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Distance magic labelings of a union of graphs

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Abstract

A 1-vertex-magic vertex labeling of a graph $G(V, E)$ with p vertices is a bijection f from the vertex set $V(G)$ to the integers $1, 2, \ldots, p$ with the property that there is a constant k such that at any vertex x the sum $\sum f(x)$ taken over all neighbors of x is k.

In this paper, we study the 1-vertex-magic vertex labelings of two families of disconnected graphs, namely a disjoint union of m copies of complete p-partite graph and a disjoint union of m copies of $2n$-regular graph $C_p[\overline{K_n}]$.

Keywords: distance magic labeling, 1-vertex-magic vertex labeling, union of graphs.

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1. Introduction

Throughout this paper, let $G(V, E)$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = p$ and $|E(G)| = q$. The degree of a vertex x, $d(x)$, is the number of edges that have x as an endpoint and the set of edges incident to x is denoted by $N_E(x)$. We use $\overline{G}$ to denote the complement of a graph G. We refer the reader to [4] or [5] for all other terms and notations not provided in this paper.

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive or non-negative integers). Specially, if we have a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ then the labeling is called a *vertex labeling*.

The 1-*vertex-weight* $w(x)$ of a vertex x in G under a vertex labeling is the sum of vertex labels of the vertices adjacent to x, i.e. the vertices on distance 1 from x.

A 1-*vertex-magic vertex labeling* of graph $G(V, E)$ with p vertices is a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ with the property that there is a constant k such that at any vertex x

$$w(x) = \sum_{y \in N_V(x)} f(y) = k,$$

where $N_V(x)$ is the set of vertices adjacent to x in V.

The definition of the 1-vertex-magic vertex labeling was introduced by Miller et al. in [2], where 1-vertex-magic vertex labeling is called *distance magic labeling*. Using similar magic condition, Sedláček [3] defined other magic labeling of a graph $G(V, E)$ as a bijection g from the edge set $E(G)$ to a set of positive integers such that

(*i*) $g(e_i) \neq g(e_j)$ for all distinct $e_i, e_j \in E(G)$, and

(*ii*) $\sum_{e \in N_E(x)} g(e)$ is the same for every $x \in V(G)$.

Miller et al. [2] completely solved the existence problem of 1-vertex-magic vertex labelings for all complete bipartite, tripartite and regular multipartite graphs and obtained some non-existence results for various families of graphs.

In the present paper we deal with 1-vertex-magic vertex labelings of two families of disconnected graphs, namely a disjoint union of m copies of complete p-partite graph and a disjoint union of m copies of $2n$-regular graph $C_p[\overline{K_n}]$.

2. Known Results

In this section we focus on known results that will subsequently be very useful. We start by a necessary condition for the existence of a 1-vertex-magic vertex labeling.

Lemma 2.1. [2] *A necessary condition for the existence of a* 1*-vertex-magic vertex labeling* f *of a graph* $G(V, E)$ *is* $k|V(G)| = \sum_{x \in V(G)} d(x)f(x)$.

If G is an r-regular graph, as most of the graphs studied in this paper, then we have a similar necessary condition $k|V(G)| = r(1 + 2 + \cdots + |V(G)|)$ or $k = \frac{r(|V(G)|+1)}{2}$.

Let us present the following families of graphs, all of which have no 1-vertex-magic vertex labeling.

Lemma 2.2. [2] *If* $G(V, E)$ *contains two vertices* x *and* y *such that* $|N_V(x) \cap N_V(y)| = d(x) - 1 = d(y) - 1$ *then* G *has no* 1*-vertex-magic vertex labeling.*

Let P_n denote the path on n vertices, C_n the cycle of length n and let the wheel W_n, $n \geq 3$, denote the graph obtained by joining all vertices of cycle C_n to a further vertex called the center.

Theorem 2.3. [2] *(i) There exists a* 1*-vertex-magic vertex labeling of* P_n *if and only if* $n \in \{1, 3\}$.
(ii) There exists a 1*-vertex-magic vertex labeling of* C_n *if and only if* $n = 4$.
(iii) There exists a 1*-vertex-magic vertex labeling of* K_n *if and only if* $n = 1$.
(iv) There exists a 1*-vertex-magic vertex labeling of* W_n *if and only if* $n = 4$.

3. Results

Let G and H be two graphs where $\{x_1, x_2, \ldots, x_p\}$ are vertices of G. Based upon the graph G, an isomorphic copy H_j of H replaces every vertex x_j, for $j = 1, 2, \ldots, p$, in such a way that each vertex in H_j is joined to all vertices corresponding to the neighbors of the original vertex x_j of G. Let $G[H]$ denote the resulting.

Theorem 3.1. *Let $r \geq 1$ and $n \geq 3$. If G is an r-regular graph and C_n the cycle of length n then $G[C_n]$ admits a 1-vertex-magic vertex labeling if and only if $n = 4$.*

Proof. Let G be an r-regular graph. Suppose, on the contrary, that $G[C_n]$ has a 1-vertex-magic vertex labeling f and $n \neq 4$. Then $G[C_n]$ contains two vertices x and y, which both belong to a j-th copy of cycle C_n, such that $|N_{V(G[C_n])}(x) \cap N_{V(G[C_n])}(y)| = d(x) - 1 = d(y) - 1$. According to Lemma 2.2 we have a contradiction.

To prove the sufficiency, suppose that G is an r-regular graph on p vertices $\{x_1, x_2, \ldots, x_p\}$ and $n = 4$. Clearly, $G[C_4]$ is the $(4r + 2)$-regular. For $1 \leq i \leq 4$ and $1 \leq j \leq p$, let $x_{i,j}$ be the vertices of $G[C_4]$ that replace x_j, $1 \leq j \leq p$ in G. Label the vertices in the following way

$$f(x_{i,j}) = \begin{cases} j, & \text{for} \quad 1 \leq j \leq p \quad \text{and} \quad i = 1 \\ 2p - j + 1, & \text{for} \quad 1 \leq j \leq p \quad \text{and} \quad i = 2 \\ 4p - j + 1, & \text{for} \quad 1 \leq j \leq p \quad \text{and} \quad i = 3 \\ 2p + j, & \text{for} \quad 1 \leq j \leq p \quad \text{and} \quad i = 4. \end{cases}$$

It is easy to verify that the labeling f uses each integer from the set $\{1, 2, \ldots, 4p\}$ exactly once. The sum of the labels of vertices in the j-th copy of C_4 is $8p+2$ which is independent of j. Additionally for $1 \leq j \leq p$

$$f(x_{1,j}) + f(x_{3,j}) = f(x_{2,j}) + f(x_{4,j}) = 4p + 1.$$

Therefore, for every $x \in G[C_4]$

$$w(x) = r(8p + 2) + (4p + 1)$$

This proves that $G[C_4]$ admits a 1-vertex-magic vertex labeling. □

In the case when G is non-regular we do not have any answer. So, we propose the following:

Problem 3.2. *If G is non-regular graph, determine if there is a 1-vertex-magic vertex labeling of $G[C_4]$.*

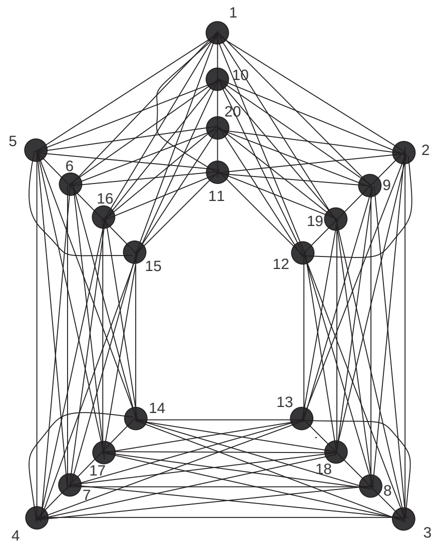

Figure 1: Labeling for $C_5[C_4]$

Let $K_{p[n]}$, $n \geq 1$ and $p \geq 1$, denote the complete p-partite graph whose every part has n vertices. Note that $K_{p[1]}$ is the complete graph on p vertices. From Theorem 2.3 it follows that $K_{p[1]}$ does not have a 1-vertex-magic vertex labeling.

We will study a disjoint union of m copies of complete p-partite graph, denoted by $mK_{p[n]}$, with the vertex set $V(mK_{p[n]}) = \bigcup_{l=1}^{m} \bigcup_{j=1}^{p} \{x_{i,j}^l : 1 \leq i \leq n\}$ and the edge set $E(mK_{p[n]}) = \bigcup_{l=1}^{m} \bigcup_{j=1}^{p-1} \bigcup_{i=1}^{n} \{x_{i,j}^l x_{s,j+t}^l : 1 \leq t \leq p-j,\ 1 \leq s \leq n\}$. Thus $|V(mK_{p[n]})| = mnp$ and every vertex in $mK_{p[n]}$ has degree $n(p-1)$.

Miller et al. [2] proved that the complete p-partite graph $K_{p[n]}$, $n > 1$ and $p > 1$, admits a 1-vertex-magic vertex labeling if and only if either n is even or both n and p are odd. The next theorem gives a result for $mK_{p[n]}$, $m > 1$.

Theorem 3.3. *(i) If n is even or mnp is odd, $m \geq 1$, $n > 1$ and $p > 1$, then $mK_{p[n]}$ has a 1-vertex-magic vertex labeling.*
(ii) If np is odd, $p \equiv 3 \pmod 4$, and m is even, then $mK_{p[n]}$ does not have a 1-vertex-magic vertex labeling.

Proof. Suppose $mK_{p[n]}$ has a 1-vertex-magic vertex labeling. According to Lemma 2.1 we have

$$k(mnp) = n(p-1)(1 + 2 + \cdots + mnp)$$

and

$$k = \frac{n(p-1)(mnp+1)}{2}. \tag{1}$$

To guarantee that k is integer, either n has to be even or p is odd or mnp must be odd. We discuss three cases.

Case 1. If n is even.

Let $f : V(mK_{p[n]}) \to \{1, 2, \ldots, mnp\}$ be a vertex labeling such that for $1 \le l \le m$ and $1 \le j \le p$

$$f(x^l_{i,j}) = \begin{cases} mp(i-1) + p(l-1) + j, & if \quad i \quad \text{is } odd, \quad 1 \le i \le n-1 \\ mpi + 1 - p(l-1) - j, & if \quad i \quad \text{is } even, \quad 2 \le i \le n. \end{cases}$$

It is a matter for routine checking to see that the sum of the vertex labels in each part of $mK_{p[n]}$ is

$$\sum_{i=1}^{n} f(x^l_{i,j}) = \frac{n(mnp+1)}{2}$$

which is independent of j and l.

Since each vertex $x^l_{i,j} \in V(mK_{p[n]})$ is adjacent to all vertices $x^l_{s,t}$, $1 \le s \le n$, $1 \le t \le p$ and $t \ne j$, i.e. to all vertices in $(p-1)$ parts, then its 1-vertex-weight is

$$w(x^l_{i,j}) = (p-1)\sum_{s=1}^{n} f(x^l_{s,t}) = (p-1)\frac{n(mnp+1)}{2} = k.$$

Therefore, under the labeling f, every vertex obtain the same 1-vertex-weight.

Case 2. If mnp is odd.

We define a matrix $A = [a_{i,t}]$ of order $n \times mp$ in the following way

$$a_{1,t} = \begin{cases} 2(t-1) + 1, & for \quad 1 \le t \le \frac{mp+1}{2} \\ 2t - (mp+1), & for \quad \frac{mp+3}{2} \le t \le mp \end{cases}$$

$$a_{i,t} = \begin{cases} imp + 1 - t, & for \quad 1 \le t \le mp \quad and \quad i \in \{2, 4, \ldots, n-1\} \\ (i-1)mp + t, & for \quad 1 \le t \le mp \quad and \quad i \in \{3, 5, \ldots, n\}. \end{cases}$$

It is easily verified that the sums of column entries in the matrix A receive consecutive integers. The aim is to construct a matrix having all the sums of column entries the same.

Therefore we construct a new matrix $B = [b_{i,t}]$ as follows

$$b_{i,t} = \begin{cases} a_{i,t}, & \textit{for} \quad 1 \le t \le mp \quad \textit{and} \quad i \ne 3 \\ a_{3,\frac{mp+3}{2}-t}, & \textit{for} \quad i = 3 \quad \textit{and} \quad 1 \le t \le \frac{mp+1}{2} \\ a_{3,\frac{3mp+3}{2}-t}, & \textit{for} \quad i = 3 \quad \textit{and} \quad \frac{mp+3}{2} \le t \le mp \end{cases}$$

where the sum of column entries of each column is $\frac{n(mnp+1)}{2}$.

Let us define the vertex labeling $g : V(mK_{p[n]}) \to \{1, 2, \ldots, mnp\}$ by the matrix B, where for $1 \le i \le n$ and $1 \le l \le m$

$$g(x_{i,j}^l) = b_{i,(l-1)p+j} \quad \textit{if} \quad 1 \le j \le p.$$

From the properties of matrix B it follows that $\sum_{i=1}^{n} g(x_{i,j}^l) = \frac{n(mnp+1)}{2}$ for every $1 \le j \le p$ and $1 \le l \le m$. Since each vertex in $mK_{p[n]}$ is adjacent to all the vertices of $(p-1)$ parts present in that copy, the 1-vertex-weight of each vertex is $k = \frac{n(p-1)(mnp+1)}{2}$, which implies that g is a required 1-vertex-magic vertex labeling.

Case 3. If np is odd, $p \equiv 3 \pmod 4$, and m is even.

From (1) it follows that k is odd. Suppose, to the contrary, that h is a 1-vertex-magic vertex labeling of $mK_{p[n]}$. We consider a l-th copy of the $K_{p[n]}$ in the graph $mK_{p[n]}$ with vertices $x_{i,j}^l$, where $1 \le i \le n$ and $1 \le j \le p$.

Let $a_j = \sum_{i=1}^{n} h(x_{i,j}^l)$ be the sum of the vertex labels in the j-th part, for $j = 1, 2, \ldots, p$. Since

$$w(x_{i,1}^l) = \sum_{j=2}^{p} a_j = k = a_1 + \sum_{j=3}^{p} a_j = w(x_{i,2}^l),$$

we have that $a_1 = a_2$. In general, it is true that $a_1 = a_2 = \cdots = a_p = a$. So, the 1-vertex-weight of each vertex of l-th copy in $mK_{p[n]}$, $1 \le l \le m$, is $(p-1)a$ which is an even number. This contradicts the fact that k is odd. □

We have tried to find a 1-vertex-magic vertex labeling of $mK_{p[n]}$ for np odd, $p \equiv 1 \pmod 4$, and m even but so far without success. So, we propose the following open problem.

Problem 3.4. *For the graph $mK_{p[n]}$, np is odd, $p \equiv 1 \pmod 4$, and m is even, $p > 1$, $m \ge 2$, determine if there is a 1-vertex-magic vertex labeling.*

Next, we will concentrate on the disjoint union of m copies of $C_p[\overline{K_n}]$. For $m \ge 1$, $p \ge 3$ and $n > 1$. Let $V(mC_p[\overline{K_n}]) = \bigcup_{l=1}^{m} \bigcup_{j=1}^{p} \{x_{i,j}^l : 1 \le i \le n\}$ be the vertex set and

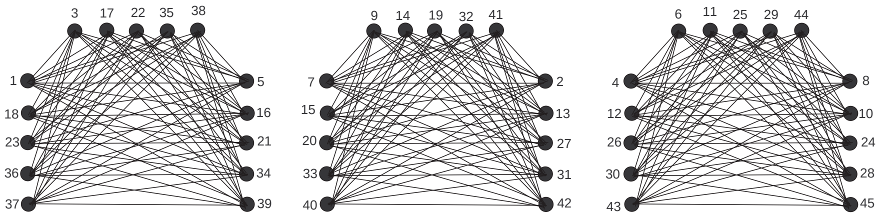

Figure 2: Labeling for $3K_{3[5]}$

$E(mC_p[\overline{K_n}]) = \bigcup_{l=1}^{m}\bigcup_{j=1}^{p-1}\bigcup_{i=1}^{n}\{x^l_{i,j}x^l_{s,j+1} : 1 \leq s \leq n\} \cup \bigcup_{l=1}^{m}\bigcup_{i=1}^{n}\{x^l_{i,p}x^l_{s,1} : 1 \leq s \leq n\}$ be the edge set of $mC_p[\overline{K_n}]$.

Theorem 3.5. *Let $m \geq 1$, $n > 1$ and $p \geq 3$. $mC_p[\overline{K_n}]$ has a 1-vertex-magic vertex labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 0$ (mod 4).*

Proof. Assume that for $m \geq 1$, $n > 1$ and $p \geq 3$ the graph $mC_p[\overline{K_n}]$ has a 1-vertex-magic vertex labeling. From Lemma 2.1, it follows that

$$k = n(mnp + 1). \tag{2}$$

We discuss the following four cases.

Case 1. If n is even.

Since the graph $mC_p[\overline{K_n}]$ has the same vertex set like the graph $mK_{p[n]}$, we can consider the labeling f from Theorem 3.3 (*Case 1.*) as a vertex labeling of the graph $mC_p[\overline{K_n}]$. According to Theorem 3.3 we have that, under the labeling f, the sum of the vertex labels in each part is equal to $\frac{n(mnp+1)}{2}$.

Each vertex $x^l_{i,j} \in V(mC_p[\overline{K_n}])$ is adjacent to all vertices $\{x^l_{i,j+1} : 1 \leq i \leq n\}$ and $\{x^l_{i,j-1} : 1 \leq i \leq n\}$, for $1 \leq l \leq m$ and $1 \leq j \leq p$, where $x^l_{i,p+1} = x^l_{i,1}$ and $x^l_{i,0} = x^l_{i,p}$. It means that the 1-vertex-weight of each vertex of $mC_p[\overline{K_n}]$ is $2\frac{n(mnp+1)}{2} = k$, which is the required property.

Case 2. If mnp is odd.

Consider the matrix $B = [b_{i,j}]$ of order $n \times mp$ from Theorem 3.3. The graph $mC_p[\overline{K_n}]$ consists of mp parts of the vertices and we can use the labeling $g : V(mC_p[\overline{K_n}]) \to \{1, 2, \ldots, mnp\}$ defined by the matrix B in Theorem 3.3.

We know that each vertex in $V(mC_p[\overline{K_n}])$ is adjacent to all the vertices of two parts in the same copy of the $C_p[\overline{K_n}]$ in the graph $mC_p[\overline{K_n}]$.

Since the sum of column entries of each column in the matrix B is $\frac{n(mnp+1)}{2}$, then

$\sum_{i=1}^{n} g(x_{i,j}^{l}) = \frac{n(mnp+1)}{2}$, for every $1 \leq j \leq p$ and $1 \leq l \leq m$, and the 1-vertex-weight of each vertex of the graph $mC_p[\overline{K_n}]$ is $k = n(mnp+1)$ which implies that g is a desired 1-vertex-magic vertex labeling.

Case 3. If n is odd and $p \equiv 0 \pmod 4$.

We construct a matrix $\mathbb{A} = [a_{i,t}]$ of order $n \times mp$ in the following way

$$a_{1,t} = \begin{cases} 2(t-1)+1, & \text{for} \quad 1 \leq t \leq \frac{mp}{2} \\ 2t - mp, & \text{for} \quad \frac{mp}{2}+1 \leq t \leq mp \end{cases}$$

$$a_{i,t} = \begin{cases} imp + 1 - t, & \text{for} \quad 1 \leq t \leq mp \quad \text{and} \quad i \in \{2,4,\ldots,n-1\} \\ (i-1)mp + t, & \text{for} \quad 1 \leq t \leq mp \quad \text{and} \quad i \in \{3,5,\ldots,n\}. \end{cases}$$

The sums of column entries in matrix $\mathbb{A}$ receive consecutive integers. To obtain the adequate sums we construct a new matrix $\mathbb{B} = [b_{i,t}]$ as follows

$$b_{i,t} = \begin{cases} a_{i,t}, & \text{for} \quad 1 \leq t \leq mp \quad \text{and} \quad i \neq 3 \\ a_{3,\frac{mp+2}{2}-t}, & \text{for} \quad i = 3 \quad \text{and} \quad 1 \leq t \leq \frac{mp}{2} \\ a_{3,\frac{3mp+2}{2}-t}, & \text{for} \quad i = 3 \quad \text{and} \quad \frac{mp}{2}+1 \leq t \leq mp. \end{cases}$$

It can be seen that $\sum_{i=1}^{n} b_{i,t} = \lfloor \frac{k}{2} \rfloor$ for all $1 \leq t \leq \frac{mp}{2}$ and $\sum_{i=1}^{n} b_{i,t} = \lceil \frac{k}{2} \rceil$ for all $\frac{mp}{2}+1 \leq t \leq mp$.

Let us define the vertex labeling $f : V(mC_p[\overline{K_n}]) \to \{1,2,\ldots,mnp\}$ by the matrix $\mathbb{B}$, where for $1 \leq i \leq n$ and $1 \leq l \leq m$

$$f(x_{i,1}^{l}) = b_{i,1+(l-1)\frac{p}{2}}$$

$$f(x_{i,2}^{l}) = b_{i,2+(l-1)\frac{p}{2}}$$

$$f(x_{i,3}^{l}) = b_{i,1+(m+l-1)\frac{p}{2}}$$

$$f(x_{i,4}^{l}) = b_{i,2+(m+l-1)\frac{p}{2}}$$

$$f(x_{i,5}^{l}) = b_{i,3+(l-1)\frac{p}{2}}$$

$$f(x_{i,6}^{l}) = b_{i,4+(l-1)\frac{p}{2}}$$

$$f(x_{i,7}^{l}) = b_{i,3+(m+l-1)\frac{p}{2}}$$

$$f(x_{i,8}^{l}) = b_{i,4+(m+l-1)\frac{p}{2}}$$

$$\cdots$$

$$f(x^l_{i,p-3}) = b_{i,\frac{p}{2}-1+(l-1)\frac{p}{2}}$$

$$f(x^l_{i,p-2}) = b_{i,\frac{p}{2}+(l-1)\frac{p}{2}}$$

$$f(x^l_{i,p-1}) = b_{i,\frac{p}{2}-1+(m+l-1)\frac{p}{2}}$$

$$f(x^l_{i,p}) = b_{i,\frac{p}{2}+(m+l-1)\frac{p}{2}}.$$

Every vertex $x^l_{i,j}$ is adjacent to each vertex in $(j-1)$-th part and also to each vertex in $(j+1)$-th part, where the sum of the vertex labels in $(j-1)$-th part is $\lfloor \frac{k}{2} \rfloor$ (respectively $\lceil \frac{k}{2} \rceil$) and the sum of the vertex labels in $(j+1)$-th part is $\lceil \frac{k}{2} \rceil$ (respectively $\lfloor \frac{k}{2} \rfloor$).

Thus, the 1-vertex-weight of each vertex is

$$w(x^l_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil \quad for \quad 1 \le i \le n,\ 1 \le j \le p \quad and \quad 1 \le l \le m.$$

Case 4. If n is odd and $p \equiv 2 \pmod 4$.

Suppose that h is a 1-vertex-magic vertex labeling of $mC_p[\overline{K_n}]$. Let $a_j = \sum_{i=1}^{n} h(x^l_{i,j})$ be the sum of the vertex labels in the j-th part, for $j = 1, 2, \ldots, p$.

Since $w(x^l_{i,2}) = a_1 + a_3 = k$ and $w(x^l_{i,4}) = a_3 + a_5 = k$, for every $1 \le i \le n$, there is $a_1 = a_5$. Similarly we can show that $a_5 = a_9 = \cdots = a_{p-1}$. The 1-vertex-weight of each vertex $x^l_{i,p}$, for $1 \le i \le n$, $1 \le l \le m$, is $w(x^l_{i,p}) = a_1 + a_{p-1} = k = 2a_1$, which is a contradiction to the fact that for n odd and $p \equiv 2 \pmod 4$ the constant k is odd. In this case the graph $mC_p[\overline{K_n}]$ does not have a 1-vertex-magic vertex labeling. □

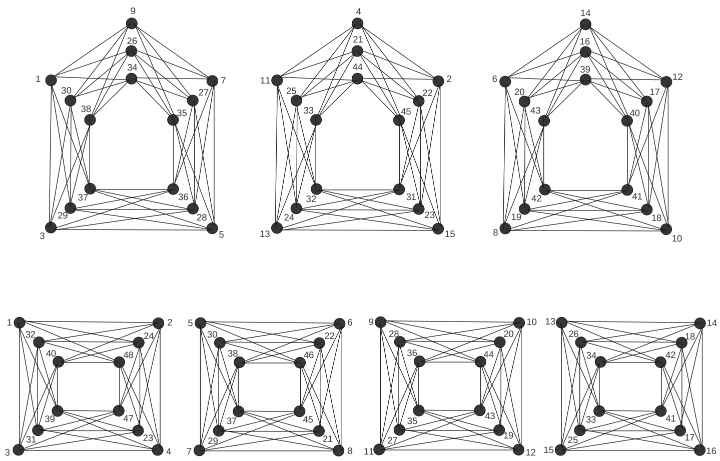

Figure 3: Labelings for $3C_5[\overline{K_3}]$ and $4C_4[\overline{K_3}]$.

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References

[1] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461.

[2] M. Miller, C. Rodger, and R. Simanjuntak, On distance magic graphs, *Australas. J. Combin.*, **28** (2003), 305–315.

[3] J. Sedláček, Problem 27, *In: Theory and Its Applications, Proc. Symp. Smolenice*, 1963, 163–169.

[4] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston - Basel - Berlin, 2001.

[5] D.B. West, *An Introduction to Graph Theory*, Prentice-Hall, 1996.