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Minimax Open and Closed Neighborhood Sums

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Abstract

For a graph G of order $|V(G)| = n$ and a real-valued mapping $f : V(G) \to \mathbb{R}$, if $S \subset V(G)$ then $f(S) = \sum_{w \in S} f(w)$ is called the weight of S under f. The closed (respectively, open) neighborhood sum of f is the maximum weight of a closed (respectively, open) neighborhood under f, that is, $NS[f] = max\{f(N[v])|v \in V(G)\}$ and $NS(f) = max\{f(N(v))|v \in V(G)\}$. We study the closed and open neighborhood sum parameter, $NS[G] = min\{NS[f]|f : V(G) \to \{1, 2, \ldots, n\}$ is a bijection$\}$ and $NS(G) = min\{NS(f)|f : V(G) \to \{1, 2, \ldots, n\}$ is a bijection$\}$.

Keywords: neighborhood sum, open neighborhood sum.

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1. Introduction

Suppose $W = \{w_1, w_2, \ldots, w_n\} \subset \mathbb{R}$ is a set (or, more generally, a multiset) of reals of order n and we wish to equitably assign these weights/values to the vertices of a graph G of order $|V(G)| = n$. A measure of equitability is to minimize the maximum total resulting weight in any closed/open neighborhood. If $f : V(G) \to W$ is a bijection, the closed neighborhood sum and open neighborhood sum of f, respectively denoted by $NS[f]$ and $NS(f)$, are defined in Schneider and Slater [3] to be the maximum weight of a closed and open neighborhood under f. For $S \subset V(G)$ the *weight* of S under f is $f(S) = \sum_{w \in S} f(w)$. Thus, $NS[f] = max\{f(N[v])|v \in V(G)\}$ and $NS(f) = max\{f(N(v))|v \in V(G)\}$. The W-valued (closed) neighborhood sum and W-valued open neighborhood sum parameters are defined by $NS_W[G] = min\{NS[f]|f : V(G) \to W$ is a bijection$\}$ and $NS_W(G) = min\{NS(f)|f : V(G) \to W$ is a bijection$\}$. For $W = [n] = \{1, 2, \ldots, n\}$ we call $NS[G] = NS_{[n]}[G]$ the *neighborhood sum* of G and $NS(G) = NS_{[n]}(G)$ the *open neighborhood sum* of G. For example, in Figure 1 we show how to achieve $NS[H] = f(N[v_2]) = f(N[v_3]) = 11$ and $NS(H) = g(N(v_2)) = g(N(v_5)) = 8$, where $(f(v_1), \ldots, f(v_5)) = (3, 2, 5, 4, 1)$ and $(g(v_1), \ldots, g(v_5)) = (1, 4, 5, 3, 2)$.

We note that a graph G of order n for which there exists a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ with $f(N(u)) = f(N(v))$ for all $u, v \in V(G)$ is called a sigma-labeled graph, as in [6, 7]. See also [2, 4].

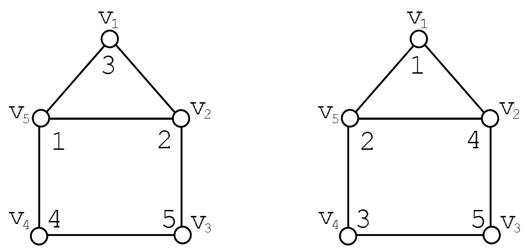

Figure 1: $NS[H] = 11$ and $NS(H) = 8$

We can obtain lower bounds for $NS[G]$ and $NS(G)$ based on the degree sequence of G. With $V(G) = \{v_1, v_2, \ldots, v_n\}$ we can assume the vertex degrees satisfy $deg(v_i) \geq deg(v_{i+1})$ for $1 \leq i \leq n-1$. The degree sequence of G is $(d_1, d_2, \ldots, d_n)$ where $d_i = deg(v_i)$ for $1 \leq i \leq n$ and $d_1 \geq d_2 \geq \ldots \geq d_n$.

Theorem 1. [3] *Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$; let $f : V(G) \to [n]$ be a bijection; and let $t_i = f(N[v_i])$. Then $\sum_{i=1}^{n} t_i \geq 1(d_1+1) + 2(d_2+1) + 3(d_3+1) + \ldots + n(d_n+1)$. Also, $NS[G] \geq \lceil (\sum_{i=1}^{n} i(d_i+1))/n \rceil$.*

A similar result for open neighborhoods is the following.

Theorem 2. *Let G be a graph with degree sequence$(d_1, d_2, \ldots, d_n)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$; let $f : V(G) \to [n]$ be a bijection; and let $s_i = f(N(v_i))$. Then $\sum_{i=1}^{n} s_i \geq 1d_1 + 2d_2 + 3d_3 + \ldots + nd_n$. Also, $NS(G) \geq \lceil (\sum_{i=1}^{n} id_i)/n \rceil$.*

Proof. Let $f : V(G) \to [n] = \{1, 2, \ldots, n\}$ be a bijection that minimizes $\sum_{i=1}^{n} s_i$. We note that $f(v_i)$ appears as a summand in $f(N(w))$ if and only if $w \in N(v_i)$. Hence, $f(v_i)$ appears as a summand d_i times. Thus, $\sum_{i=1}^{n} f(v_i)d_i = \sum_{i=1}^{n} f(N(v_i))$.

Suppose that $f(v_i) > f(v_{i+1})$ for some i with $1 \leq i \leq n-1$. Let $g : V(G) \to [n]$ be the bijection with $g(v_i) = f(v_{i+1})$,$g(v_{i+1}) = f(v_i)$ and $g(v_j) = f(v_j)$ when $j \notin \{i, i+1\}$. Then $\sum_{i=1}^{n} g(N(v_i)) - \sum_{i=1}^{n} f(N(v_i)) = g(v_{i+1})d_{i+1} + g(v_i)d_i - f(v_i)d_i - f(v_{i+1})d_{i+1} = f(v_i)d_{i+1} + f(v_{i+1})d_i - f(v_i)d_i - f(v_{i+1})d_{i+1} = (f(v_i) - f(v_{i+1}))(d_{i+1} - d_i) \geq 0$ because f minimizes $\sum_{i=1}^{n} s_i$. Now, $f(v_i) > f(v_{i+1})$ implies $d_{i+1} - d_i \geq 0$. But $d_i \geq d_{i+1}$, and so $d_i = d_{i+1}$. It follows that, for the bijection $f : V(G) \to [n]$ that minimizes $\sum_{i=1}^{n} f(N(v_i))$, we can assume $f(v_i) \leq f(v_{i+1})$. In particular, $f(v_i) = i$ and $\sum_{i=1}^{n} f(N(v_i)) = \sum_{i=1}^{n} id_i$.

It remains to show that $NS(G) \geq \lceil (\sum_{i=1}^{n} id_i)/n \rceil$. Let $f : V(G) \to [n]$ be a bijection and $NS(G) = NS(f)$. Then the average value of $f(N(v_i))$ satisfies $\sum_{i=1}^{n} f(N(v_i))/n \geq (\sum_{i=1}^{n} id_i)/n$. Thus $NS(G) = max\{f(N(v_i))|1 \leq i \leq n\} \geq \sum_{i=1}^{n} f(N(v_i))/n \geq (\sum_{i=1}^{n} id_i)/n$. Because $NS(G)$ is integer valued, the result follows. □

Corollary 3. *If G is regular of degree r, then $NS[G] \geq \lceil (r+1)(n+1)/2 \rceil$ and $NS(G) \geq \lceil r(n+1)/2 \rceil$.*

2. Cycles and $K_{s,s}$

In particular, for a cycle C_n, we have $NS[C_n] \geq (3/2)n + 3/2$. As noted in [3], determining $NS[C_n]$ is apparently difficult. The best we have been able to achieve for $NS[C_n]$ is approximately $(3/2)n + (1/18)n$.

For $NS(C_n)$ we have $NS(C_n) \geq n+1$ by Corollary 3. Note that if $n \geq 5$ and $f(v_i) = n$, then $f(N(v_{i-1})) = f(v_{i-2})+n$ and $f(N(v_{i+1})) = f(v_{i+2})+n \neq f(N(v_{i-1}))$. So, $NS(C_n) \geq n+2$ for $n \geq 5$.

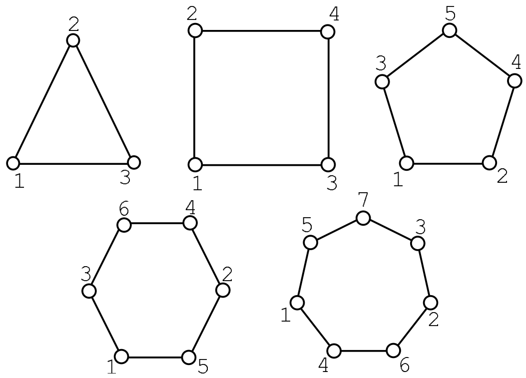

Figure 2: Labelings achieving $NS(C_n)$ for $3 \leq n \leq 7$

Theorem 4. *$NS(C_3) = 5$ and $NS(C_4) = 5$. If $n \geq 5$ then $NS(C_n) = n+2$ if $n \equiv 0, 1, 3(mod\ 4)$ and $NS(C_n) = n+3$ for $n \equiv 2(mod\ 4)$.*

Proof. As noted, for $n \geq 5$ we have $NS(C_n) \geq n+2$. Hence, for $n \equiv 0, 1, 3(mod\ 4)$ it suffices to show a labeling achieving $n+2$.

For $n = 4k$, consider the $4k$ vertices to alternately be dark and light. If v_i is dark (respectively, light), then $f(N(v_i)) = f(v_{i-1}) + f(v_{i+1})$ is the sum of two consecutive light (respectively, dark) vertex weights. Thus, we seek a labeling of two disjoint C_{2k}'s such that the maximum sum of any two consecutive vertices on either cycle is minimized at $n+2 = 4k+2$. Figure 3 shows how to achieve this for $n = 20$ and $n = 24$. In general, using $\{1, 2, \ldots, k, 3k+1, 3k+2, \ldots, 4k\}$ we can label the dark vertices consecutively by $(k, 3k+2, k-2, 3k+4, k-4, \ldots, 3, 4k-1, 1, 4k, 2, 4k-2, 4, 4k-4, \ldots, 3k+3, k-1, 3k+1)$ if k is odd and $(3k+1, k-1, 3k+3, k-3, \ldots, 3, 4k-1, 1, 4k, 2, 4k-2, 4, 4k-4, \ldots, 3k+2, k)$ if k is even. The light vertices are labeled using $(k+1, k+2, \ldots, 3k)$ consecutively as $(3k, k+2, 3k-2, k+4, \ldots, 2k+1, 2k, 2k+2, 2k-2, 2k+4, 2k-4, \ldots, k+1)$ for odd k and $(3k, k+2, 3k-2, k+4, \ldots, 2k+2, 2k, 2k+1, 2k-1, 2k+3, 2k-3, \ldots, 3k-1, k+1)$ for even k.

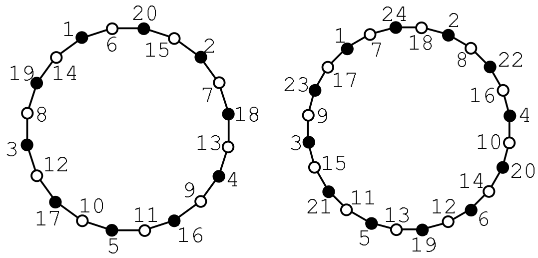

Figure 3: $NS(C_{20}) = 22$ and $NS(C_{24}) = 26$

For the odd cycles we can consecutively label every second vertex of C_{4k+1} as $(4k+1, 2, 4k-1, 4, 4k-3, 6, 4k-5, \ldots, 2k-2, 2k+3, 2k, 2k+1, 2k+2, 2k-1, 2k+4, 2k-3, \ldots, 4k-4, 5, 4k-2, 3, 4k, 1)$ and the vertices of C_{4k+3} as $(4k+3, 2, 4k+1, 4, 4k-1, 6, \ldots, 2k+5, 2k, 2k+3, 2k+2, 2k+1, 2k+4, 2k-1, 2k+6, \ldots, 5, 4k, 3, 4k+2, 1)$. We emphasize that two adjacent values in these sequences are being placed on two vertices at distance two on the cycle. See Figure 4 for $n = 17$ and $n = 19$. This shows that $NS(C_{2k+1}) = 2k+3 = n+2$.

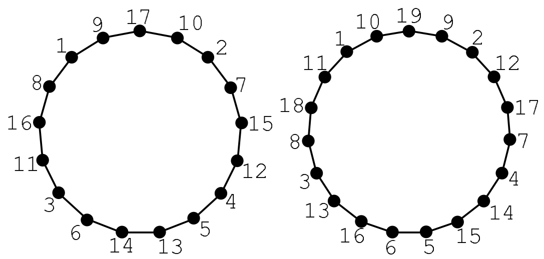

Figure 4: $NS(C_{17}) = 19 = n+2$ and $NS(C_{19}) = 21 = n+2$

To show that $NS(C_{4k+2}) = n+3 = 4k+5$, we first show that $NS(C_{4k+2}) \geq 4k+5$. Suppose $V(C_{4k+2}) = \{v_1, v_2, \ldots, v_{4k+2}\}$ with edges $v_i v_{i+1} (mod\ 4k+2)$ for $1 \leq i \leq 4k+2$. Consider v_i with i odd to be a dark vertex and v_i with i even to be light. Note that the sum of the values on two consecutive dark vertices (respectively, light vertices) gives the open neighborhood weight of the light (respectively, dark) vertex between them. Because we have $2k+1$ values in $\{2k+2, 2k+3, \ldots, 4k+2\}$, some $k+1$ of them are assigned to vertices of the same shade for any bijection $f : V(C_{4k+2}) \to \{1, 2, \ldots, 4k+2\}$, without loss of generality say dark. Thus some two consecutive dark vertices, say v_{2i+1} and v_{2i+3}, have their weights in $\{2k+2, 2k+3, \ldots, 4k+2\}$. Hence, $f(N(v_{2i+2})) = f(v_{2i+1}) + f(v_{2i+3}) \geq (2k+2) + (2k+3) = 4k+5$. Hence, $NS(C_{4k+2}) \geq 4k+5$.

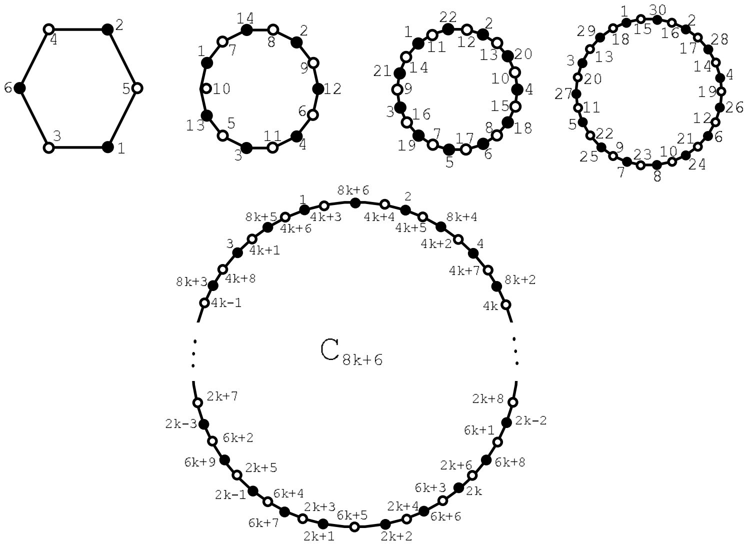

Figure 5: $NS(C_6) = 9, NS(C_{14}) = 17, NS(C_{22}) = 25, NS(C_{30}) = 33, NS(C_{8k+6}) = 8k+9$

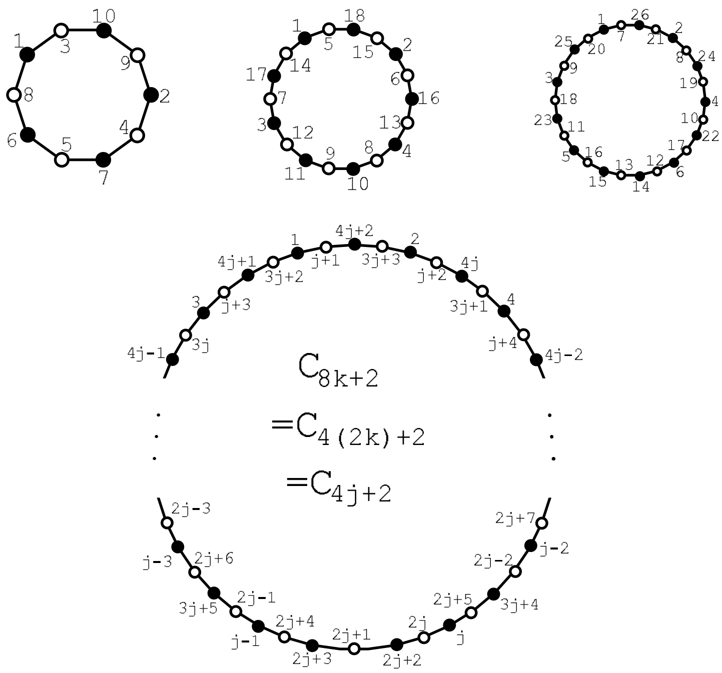

Figure 6: $NS(C_{10}) = 13, NS(C_{18}) = 21, NS(C_{26}) = 29, NS(C_{8k+2}) = NS(C_{4j+2}) = 8k+5$ for $j = 2k$

To see that $NS(C_{4k+2}) = 4k + 5$, one can verify that the labelings in Figures 5 and 6 achieve this bound for $n \equiv 6(mod\ 8)$ and $n \equiv 2(mod\ 8)$, respectively. $\square$

For a complete bipartite graph $K_{s,s}$ it is easy to compute $NS(K_{s,s})$. When $s = 2k$ and so $n = |V(K_{s,s})| = 2s = 4k$ one can assign weights $\{1, 2, \ldots, k-1, k, 3k+1, 3k+2, \ldots, 4k\}$ to one of the partite sets and $\{k + 1, k + 2, \ldots, 3k\}$ to the other. So exactly half of the total weight $1 + 2 + \ldots + 4k = 8k^2 + 2k$ can go on each partite set. When $s = 2k + 1$ and so $n = 2s = 4k + 2$ the total weight is $1 + 2 + \ldots + (4k + 2) = 8k^2 + 10k + 3$. Assigning weights $\{1, 2, \ldots, k - 1, k, 2k + 1, 3k + 3, 3k + 4, \ldots, 4k + 2\}$ and $\{k + 1, k + 2, \ldots, 2k - 1, 2k, 2k + 2, 2k + 3, 2k + 4, \ldots, 3k + 2\}$ to the respective partite sets gives us $4k^2 + 5k + 1$ and $4k^2 + 5k + 2$, respectively. We thus have the following result.

Proposition 5. *$NS(K_{2k,2k}) = 4k^2 + k = (2k)^2 + k = (1/4)(n^2 + n)$ (that is, $K_{2k,2k}$ is a sigma-labeled graph), and $NS(K_{2k+1,2k+1}) = 4k^2 + 5k + 2 = (2k + 1)^2 + k + 1 = (1/4)(n^2 + n + 2)$.*

For $K_{s,s}$ of order $n = 2s$, to achieve $NS[K_{s,s}]$ we do not quite equally divide the total weight $1 + 2 + \ldots + 2s = n(n + 1)/2$ over the two partite sets V_1 and V_2. Without loss of generality, we will assume that a vertex of V_1 gets weight $n = 2s$ under the bijection $f : V(K_{s,s}) \to \{1, 2, \ldots, 2s\}$. We let $S_1 = \sum_{v\in V_1} f(v)$, $S_2 = \sum_{v\in V_2} f(v)$, and $M = max\{f(v)|v \in V_2\}$. Note that we then have $NS[f] = max\{n + S_2, M + S_1\}$.

As an example, for $K_{8,8}$ if we let $f(V_1) = \{1, 2, 3, 4, 13, 14, 15, 16\}$ and $f(V_2) = \{5, 6, \ldots, 11, 12\}$, then $M = 12$ and $S_1 = S_2 = 68$, so $NS[f] = max\{16+68, 12+68\} = 84$. If we change f so that $f(V_1) = \{1, 2, 4, 5, 13, 14, 15, 16\}$ and $f(V_2) = \{3, 6, 7, 8, \ldots, 12\}$, then $S_1 = 70$ and $S_2 = 66$ and $NS[f] = max\{16 + 66, 12 + 70\} = 82 = NS[K_{8,8}]$.

Theorem 6. *For complete bipartite graph $K_{s,s}$ of order $n = 2s$, we have*

1. $NS[K_{4j,4j}] = 16j^2 + 9j = n^2/4 + (9/8)n$
2. $NS[K_{4j+1,4j+1}] = 16j^2 + 17j + 4 = n^2/4 + (9/8)n + 3/4$
3. $NS[K_{4j+2,4j+2}] = 16j^2 + 25j + 9 = n^2/4 + (9/8)n + 1/2$
4. $NS[K_{4j+3,4j+3}] = 16j^2 + 33j + 16 = n^2/4 + (9/8)n + 1/4$

Proof. For $K_{4j,4j}$ with $n = 8j$ we have $S_1 + S_2 = 8j(8j + 1)/2 = 32j^2 + 4j$. Let $f(V_1) = \{1, 2, \ldots, j - 1, j, j + 2, j + 3, \ldots, 2j + 1, 6j + 1, 6j + 2, \ldots, 8j\}$ and $f(V_2) = \{j + 1, 2j + 2, 2j + 3, 2j + 4, \ldots, 6j\}$. Then $S_1 = 16j^2 + 3j$ and $S_2 = 16j^2 + j$. Hence, $NS[f] = max\{8j + S_2, 6j + S_1\} = 16j^2 + 9j$, and $NS[K_{4j,4j}] \le 16j^2 + 9j$.

Suppose bijection $f : V(K_{4j,4j}) \to \{1, 2, \ldots, 8j\}$ has $NS[f] < 16j^2 + 9j$. Then $NS[f] \ge 8j+S_2$ implies that $S_2 < 16j^2+j$. Say $S_2 = 16j^2+j-t$ where $t \ge 1$, and so $S_1 = 16j^2+3j+t$. Now $NS[f] \ge S_1 + M$ where $M = max\{f(v)|v \in V_2\}$ implies that $M \le 6j - t - 1$. Thus $S_2 \le (2j - t) + (2j - t + 1) + \ldots + (6j - t - 2) + (6j - t - 1) = 4j(2j - t) + (0 + 1 + 2 +$

$\ldots + (4j-1)) = 16j^2 - 2j - 4jt$. Hence, $S_2 = 16j^2 + j - t \leq 16j^2 - 2j - 4jt$, which implies that $3j \leq t(1-4j) < 0$, a contradiction. Hence, $NS[K_{4j,4j}] = 16j^2 + 9j$.

The other three cases can be proven similarly. □

3. Related Problems

In this section we show some of the relations between the problem of determining $NS_W[G]$ and other well-studied problems. As expected, determining $NS_W[G]$ is inherently difficult in general.

The following problem, as noted in Garey and Johnson [1], is NP-complete.
PARTITION
INSTANCE: Finite $W = \{w_1, w_2, \ldots, w_{2k}\}$ of positive integers.

QUESTION: Is there a subset $A \subset W$ such that $|A| = k$ and $\sum\{w_i | w_i \in A\}$
$= (\sum\{w_i | w_i \in W\})/2$?

Note that this problem is equivalent to deciding if $NS[2K_k] = (\sum\{w_i | w_i \in W\})/2$ where $2K_k$ consists of two disjoint complete graphs on k vertices each.

Let $\{1^k, 0^{n-k}\}$ denote the multiset $W_{k,n}$ consisting of k 1's and $n-k$ 0's. Letting $\rho(G)$ denote the packing number of G, we see that $\rho(G) \geq k$ if and only if $NS_{W_{k,n}}[G] \leq 1$.

$NS_w[G]$ and $NS_w(G)$ are parameters seeking to minimize the maximum weight of a closed/open neighborhood. Two similar parameters involve maximizing the minimum weight of a closed/open neighborhood, see [5]. For a bijection $f : V(G) \rightarrow W$ the lower closed neighborhood sum and lower open neighborhood sum of f are $NS^-[f] = min\{f(N[v]) | v \in V(G)\}$ and $NS^-(f) = min\{f(N(v) | v \in V(G)\}$. The W-valued lower closed and open neighborhood sum parameters are $NS^-_W[G] = max\{NS^-[f] | f : V(G) \rightarrow W$ is a bijection$\}$ and $NS^-_W(G) = max\{NS^-(f) | f : V(G) \rightarrow W$ is a bijection$\}$. The lower neighborhood sum of G is $NS^-[G] = NS^-_{[n]}[G]$, and $NS^-(G) = NS^-_{[n]}(G)$ is the lower open neighborhood sum of G.

We see that the domination number satisfies $\gamma(G) \leq k$ if and only if $NS^-_{W_{k,n}}[G] \geq 1$, and the open (or total) domination number satisfies $\gamma_t(G) \leq k$ if and only if $NS^-_{W_{k,n}}(G) \geq 1$.

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