n and $Pl_{m,n}$

In this section, we show that two classes of planar graphs whose definitions are based on complete graphs and complete bipartite graphs are shown to be cordial. We first define the graphs below.

In [4], Babujee defines a class of planar graphs as graphs obtained by removing certain edges from the corresponding complete graphs. The class of planar graphs so obtained are denoted by Pl_n and contain the maximum number of edges possible in a planar graph on n vertices. We reproduce the definition from [4].

Definition 2.1. [4] *The class of graphs* $Pl_n = (V_n, E_n)$ *has the vertex set* $V_n = \{v_1, v_2, \cdots, v_n\}$ *and edge set* $E_n = E(K_n) \setminus \{(v_k, v_\ell) : 1 \leq k \leq n-4, k+2 \leq \ell \leq n-2\}$.

The embedding we use for Pl_n is described as follows. Place the vertices $v_1, v_2, \cdots, v_{n-2}$ along a vertical line in that order with v_1 at the bottom and v_{n-2} at the top as shown in Figure 1. Now place the vertices v_{n-1} and v_n as the end points of a horizontal line segment (perpendicular to the line segment used for placing the other $n-2$ points) with v_{n-1} to the left of v_n so that the vertices v_n, v_{n-1}, and v_{n-2} form a triangular face. See Figure 1 for an illustration. The edges of the graph Pl_n can now be drawn without any crossings. All the faces of this graph are of length 3.

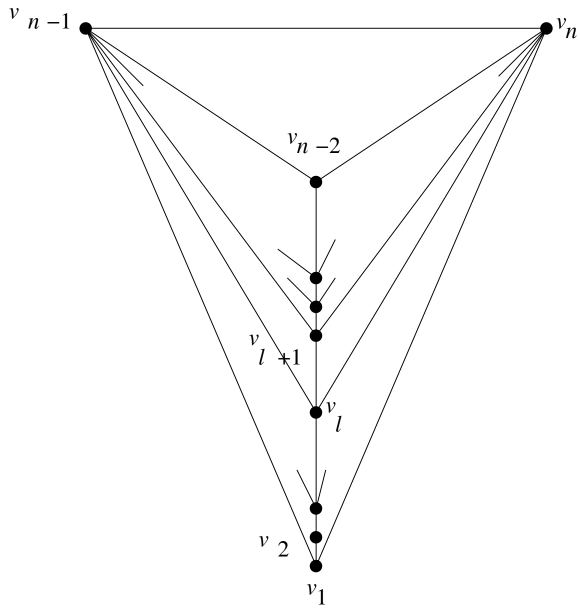

Figure 1: The Class Pl_n.

In [12], we defined another class of planar graphs which is obtained from the complete bipartite graph $K_{m,n}, m, n \geq 3$ by removing some edges to make it planar graph, which is called a bipartite planar class and it is denoted by $Pl_{m,n}$. The graph $Pl_{m,n}$ has the maximum number of edges permissible in a planar bipartite graph.

Definition 2.2. [12] *Let* $K_{m,n}(V_m, U_n)$ *be the complete bipartite graph on* $V_m = \{v_1, v_2, \cdots, v_m\}$ *and* $U_n = \{u_1, u_2, \cdots, u_n\}$. *The class of graphs* $Pl_{m,n}(V, E)$ *has the vertex set* $V_m \cup U_n$ *and the edge set* $E = E(K_{m,n}(V_m, U_n)) \setminus \{(v_\ell, u_p) : 3 \leq \ell \leq m$ *and* $2 \leq p \leq n-1\}$.

This graph, shown in Figure 2, is a bipartite planar graph with $2m + 2n - 4$ edges and $m + n$ vertices. We now describe the embedding used for our proofs. Place the vertices

$u_1, u_2, \cdots, u_n$ in that order along a horizontal line segment with u_1 as the left endpoint and u_n as the right endpoint as shown in Figure 2. Place the vertices $v_m, v_{m-1}, \cdots, v_3, v_1$ in that order along a vertical line segment with v_m as the top end point and v_1 as the bottom end point so that this entire line segment is above the horizontal line segment where the vertices u_1 through u_n are placed. Finally, place v_2 below the horizontal line segment so that the vertices v_1, u_k, v_2, u_{k+1} form a face of length 4 for $1 \leq k \leq n-1$. Notice that though we talk about placement along a line segment, no edges other than those mentioned in the definition are to be added.

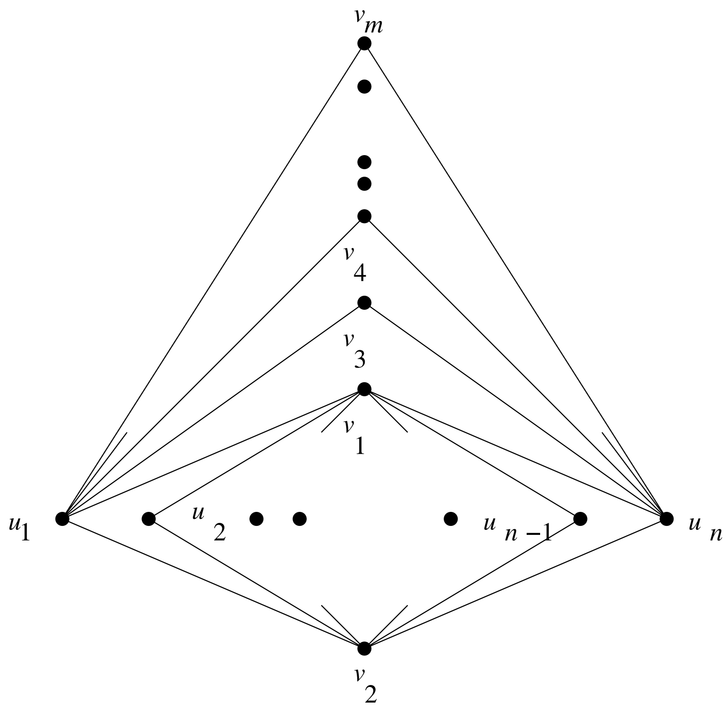

Figure 2: The Class $Pl_{m,n}$.

2.1. Cordial labelings of Pl_n

In this section, we show that the graph Pl_n allows a cordial labeling. The proof involves a case distinction based on n to come up with such a labeling.

Theorem 2.3. *The graph Pl_n, $n \geq 5$, is a cordial graph if $n \not\equiv 0 \bmod 4$.*

Proof. Consider the planar graph $Pl_n(V, E)$ with n vertices $v_1, v_2, \cdots, v_n$ and $3(n-2)$ edges and $n \not\equiv 0 \bmod 4$.

Case 1. n is even.

Then n is of the form $4m + 2$, where m is a positive integer. Define f from $V \cup E$ to $\{0, 1\}$ as follows.

$$\begin{array}{ll} f(v_n) = 1; f(v_{n-1}) = 0 & \\ f(v_{4i}) = 1; f(v_{4i-1}) = 1, & \text{for } i = 1, 2, 3, \cdots, m \\ f(v_{4i-2}) = 0; f(v_{4i-3}) = 0, & \text{for } i = 1, 2, 3, \cdots, m. \end{array}$$

Since $f(xy) = \mid f(x) - f(y) \mid$, for $xy \in E$ and $x, y \in V$, we have

$$\begin{array}{ll} f(v_n, v_{n-1}) = 1, & \\ f(v_{n-1}, v_{4i}) = 1; f(v_n, v_{4i}) = 0, & \text{for } i = 1, 2, \cdots, m, \\ f(v_{n-1}, v_{4i-1}) = 1; f(v_n, v_{4i-1}) = 0, & \text{for } i = 1, 2, \cdots, m, \\ f(v_{n-1}, v_{4i-2}) = 0; f(v_n, v_{4i-2}) = 1, & \text{for } i = 1, 2, \cdots, m, \\ f(v_{n-1}, v_{4i-3}) = 0; f(v_n, v_{4i-3}) = 1, & \text{for } i = 1, 2, 3, \cdots, m, \\ f(v_i, v_{i+1}) = 0, & \text{for } i = 1, 3, 5, \cdots, 4m-1, \text{ and} \\ f(v_i, v_{i+1}) = 1, & \text{for } i = 2, 4, \cdots, 4m-2, \end{array}$$

The vertices labeled 1 are $v_3, v_4, v_7, v_8, \cdots, v_{4m-1}, v_{4m},$ and v_{4m+2}. The vertices labeled 0 are $v_1, v_2, v_5, v_6, \cdots, v_{4m-3}, v_{4m-2},$ and v_{4m+1}.

The number of vertices labeled 0 is $2m+1$. The number of vertices labeled 1 is $2m+1$. The number of edges labeled 0 is $6m$ and the number of edges labeled 1 is $6m$. Hence f is a cordial labeling.

Case 2. n is of the form $4m+1$, where m is any positive integer.

Define f from $V \cup E$ to $\{0, 1\}$ as follows.

$$\begin{array}{ll} f(v_n) = 1, f(v_{n-1}) = 0, f(v_{n-2}) = 1. & \\ f(v_{4i}) = 1, f(v_{4i-1}) = 1, & \text{for } i = 1, 2, 3, \cdots, (m-1) \\ f(v_{4i-2}) = 0, f(v_{4i-3}) = 0, & \text{for } i = 1, 2, 3, \cdots, m \\ f(v_i, v_{i+1}) = 0, & \text{for } i = 1, 3, 5, \cdots, 4m-3, \\ f(v_i, v_{i+1}) = 1, & \text{for } i = 2, 4, 6, \cdots, 4m-2, \\ f(v_n, v_{n-1}) = 1, & \\ f(v_{n-1}, v_{4i}) = 1; f(v_n, v_{4i}) = 0, & \text{for } i = 1, 2, \cdots, m-1, \\ f(v_{n-1}, v_{4i-1}) = 1; f(v_n, v_{4i-1}) = 0, & \text{for } i = 1, 2, 3, \cdots, m-1, \end{array}$$

$$\begin{array}{lll} f(v_{n-1}, v_{4i-2}) = 0, & f(v_n, v_{4i-2}) = 1, & \text{for } i = 1, 2, 3, \cdots, m-1, \\ f(v_{n-1}, v_{4i-3}) = 0, & f(v_n, v_{4i-3}) = 1, & \text{for } i = 1, 2, 3, \cdots, m-1, \\ f(v_n, v_{n-2}) = 0, & f(v_{n-1}, v_{n-2}) = 1, & \\ f(v_n, v_{n-3}) = 1, & f(v_{n-1}, v_{n-3}) = 0, & \\ f(v_n, v_{n-4}) = 1, & f(v_{n-1}, v_{n-4}) = 0, & \end{array}$$

The vertices labeled 1 are $v_3, v_4, v_7, v_8, \cdots, v_{4m-5}, v_{4m-4}, v_{4m-1},$ and v_{4m+1}. The vertices labeled 0 are $v_1, v_2, v_5, v_6, \cdots, v_{4m-3}, v_{4m-2},$ and v_{4m}. The number of vertices labeled 0 is $2m+1$ and the number of vertices labeled 1 is $2m$. The number of edges labeled 0 is $6m-2$ and the number of edges labeled 1 is $6m-1$. Hence f is a cordial labeling.

Case 3. n is of the form $4m+3$, where m is a positive integer.

Define f from $V \cup E$ to $\{0, 1\}$ as follows.

$$\begin{aligned}
&f(v_n) = 1,\\
&f(v_{n-1}) = f(v_{n-2}) = 0,\\
&f(v_{4i}) = 1, f(v_{4i-1}) = 1, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_{4i-2}) = 0, f(v_{4i-3}) = 0, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_i, v_{i+1}) = 0, \text{ for } i = 1, 3, 5, \cdots, 4m-1,\\
&f(v_i, v_{i+1}) = 1, \text{ for } i = 2, 4, 6, \cdots, 4m,\\
&f(v_n, v_{n-1}) = 1,\\
&f(v_{n-1}, v_{4i}) = 1, f(v_n, v_{4i}) = 0, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_{n-1}, v_{4i-1}) = 1, f(v_n, v_{4i-1}) = 0, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_{n-1}, v_{4i-2}) = 0, f(v_n, v_{4i-2}) = 1, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_{n-1}, v_{4i-3}) = 0, f(v_n, v_{4i-3}) = 1, \text{ for } i = 1, 2, 3, \cdots, m,\\
&f(v_n, v_{n-2}) = 1, f(v_{n-1}, v_{n-2}) = 0.
\end{aligned}$$

The vertices labeled 1 are $v_3, v_4, v_7, v_8, \cdots, v_{4m-1}, v_{4m},$ and v_{4m+3}. The vertices labeled 0 are $v_1, v_2, v_5, v_6, \cdots, v_{4m-3}, v_{4m-2}, v_{4m+1},$ and v_{4m+2}. The number of vertices labeled 0 is $2m + 2$ and the number of vertices labeled 1 is $2m + 1$. The number of edges labeled 0 is $6m + 1$ and the number of edges labeled 1 is $6m + 2$. Hence f is a cordial labeling.

So the graph Pl_n, $n \geq 5$, is a cordial graph if $n \not\equiv 0 \bmod 4$. □

2.2 Cordial labelings of $Pl_{m,n}$

In this section, we show that the graph $Pl_{m,n}$ is cordial. This also is done in a constructive manner by exhibiting such a labeling using a case distinction based on n.

Theorem 2.4. *The graph $Pl_{m,n}$, $m, n \geq 3$, is a cordial graph.*

Proof. Consider the graph $Pl_{m,n}(V, E)$, where $V = \{v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_n\}$

Case 1. n is even.

Define f from V to $\{0, 1\}$ as follows

$$\begin{aligned}
&f(v_i) = 0, \text{ if } i \text{ is odd }, f(v_i) = 1, \text{ if } i \text{ is even, for } i \in \{1, 2, \cdots, m\},\\
&f(u_i) = 0, \text{ if } i \text{ is odd }, f(u_i) = 1, \text{ if } i \text{ is even, for } i \in 1, 2, \cdots, n.
\end{aligned}$$

Since $f(xy) = | f(x) - f(y) |$ for $xy \in E$, we have

$$\begin{aligned}
&f(v_1, u_i) = 0, \text{ if } i \text{ is odd}, f(v_1, u_i) = 1, \text{ if } i \text{ is even, and } i \in \{1, 2, \cdots, n\},\\
&f(v_2, u_i) = 1, \text{ if } i \text{ is odd}, f(v_2, u_i) = 0, \text{ if } i \text{ is even, and } i \in \{1, 2, \cdots, n\}\\
&f(u_1, v_i) = 0, \text{ and } f(u_n, v_i) = 1, \text{ if } i \text{ is odd, and } i \in \{3, 4, \cdots, m\}\\
&f(u_1, v_i) = 1, \text{ and } f(u_n, v_i) = 0, \text{ if } i \text{ is even, and } i \in \{3, 4, \cdots, m\}.
\end{aligned}$$

Clearly f is a cordial labeling.

Case 2. n is odd.

Define f from V to $\{0,1\}$ as follows

$$\begin{array}{l} f(u_n) = 1, \\ f(u_i) = 0, \text{ if } i \text{ is odd}, \quad f(u_i) = 1, \text{ if } i \text{ is even, for } i \in \{1, 2, \cdots, n-1\} \\ f(v_i) = 0, \text{ if } i \text{ is odd}, \quad f(v_i) = 1, \text{ if } i \text{ is even, for } i \in \{1, 2, \cdots, m\}. \end{array}$$

Since $f(xy) = \mid f(x) - f(y) \mid$ for $xy \in E$, we have

$$\begin{array}{l} f(v_1, u_i) = 0, \text{ if } i \text{ is odd}, \quad f(v_1, u_i) = 1, \text{ if } i \text{ is even, for } i \in \{1, 2 \cdots, n-1\} \\ f(v_2, u_i) = 1, \text{ if } i \text{ is odd}, \quad f(v_2, u_i) = 0, \text{ if } i \text{ is even, for } i \in \{1, 2 \cdots, n-1\} \\ f(u_1, v_i) = 1, \text{ if } i \text{ is even}, \quad f(u_1, v_i) = 0, \text{ if } i \text{ is odd, for } i \in \{3, 4, \cdots, m\} \\ f(u_n, v_i) = 0, \text{ if } i \text{ is even}, \quad f(u_n, v_i) = 1, \text{ if } i \text{ is odd, for } i \in \{1, 2, \cdots, m\}. \end{array}$$

Clearly f is a cordial labeling. Hence $Pl_{m,n}$, $m, n \geq 3$, is a cordial graph. □

3. Total product cordial labeling of planar graph $Pl_{m,n}$

In this section, we show that the class of graphs $Pl_{m,n}$ is total product cordial (see Definitions 1.2, 1.3), under certain mild conditions. This is done by exhibiting such a labeling.

Theorem 3.1. *The graph $Pl_{m,n}$, $m, n \geq 3$, is a total product cordial graph except for either m even and $n \not\equiv 2 \bmod 4$, or m odd and $n \not\equiv 1 \bmod 4$.*

Proof. Consider the graph $Pl_{m,n}(V, E)$ with $m+n$ vertices $v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_n$ and $2m + 2n - 4$ edges.

Case 1. m is even and n is odd.

Define a function f from $V \cup E$ to $\{0,1\}$ as follows.

$$\begin{array}{l} f(v_1) = f(v_2) = 1, \quad f(v_3) = f(v_4) = 0, \\ f(v_i) = 1 \text{ if } i \text{ is odd}, \quad f(v_i) = 0, \text{ if } i \text{ is even, for } i = 5, 6, \cdots, m, \\ f(u_j) = 1, \text{ if j is odd}, \quad f(u_j) = 0 \text{ if j is even, for } j = 1, 2, \cdots, n. \end{array}$$

Since $f(ab) = f(a)f(b)$, for $ab \in E$, we have

$$\begin{aligned}
&f(v_1, u_j) = 1, \text{ if } j \text{ is odd,}\\
&f(v_1, u_j) = 0, \text{ if } j \text{ is even, for } j = 1, 2, \cdots, n,\\
&f(v_2, u_j) = 1, \text{ if } j \text{ is odd, and}\\
&f(v_2, u_j) = 0, \text{ if } j \text{ is even, for } j = 1, 2, \cdots, n,\\
&f(u_1, v_3) = f(u_1, v_4) = 0,\\
&f(u_n, v_3) = f(u_n, v_4) = 0,\\
&f(u_1, v_i) = 1, \text{ if } i \text{ is odd, and}\\
&f(u_1, v_i) = 0, \text{ if } i \text{ is even, for } i \in \{5, 6, \cdots, m\},\\
&f(u_n, v_i) = 1, \text{ if } i \text{ is odd, and}\\
&f(u_n, v_i) = 0, \text{ if } i \text{ is even, for } i \in \{5, 6, \cdots, m\}.
\end{aligned}$$

Then the number of vertices labeled with 0 is $(m + n - 1)/2$ and the number of vertices labeled with 1 is $(m + n + 1)/2$. The number of edges labeled with 0 is $(m + n - 1)$ and the number of edges labeled with 1 is $(m+n-3)$. The total number of vertices and edges labeled with 0 is $(3m + 3n - 3)/2$. The total number of vertices and edges labeled with 1 is $(3m + 3n - 5)/2$. Hence f is a total product cordial labeling.

Case 2. m is odd and n is even.

Define a function f from $V \cup E$ to $\{0, 1\}$ as follows.

$$\begin{aligned}
&f(v_1) = f(v_2) = 1,\\
&f(v_i) = 0, \text{ if } i \text{ is odd,}\\
&f(v_i) = 1, \text{ if } i \text{ is even, for } i \in \{3, 4, \cdots, m\},\\
&f(u_1) = 1, f(u_2) = 0,\\
&f(u_j) = 0, \text{ if } j \text{ is odd,}\\
&f(u_j) = 1, \text{ if } j \text{ is even, for } j \in \{3, 4, \cdots, n\}.
\end{aligned}$$

Since $f(ab) = f(a)f(b)$, for $ab \in E$, we have the edge labels as

$$\begin{aligned}
&f(v_1, u_j) = 0, \text{ if } j \text{ is odd, and}\\
&f(v_1, u_j) = 1, \text{ if } j \text{ is even, for } j \in \{3, 4, \cdots, n\},\\
&f(v_2, u_j) = 0, \text{ if } j \text{ is odd, and}\\
&f(v_2, u_j) = 1, \text{ if } j \text{ is even, for } j \in \{3, 4, \cdots, n\},
\end{aligned}$$

$$\begin{aligned}
&f(v_1, u_1) = 1, f(v_1, u_2) = 0,\\
&f(v_2, u_1) = 1, f(v_2, u_2) = 0,\\
&f(u_1, v_i) = 0, \text{ if } i \text{ is odd, and}\\
&f(u_1, v_i) = 1, \text{ if } i \text{ is even, for } i \in \{3, 4, \cdots, m\},\\
&f(u_n, v_i) = 0, \text{ if } i \text{ is odd, and}\\
&f(u_n, v_i) = 1, \text{ if } i \text{ is even, for } i \in \{3, 4, \cdots, m\}.
\end{aligned}$$

Then the number of vertices labeled with 0 is $(m + n - 1)/2$ and the number of vertices labeled with 1 is $(m + n + 1)/2$. The number of edges labeled with 0 is $(m + n - 1)$ and the number of edges labeled with 1 is $(m+n-3)$. The total number of vertices and edges labeled with 0 is $(3m+3n-3)/2$. The total number of vertices and edges labeled with 1 is $(3m+3n-5)/2$. Hence f is a total product cordial labeling.

Case 3. m is even and n is a multiple of 4

Define a function f from $V \cup E$ to $\{0,1\}$ as follows.

$$\begin{aligned}
&f(v_i) = 1, \text{ if } i \text{ is odd},\\
&f(v_i) = 0, \text{ if } i \text{ is even, for } i \in \{1,2,\cdots,m\},\\
&f(u_i) = 0, \text{ if } i \equiv 3 \bmod 4,\\
&f(u_i) = 1, \text{ if } i \not\equiv 3 \bmod 4, \text{ for } i \in \{1,2,\cdots,n\}.
\end{aligned}$$

Since $f(ab) = f(a)f(b)$, for $ab \in E$, we have the edge labels as

$$\begin{aligned}
&f(v_1,u_i) = 0, \text{ if } i \equiv 3 \bmod 4, \text{ and}\\
&f(v_1,u_i) = 1, \text{ if } i \not\equiv 3 \bmod 4, \text{ for } i \in \{1,2,\cdots,n\},\\
&f(v_2,u_i) = 0, \text{ for } i = 1,2,\cdots,n,\\
&f(u_1,v_i) = 1, \text{ if } i \text{ is odd, and}\\
&f(u_1,v_i) = 0, \text{ if } i \text{ is even, for } i \in \{3,4,\cdots,m\},\\
&f(u_n,v_i) = 1, \text{ if } i \text{ is odd, and}\\
&f(u_n,v_i) = 0 \text{ if } i \text{ is even, for } i \in \{3,4,\cdots,m\}.
\end{aligned}$$

Then the number of vertices labeled with 0 is $(2m+n)/4$ and the number of vertices labeled with 1 is $(2m+3n)/4$. The number of edges labeled with 0 is $(4m+5n-8)/4$ and the number of edges labeled with 1 is $(4m+3n-8)/4$. The total number of vertices and edges labeled with 0 is $(6m+6n-8)/4$. The total number of vertices and edges labeled with 1 is $(6m+6n-8)/4$. Hence f is a total product cordial labeling.

Case 4. m is odd and n is of the form $4k+3$, for $k = 0,1,2,3\cdots$,

Define a function f from $V \cup E$ to $\{0,1\}$ as follows.

$$\begin{aligned}
&f(v_i) = 0, \text{ if } i \text{ is odd},\\
&f(v_i) = 1, \text{ if } i \text{ is even, for } i \in \{1,2,\cdots,m\},\\
&f(u_1) = 1, \quad f(u_2) = 1, \text{ and } \quad f(u_3) = 1,\\
&f(u_i) = 0, \text{ if } i \equiv 2 \bmod 4,\\
&f(u_i) = 1, \text{ if } i \not\equiv 2 \bmod 4, \text{ for } i \in \{4,5,\cdots,n\}.
\end{aligned}$$

Since $f(ab) = f(a)f(b)$, for $ab \in E$, we have the edge labels as

$$\begin{aligned}
&f(v_1,u_i) = 0, \text{ for } i = 1,2,\cdots,n,\\
&f(v_2,u_i) = 0, \text{ if } i \equiv 2 \bmod 4,\\
&f(v_2,u_i) = 1, \text{ if } i \not\equiv 2 \bmod 4, \text{ for } i \in \{4,5,\cdots,n\},\\
&f(u_1,v_i) = 0, \text{ if } i \text{ is odd},\\
&f(u_1,v_i) = 1, \text{ if } i \text{ is even, for } i \in \{3,4,\cdots,m\}\\
&f(u_n,v_i) = 0, \text{ if } i \text{ is odd},\\
&f(u_n,v_i) = 1, \text{ if } i \text{ is even, for } i \in \{3,4,\cdots,m\}.
\end{aligned}$$

Then the number of vertices labeled with 0 is $(2m+n-1)/4$ and the number of vertices labeled with 1 is $(2m+3n+1)/4$. The number of edges labeled with 0 is $(4m+5n-7)/4$ and the number of edges labeled with 1 is $(4m+3n-9)/4$. The total number of vertices and edges labeled with 0 is $(6m+6n-8)/4$. The total number of vertices and edges labeled with 1 is $(6m+6n-8)/4$. Hence f is a total product cordial labeling. $\square$

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