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On super vertex-antimagic total labelings of disjoint union of paths

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Abstract

Let $G = (V, E)$ be a (p, q)-graph of order p and size q and f be a bijection from the set $V \cup E$ to the set of the first $p + q$ natural numbers. The weight of a vertex is the sum of its label and the labels of all adjacent edges. We say f is an (a, d)-vertex-antimagic total labeling if the vertex-weights form an arithmetic progression with the initial term a and the common difference d. Such a labeling is called super if the smallest possible labels appear on the vertices.

In this paper, we study super (a, d)-vertex-antimagic total properties of disjoint union of paths.

Keywords: (a, d)-vertex-antimagic total labeling, super (a, d)-vertex-antimagic total labeling, union of paths

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1. Introduction

All graphs in this paper are finite, undirected, and simple. For a graph G, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set, respectively. A (p, q)-graph G is a graph such that $|V(G)| = p$ and $|E(G)| = q$. The degree of a vertex x is the number of edges incident to x, and the set of neighbors of x is denoted by $N(x)$.

A labeling of a graph G is a mapping from the set of vertices, edges, or both vertices and edges to the set of labels. Based on the domain we distinguish vertex labeling, edge labeling and total labeling.

The *vertex-weight*, $w(x)$, of a vertex $x \in V$, under a labeling $f\colon V \cup E \to \{1, 2, \dots, p+q\}$, is the sum of the labels assigned to all edges incident to a given vertex x together with the value assigned to x itself:

$$w(x) = f(x) + \sum_{y \in N(x)} f(xy).$$

A one-to-one mapping $f\colon V \cup E \to \{1, 2, \dots, p+q\}$ is called an (a,d)-*vertex-antimagic total* (in short, (a,d)-VAT) *labeling* of G if the set of vertex-weights of all vertices in G is $\{a, a+d, a+2d, \dots, a+(p-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a,d)-VAT graph. Furthermore, f is a *super* (a,d)*-VAT labeling* of G if the vertex labels are the integers $1, 2, \dots, p$. Thus, a super (a,d)-VAT graph is a graph that admits a super (a,d)-VAT labeling.

These labelings were introduced in [1]. If $d = 0$ then we call f a *vertex-magic total labeling* (simply VMTL) or super VMTL, respectively. The concept of the VMTL and super VMTL was introduced by MacDougall et al. in [12]. In [13], it is shown that wheels, fans and friendship graphs have no VMTLs except for the certain range of n and for all n in this range the VMTLs are found. A VMTL for K_n, for odd n, can be found in [11], [12] and [14], and for K_n, with n even, is given in [4] and [6]. A construction for VMTL of complete bipartite graphs $K_{m,m}$ is presented in [12]. In [7], it is completely determined which complete bipartite graphs have VMTLs. The constructions of VMTLs of certain regular graphs are given in [9], [10] and [16]. For further results on VMTLs see [3] and [17].

The basic properties of (a,d)-VAT labelings are investigated in [1] and super (a,d)-VAT labelings are studied in [15]. In [15], it is shown how to construct the super (a,d)-VAT labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs.

In this paper we concentrate on vertex-antimagicness for disconnected graphs. Certain results for vertex-magicness of disconnected graphs are known. For instance, a super VMTL for the disjoint union of m cycles of length n, mC_n, for m and n odd, is given in [2]. Gray et al. [8] explore VMTLs for a disjoint union of stars and prove that mP_3 ($mK_{1,2}$) has a VMTL. Gómez [5] studies the super VMTLs for the disjoint union of regular graphs.

We will investigate the existence of super (a,d)-VAT labeling for a disjoint union of m copies of path.

2. Preliminary Results

Suppose that a (p,q)-graph G has a super (a,d)-VAT labeling $f\colon V \cup E \to \{1, 2, \dots, p+q\}$ with the set of the vertex-weights $W = \{a, a+d, \dots, a+(p-1)d\}$. The sum of the vertex-weights over all vertices is

$$\sum_{x \in V(G)} w_f(x) = ap + \frac{pd(p-1)}{2}. \tag{1}$$

This sum is equal to the sum of all vertex labels and all edge labels under the labeling f, where each vertex label is used once and each edge label is used twice:

$$\sum_{x \in V(G)} f(x) + 2 \sum_{e \in E(G)} f(e) = \frac{p(p+1)}{2} + 2pq + q(q+1). \tag{2}$$

Combining (1) and (2), we obtain the minimum vertex-weight

$$a = \frac{1}{2}(p+1-(p-1)d) + 2q + \frac{q(q+1)}{p}. \tag{3}$$

If δ is the smallest degree in G then the minimum possible vertex-weight is at least $1 + (p+1) + (p+2) + \cdots + (p+\delta)$. Then

$$a \geq 1 + p\delta + \frac{\delta(\delta+1)}{2}. \tag{4}$$

If Δ is the largest degree in G then the maximum possible vertex-weight is no more than $p + (p+q) + (p+q-1) + \cdots + (p+q-\Delta+1)$. Thus

$$a + (p-1)d \leq p + \sum_{i=0}^{\Delta-1}(p+q-i). \tag{5}$$

Combining (4) and (5) we get the upper bound on feasible value of difference d:

$$d \leq 1 + \frac{\Delta(2p+2q-\Delta+1) - \delta(2p+\delta+1)}{2(p-1)}. \tag{6}$$

Let us now consider a disjoint union of m copies of the path P_n and denote it by mP_n. The graph mP_n, $m > 1$, is disconnected with vertex set $V(mP_n) = \{x_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and edge set $E(mP_n) = \{x_i^j x_{i+1}^j : 1 \leq i \leq n-1,\ 1 \leq j \leq m\}$. The next theorem presents a result for a particular case when $n = 2$.

Theorem 2.1. *The graph mP_2, $m \geq 1$, has a super (a,d)-VAT labeling if and only if m is odd and $d = 1$.*

Proof. Assume that mP_2 has a super (a,d)-VAT labeling f. From (6), it follows that, for $\delta = \Delta = 1$, $d \leq \frac{3m-2}{2m-1} < \frac{3}{2}$. For $d = 0$ we suppose, to the contrary, that f is a super VMTL with common vertex-weight k. Clearly, $f(x_1^j) + f(x_1^j x_2^j) = k = f(x_1^j x_2^j) + f(x_2^j)$ and $f(x_1^j) = f(x_2^j)$, for every $1 \leq j \leq m$. This produces a contradiction. Thus, mP_2 does not have any super VMTL.

From (3) we have that for $d = 1$ the smallest vertex-weight is $a = \frac{5m+3}{2}$. If m is even this contradicts the fact that a is an integer.

It remains to investigate whether mP_2, for m odd, admits a super $(\frac{5m+3}{2}, 1)$-VAT labeling. We construct a total labeling f_1 as follows:

$$f_1(x_1^j) = \begin{cases} \frac{m-1}{2} + j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ j - \frac{m+1}{2}, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_1(x_2^j) = \begin{cases} \frac{3m-1}{2} + j, & \text{for} \quad 1 \le j \le \frac{m+1}{2} \\ \frac{m-1}{2} + j, & \text{for} \quad \frac{m+3}{2} \le j \le m \end{cases}$$

$$f_1(x_1^j x_2^j) = 2m + j, \quad \text{for} \quad 1 \le j \le m.$$

Evidently, f_1 is super $(\frac{5m+3}{2}, 1)$-VAT labeling, for m odd, since all verifications are trivial. □

3. Results for mP_3 and mP_4

If the disjoint union of m copies of P_n, $n \ge 3$, is super (a, d)-VAT then, for $p = mn$ and $q = m(n-1)$, it follows from (6) that $d < 4$.

Theorem 3.2. *For the graph mP_3, $m \ge 1$, there is no super VMTL.*

Proof. Suppose mP_3 has a super VMTL with common vertex-weight k. The maximum possible sum of vertex-weights on the leaves is the sum of the $2m$ largest vertex labels and all edge labels:

$$\sum_{j=1}^{m} (w(x_1^j) + w(x_3^j)) \le \sum_{i=1}^{2m} (m+i) + \sum_{i=1}^{2m} (3m+i) = 2m(6m+1).$$

Since there are $2m$ leaves, then $k \le 6m + 1$.

The minimum possible sum of vertex-weights on the internal vertices of degree 2 is the sum of the m smallest vertex labels and all the edge labels:

$$\sum_{j=1}^{m} w(x_2^j) \ge \sum_{j=1}^{m} j + \sum_{i=1}^{2m} (3m+i) = \frac{m(17m+3)}{2}.$$

Since there are m internal vertices, then $k \ge \frac{17m+3}{2}$. These two inequalities imply that

$$\frac{17m+3}{2} \le k \le 6m + 1,$$

which is a contradiction. □

Theorem 3.3. *For the graph mP_3, $m > 1$, there is no super $(a, 3)$-VAT labeling.*

Proof. Assume that mP_3 has a super $(a, 3)$-VAT labeling $f\colon V(mP_3) \cup E(mP_3) \to \{1, 2, \ldots, 5m\}$ and $\{a, a+3, \ldots, a+(3m-1)3\}$ is the set of vertex-weights. The smallest possible vertex-weight is achieved by putting the label 1 on a leaf and the label $3m+1$ on the its incident edge. Thus $a = 3m + 2$.

Suppose that the first $2m$ vertex-weights $3m+2, 3m+5, \dots, 9m-1$ are created on leaves and the next m vertex-weights $9m+2, 9m+5, \dots, 12m-1$ are created on internal vertices of mP_3. The largest possible vertex-weight on a leaf can be composed as a sum of the largest vertex label $3m$ and the largest edge label $5m$. Since $8m < 9m-1$, for $m > 1$, the value $9m-1$ is the vertex-weight of an internal vertex. However, there are still m vertex-weights bigger than $9m-1$ but just $m-1$ internal vertices, and we have a contradiction. $\square$

Theorem 3.4. *If $m \equiv 1$ (mod 6), $m \geq 1$, then the graph mP_3 has a super $(a,2)$-VAT labeling.*

Proof. Let h be a positive integer and let $m = 1+6h$. We construct a labeling f_2 of mP_3 in the following way:

$$f_2(x_1^j) = \begin{cases} h+1-j, & \text{for} \quad 1 \leq j \leq h \\ 2m+h+1-j, & \text{for} \quad h+1 \leq j \leq m-h \\ 4m+1-h-j, & \text{for} \quad m-h+1 \leq j \leq m \end{cases}$$

$$f_2(x_2^j) = \begin{cases} m+2h+2-2j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ 2m+2h+2-2j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_2(x_3^j) = \begin{cases} \frac{5m+3}{2}-h-j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2}-h \\ \frac{m+3}{2}+h-j, & \text{for} \quad \frac{m+3}{2}-h \leq j \leq \frac{m+1}{2} \\ \frac{7m+3}{2}-h-j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_2(x_1^j x_2^j) = 4m+1-j, \quad \text{for} \quad 1 \leq j \leq m$$

$$f_2(x_2^j x_3^j) = \begin{cases} \frac{9m+3}{2}-j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ \frac{11m+3}{2}-j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m. \end{cases}$$

We can see that the labeling f_2 is a bijective function from $V(mP_3) \cup E(mP_3)$ onto the set $\{1, 2, \dots, 5m\}$. The vertex-weights of mP_3, under the labeling f_2, constitute the sets

$$\begin{aligned} W_{f_2}^1 &= \{w_{f_2}(x_1^j) = 4m+2+h-2j : \text{if} \quad 1 \leq j \leq h\} \\ &= \{4m-h+2, 4m-h+4, \dots, 4m+h\}, \\ W_{f_2}^2 &= \{w_{f_2}(x_3^j) = 5m+3+h-2j : \text{if} \quad \tfrac{m+3}{2}-h \leq j \leq \tfrac{m+1}{2}\} \\ &= \{4m+h+2, 4m+h+4, \dots, 4m+3h\}, \end{aligned}$$

$$
\begin{aligned}
W^3_{f_2} &= \{w_{f_2}(x^j_1) = 6m + h + 2 - 2j : \text{if} \quad h + 1 \le j \le m - h\} \\
&= \{4m + 3h + 2, 4m + 3h + 4, \ldots, 6m - h\}, \\
W^4_{f_2} &= \{w_{f_2}(x^j_1) = 8m + 2 - h - 2j : \text{if} \quad m - h + 1 \le j \le m\} \\
&= \{6m - h + 2, 6m - h + 4, \ldots, 6m + h\}, \\
W^5_{f_2} &= \{w_{f_2}(x^j_3) = 7m + 3 - h - 2j : \text{if} \quad 1 \le j \le \tfrac{m+1}{2} - h\} \\
&= \{6m + h + 2, 6m + h + 4, \ldots, 7m - h + 1\}, \\
W^6_{f_2} &= \{w_{f_2}(x^j_3) = 9m + 3 - h - 2j : \text{if} \quad \tfrac{m+3}{2} \le j \le m\} \\
&= \{7m - h + 3, 7m - h + 5, \ldots, 8m - h\}, \\
W^7_{f_2} &= \{w_{f_2}(x^j_2) = \tfrac{19m+9}{2} + 2h - 4j : \text{if} \quad 1 \le j \le \tfrac{m+1}{2}\} \\
&= \{\tfrac{15m+5}{2} + 2h, \tfrac{15m+5}{2} + 2h + 4, \ldots, \tfrac{19m+1}{2} + 2h\}, \\
W^8_{f_2} &= \{w_{f_2}(x^j_2) = \tfrac{23m+9}{2} + 2h - 4j : \text{if} \quad \tfrac{m+3}{2} \le j \le m\} \\
&= \{\tfrac{15m+5}{2} + 2h + 2, \tfrac{15m+5}{2} + 2h + 6, \ldots, \tfrac{19m+1}{2} + 2h - 2\}.
\end{aligned}
$$

Hence, the set $\bigcup_{i=1}^{8} W^i_{f_2} = \{4m-h+2, 4m-h+4, \ldots, \frac{19m+1}{2}+2h\}$ contains an arithmetic progression with the common difference 2. Thus f_2 is a super $(a, 2)$-VAT labeling. □

Theorem 3.5. *For the graph mP_4, $m \ge 1$, there is no super VMTL.*

Proof. Suppose, to the contrary, that mP_4 has a super VMTL with the common vertex-weight k. We calculate the minimum possible sum of vertex-weights on the inner vertices of degree 2; this is achieved by putting the $2m$ smallest vertex labels on the inner vertices and using all edge labels, where the m smallest labels on edges $x^j_2x^j_3$, $1 \le j \le m$, will each be added twice. This gives us

$$\sum_{j=1}^{m}(w(x^j_2) + w(x^j_3)) \ge \sum_{i=1}^{2m} i + 2\sum_{j=1}^{m}(4m + j) + \sum_{i=1}^{2m}(5m + i) = m(23m + 3).$$

Since there are $2m$ inner vertices, we must therefore have $k \ge \frac{23m+3}{2}$.

Calculating the maximum possible sum of vertex-weights on the outer vertices, we take the $2m$ largest vertex labels and $2m$ largest edge labels:

$$\sum_{j=1}^{m}(w(x^j_1) + w(x^j_4)) \le \sum_{i=1}^{2m}(2m + i) + \sum_{i=1}^{2m}(5m + i) = 2m(9m + 1).$$

Since there are $2m$ outer vertices, we have $k \le 9m + 1$.

Thus, $\frac{23m+3}{2} \le k \le 9m + 1$ and we have a contradiction. Consequently, a super VMTL cannot exist. □

Theorem 3.6. *If $m \equiv 3 \pmod 4$, $m \geq 3$, then the graph mP_4 has a super $(a, 2)$-VAT labeling.*

Proof. Let s be a nonnegative integer and let $m = 3+4s$. For $s \geq 0$, define the bijection $f_3 : V(mP_4) \cup E(mP_4) \to \{1, 2, \dots, 7m\}$ as follows:

$$f_3(x_1^j) = \begin{cases} 3m + 2 + s - j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ 4m + 2 + s - j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_3(x_2^j) = \begin{cases} \frac{3m+5}{2} + s - 2j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ \frac{5m+5}{2} + s - 2j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_3(x_3^j) = \begin{cases} \frac{5m+5}{2} + s - 2j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ \frac{7m+5}{2} + s - 2j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_3(x_4^j) = \begin{cases} 2 + s - j, & \text{for} \quad 1 \leq j \leq s+1 \\ 4m + 2 + s - j, & \text{for} \quad s+2 \leq j \leq \frac{m+1}{2} \\ m + 2 + s - j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_3(x_1^j x_2^j) = \begin{cases} \frac{11m+3}{2} - j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ \frac{13m+3}{2} - j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m \end{cases}$$

$$f_3(x_2^j x_3^j) = 5m + 1 - j, \quad \text{for} \quad 1 \leq j \leq m$$

$$f_3(x_3^j x_4^j) = \begin{cases} \frac{13m+3}{2} - j, & \text{for} \quad 1 \leq j \leq \frac{m+1}{2} \\ \frac{15m+3}{2} - j, & \text{for} \quad \frac{m+3}{2} \leq j \leq m. \end{cases}$$

Then for the vertex-weights of mP_4 we have:

$$\begin{aligned}
W^1_{f_3} &= \{w_{f_3}(x_4^j) = \tfrac{13m+7}{2} + s - 2j : \text{if} \quad 1 \leq j \leq s+1\} \\
&= \{\tfrac{13m+3}{2} - s, \tfrac{13m+3}{2} - s + 2, \dots, \tfrac{13m+3}{2} + s\}, \\
W^2_{f_3} &= \{w_{f_3}(x_4^j) = \tfrac{17m+7}{2} + s - 2j : \text{if} \quad \tfrac{m+3}{2} \leq j \leq m\} \\
&= \{\tfrac{13m+3}{2} + s + 2, \tfrac{13m+3}{2} + s + 4, \dots, \tfrac{15m+1}{2} + s\}, \\
W^3_{f_3} &= \{w_{f_3}(x_1^j) = \tfrac{17m+7}{2} + s - 2j : \text{if} \quad 1 \leq j \leq \tfrac{m+1}{2}\} \\
&= \{\tfrac{15m+1}{2} + s + 2, \tfrac{15m+1}{2} + s + 4, \dots, \tfrac{17m+3}{2} + s\}, \\
W^4_{f_3} &= \{w_{f_3}(x_1^j) = \tfrac{21m+7}{2} + s - 2j : \text{if} \quad \tfrac{m+3}{2} \leq j \leq m\} \\
&= \{\tfrac{17m+3}{2} + s + 2, \tfrac{17m+3}{2} + s + 4, \dots, \tfrac{19m+1}{2} + s\},
\end{aligned}$$

$$\begin{aligned}
W^5_{f_3} &= \{w_{f_3}(x^j_4) = \tfrac{21m+7}{2} + s - 2j : \text{if} \quad s+2 \le j \le \tfrac{m+1}{2}\} \\
&= \{\tfrac{19m+1}{2} + s + 2, \tfrac{19m+1}{2} + s + 4, \dots, \tfrac{21m-1}{2} - s\}, \\
W^6_{f_3} &= \{w_{f_3}(x^j_2) = 12m + 5 + s - 4j : \text{if} \quad 1 \le j \le \tfrac{m+1}{2}\} \\
&= \{10m + s + 3, 10m + s + 7, \dots, 12m + s + 1\}, \\
W^7_{f_3} &= \{w_{f_3}(x^j_2) = 14m + 5 + s - 4j : \text{if} \quad \tfrac{m+3}{2} \le j \le m\} \\
&= \{10m + s + 5, 10m + s + 9, \dots, 12m + s - 1\}, \\
W^8_{f_3} &= \{w_{f_3}(x^j_3) = 14m + 5 + s - 4j : \text{if} \quad 1 \le j \le \tfrac{m+1}{2}\} \\
&= \{12m + s + 3, 12m + s + 7, \dots, 14m + s + 1\}, \\
W^9_{f_3} &= \{w_{f_3}(x^j_3) = 16m + 5 + s - 4j : \text{if} \quad \tfrac{m+3}{2} \le j \le m\} \\
&= \{12m + s + 5, 12m + s + 9, \dots, 14m + s - 1\}
\end{aligned}$$

and $\bigcup_{i=1}^{9} W^i_{f_3} = \{\frac{13m+3}{2} - s, \frac{13m+3}{2} - s + 2, \dots, 14m + s + 1\}$ contains an arithmetic progression with the difference 2. This implies that f_3 is a super $(a, 2)$-VAT labeling. □

5. Conclusion

In this paper we have shown that the graph mP_3 has a super $(a, 2)$-VAT labeling for $m \equiv 1 \pmod 6$, $m \ge 1$, and the graph mP_4 has a super $(a, 2)$-VAT labeling for $m \equiv 3 \pmod 4$, $m \ge 3$. For mP_3 and mP_4 we have tried to find a super $(a, 2)$-VAT labeling also for other m and a super $(a, 1)$-VAT labeling for every $m \ge 2$, but so far without success. So, we propose the following

Problem 5.1. *For the graph mP_3 and mP_4 determine if there is a super (a, d)-VAT labeling for every $m \ge 2$ and $d \in \{1, 2\}$.*

In the case when $n \ge 5$ and $d < 4$ we do not have any answer. Therefore for further investigation we propose the following open problem.

Problem 5.2. *For the graph mP_n, $n \ge 5$ and $m > 1$, determine if there is a super (a, d)-VAT labeling for the feasible values of the difference d.*

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References

[1] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R.Simanjuntak and Slamin, Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae Graph Theory,* **23** (2003), 67–83.

[2] C. Balbuena, E. Barker, K.C. Das, Y. Lin, M. Miller, J. Ryan, Slamin, K. Sugeng, M. Tkáč, On the degrees of a strongly vertex-magic graph, *Discrete Math.,* **306** (2006), 539–551.

[3] J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics,* **14** (2007), #DS6.

[4] J. Gómez, Solution of the conjecture: If $n \equiv 0 \pmod 4$, $n > 4$, then K_n has a super vertex-magic total labeling, *Discrete Math.,* **307** (2007), 2525–2534.

[5] J. Gómez, Two new methods to obtain super vertex-magic total labelings of graphs, *Discrete Math.,* **308** (2008), 3361–3372.

[6] I.D. Gray, J.A. MacDougall and W.D. Wallis, On vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl.,* **38** (2003), 42–44.

[7] I.D. Gray, J.A. MacDougall, R.J. Simpson and W.D. Wallis, Vertex-magic total labeling of complete bipartite graphs, *Ars Combin.,* **69** (2003), 117–127.

[8] I.D. Gray, J.A. MacDougall, J.P. McSorley and W.D. Wallis, Vertex-magic labeling of trees and forests, *Discrete Math.,* **261** (2003), 285–298.

[9] I.D. Gray, Vertex-magic total labelings of regular graphs, *SIAM J. Discrete Math.,* **21** (2007), 170–177.

[10] P. Kovář, Magic labelings of regular graphs, *AKCE J. Graphs. Combin.,* **4** (2007) 261–275.

[11] Y. Lin and M. Miller, Vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl.,* **33** (2001), 68–76.

[12] J.A. MacDougall, M. Miller, Slamin and W. Wallis, Vertex-magic total labelings of graphs, *Utilitas Math.,* **61** (2002), 68–76.

[13] J.A. MacDougall, M. Miller and W. Wallis, Vertex-magic total labelings of wheels and related graphs, *Utilitas Math.,* **62** (2002), 175–183.

[14] D. McQuillan and K. Smith, Vertex-magic total labeling of odd complete graphs, *Discrete Math.,* **305** (2005), 240–249.

[15] K.A. Sugeng, M. Miller, Y. Lin and M. Bača, Super (a, d)-vertex- antimagic total labelings, *The Journal of Combinatorial Mathematics and Combinatorial Computing,* **55** (2005), 91–102.

[16] V. Swaminathan and P. Jeyanthi, Super vertex-magic labeling, *Indian J. Pure Appl. Math.,* **34** (2003), 935–939.

[17] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston - Basel - Berlin, 2001.