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On Radio $(n-4)$-chromatic Number of the Path P_n

Srinivasa Rao Kola and Pratima Panigrahi
Department of Mathematics
Indian Institute of Technology Kharagpur
Kharagpur 721302, India
e-mail: *srinu.iitkgp@gmail.com, pratima@maths.iitkgp.ernet.in*

Abstract

For a path P_n of order n, Chartrand et al. [3] have given an upper bound for radio k-chromatic number when $1 \leq k \leq n-1$. Liu and Zhu [7] have determined the exact value of radio $(n-1)$-chromatic number of P_n, namely *radio number*, $rn(P_n)$, when $n \geq 3$. Khennoufa and Togni [5] have given the exact value of radio $(n-2)$-chromatic number of P_n, namely *antipodal number*, $ac(P_n)$, when $n \geq 5$. Kola and Panigrahi [6] have given the exact value of radio $(n-3)$-chromatic number of P_n, namely *nearly antipodal number*, $ac'(P_n)$, when $n \geq 8$. In this paper, we give the exact value of radio $(n-4)$-chromatic number of P_n, $rc_{n-4}(P_n)$, when n is odd and $n \geq 11$. Consequently, the lower bound of $rc_{n-4}(P_{n+i})$, $n \geq 11$ and $i \geq 1$ is improved. We also improve the upper bound of $rc_{n-4}(P_n)$ when n is even and $n \geq 12$.

Keywords: radio k-coloring, Span of a radio k-coloring, Radio k-chromatic number.

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1. Introduction

The radio k-coloring of a graph was defined by Chartrand et al. in [2, 3]. For any connected graph G and any positive integer k, a *radio k-coloring* is an assignment f of positive integers to the vertices of G such that $|f(u) - f(v)| \geq 1 + k - d(u,v)$ for every two distinct vertices u and v of G. The maximum positive integer assigned by f is called the span of f, denoted by $rc_k(f)$. The minimum span of all radio k-colorings is called the *radio k-chromatic number* of G, denoted by $rc_k(G)$. The radio k-coloring having the span $rc_k(G)$ is called a *minimal radio k-coloring*. For some special values of k there are special names of radio k-coloring and as well as radio k-chromatic number in the literature which are given below.

k	*Name of coloring*	$rc_k(G)$
1	Usual coloring	Chromatic number $\chi(G)$
$diam(G)$	Radio coloring [1]	Radio number $rn(G)$
$diam(G)-1$	Antipodal coloring	Antipodal number $ac(G)$
$diam(G)-2$	Nearly antipodal coloring	Nearly antipodal number $ac'(G)$

Because radio k-coloring number of paths for $k \geq n$ will be helpful to find radio k-chromatic number of graphs with bigger diameter like cartesian product of graphs, Kchikech et al. [4] have given exact value of radio k-chromatic number of P_n for $k \geq n$ as below.

Theorem 1.1. [4] *For any integer $k \geq n$,*

$$rc_k(P_n) = \begin{cases} (n-1)k - \frac{1}{2}n(n-2) + 1, & if\ n\ is\ even \\ (n-1)k - \frac{1}{2}(n-1)^2 + 2, & if\ n\ is\ odd. \end{cases}$$

In theorems below, the exact value of $rn(P_n)$, $ac(P_n)$ and $ac'(P_n)$ are given, which are determined by Liu and Zhu [7], Khennoufa and Togni [5], and Kola and Panigrahi [6] respectively.

Theorem 1.2. [7] *For any integer $n \geq 3$,*

$$rn(P_n) = \begin{cases} 2p^2 - 2p + 2, & if\ n = 2p \\ 2p^2 + 3, & if\ n = 2p+1. \end{cases}$$

Theorem 1.3. [5] *For any integer $n \geq 5$,*

$$ac(P_n) = \begin{cases} 2p^2 - 4p + 5, & if\ n = 2p \\ 2p^2 - 2p + 3, & if\ n = 2p+1. \end{cases}$$

Theorem 1.4. [6] *For any integer $n \geq 8$,*

$$ac'(P_n) = \begin{cases} 2p^2 - 6p + 8, & if\ n = 2p \\ 2p^2 - 4p + 6, & if\ n = 2p+1. \end{cases}$$

In [6], Kola and Panigrahi have improved the existing lower bound of $rc_k(P_n)$ for $5 \leq k \leq n-4$ and $n \geq 9$. In this paper we give the exact value of the radio $(n-4)$-chromatic number, $rc_{n-4}(P_n)$ of P_n for n odd and $n \geq 11$. Consequently, the lower bound of $rc_{n-4}(P_{n+i})$, $n \geq 11$ and $i \geq 1$ is improved. We also improve the upper bound of $rc_{n-4}(P_n)$ when n is even and $n \geq 12$.

2. Upper bound

We first give an upper bound for $rc_{n-4}(P_n)$, $n \geq 11$, by defining radio $(n-4)$-colorings of P_n.

Theorem 2.1. *For any integer $n \geq 11$,*

$$rc_{n-4}(P_n) \leq \begin{cases} 2p^2 - 8p + 13, & if\ n = 2p \\ 2p^2 - 6p + 9, & if\ n = 2p+1. \end{cases}$$

Proof. Let P_n be the path $a_1\ a_2\ \ldots a_n$.
Case 1. In this case we take n even, say $n = 2p$.

Define a map f as

$$\begin{aligned}
f(a_{1+i}) &= p + i(2p-3), \quad 0 \le i \le 1,\\
f(a_3) &= 2p^2 - 9p + 15,\\
f(a_{4+j}) &= 5(p-1) + j(2p-3), \quad 0 \le j \le p-6,\\
f(a_{p-1+l}) &= l(2p-3) + 1, \quad 0 \le l \le 2,\\
f(a_{p+2}) &= 2p^2 - 8p + 13,\\
f(a_{p+3+m}) &= 3(2p-4) + 5 + m(2p-3), \quad 0 \le m \le p-6,\\
f(a_{2p-2+q}) &= p - 1 + q(2p-3), \quad 0 \le q \le 2.
\end{aligned}$$

We check the distance condition between the vertices a_{4+j} and a_{p+3+m}, $0 \le j, q \le p-6$ as checking the same for the other vertices is similar or easy.

For $0 \le j, q \le p-6$, $|f(a_{4+j}) - f(a_{p+3+m})| = |5(p-1) + j(2p-3) - (3(2p-3) + 5 + m(2p-3))| = |(j-m)(2p-3) - (p-2)| \ge p-2-(m-j) = 1+(n-4)-d(a_{4+j}, a_{p+3+m})$, if $m \ge j$. For $m < j$, $|(j-m)(2p-3)-(p-2)| = |(p-2)+(j-m)+(j-m)(2p-4)-2(p-2)| \ge p-2+(j-m) = 1+(n-4)-d(a_{4+j}, a_{p+3+m})$ as $(j-m)(2p-4) - 2(p-2) \ge 0$.

Therefore f is a radio $(n-4)$-coloring and $rc_{n-4}(P_n) \le 2p^2 - 8p + 13$ when $n = 2p$.

Case 2. In this case we take n odd, say $n = 2p+1$.

Define a map g as

$$\begin{aligned}
g(a_1) &= p,\\
g(a_2) &= 2p^2 - 7p + 11,\\
g(a_{3+i}) &= 3p - 2 + i(2p-3), \quad 0 \le i \le p-4,\\
g(a_{p+j}) &= j(2p-3) + 1, \quad 0 \le j \le 1,\\
g(a_{p+2}) &= 2p^2 - 6p + 9,\\
g(a_{p+3+l}) &= (l+2)(2p-3) + 2, \quad 0 \le l \le p-4,\\
g(a_{2p+m}) &= p - 1 + m(2p-3), \quad 0 \le m \le 1.
\end{aligned}$$

We check the distance condition between the vertices a_{3+i} and a_{p+3+l}, $0 \le i, l \le p-4$ as checking the same for the other vertices is similar or easy. For $0 \le i, l \le p-4$, $|g(a_{3+i}) - g(a_{p+3+l})| = |3p-2+i(2p-3) - ((l+2)(2p-3)+2)| = |(i-l)(2p-3)-(p-2)| \ge p-2-(l-i) = 1+(n-4)-d(a_{3+i}, a_{p+3+l})$, if $l \ge i$. For $l < i$, $|(i-l)(2p-3)-(p-2)| = |(p-2)+(i-l)+(i-l)(2p-4)-2(p-2)| \ge p-2+(i-l) = 1+(n-4)-d(a_{3+i}, a_{p+3+l})$ as $(i-l)(2p-4) - 2(p-2) \ge 0$.

Therefore g is a radio $(n-4)$-coloring and $rc_{n-4}(P_n) \le 2p^2-6p+9$ when $n = 2p+1$. □

Example 2.2. *Here we illustrate Theorem 2.1 by giving radio $(n-4)$-colorings for* $n = 11, 12$.

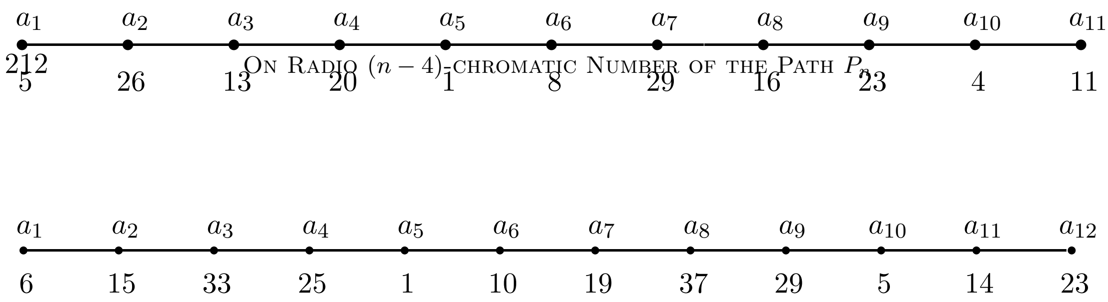

Figure 1

3. Lower bound

In this section, we prove that the upper bound of $rc_{n-4}(P_n)$ (i.e $2p^2-6p+9$), $n=2p+1$, $n \geq 11$, obtained in the previous section is also a lower bound of the same, and hence get the exact value.

The following theorem is a modification of the proof of finding the distance sum in Theorem 3 by Liu and Zhu in [7].

Theorem 3.1. *Let f be any radio k-coloring of a path P_n: $a_1\ a_2 \ldots a_n$, $n = 2p+1$ and $x_1, x_2, \ldots, x_n$ be an ordering of the vertices of P_n such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$ with $\sum_{i=2}^{n} d(x_i, x_{i-1}) = 2p^2+2p-2$. Then any one of $\{f(a_{p-1}), f(a_{p+1})\}$, $\{f(a_{p+1}), f(a_{p+3})\}$ or $\{f(a_p), f(a_{p+2})\}$ is equal to $\{0, f(x_n)\}$.*

Proof. Let $x_i = a_{\sigma(i)}$, $1 \leq i \leq n$. Then σ is a permutation of $\{1, 2, 3, \ldots, n\}$. Note that $d(x_i, x_{i-1})$ is equal to either $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive. By replacing each term $d(x_i, x_{i-1})$ in $\sum_{i=2}^{n} d(x_i, x_{i-1})$ with the corresponding $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, 3, \ldots, n\}$. All together there are $2(n-1)$ terms of the form $\pm j$ in the summation $\sum_{i=2}^{n} d(x_i, x_{i-1})$, half of them positive and half negative, and exactly two from $\{1, 2, \ldots, 2p+1\}$ occur only once and each of the remaining occur twice. To maximize the summation $\sum_{i=2}^{n} d(x_i, x_{i-1})$, one needs to minimize the absolute values for negative terms and maximize the values of positive terms. It is easy to verify that the following are the only possibilities achieving the maximum summation (see [7]).

(i) Each of the numbers in $\{p+2, p+3, p+4, \ldots, 2p+1\}$ occurs twice with a positive sign, each of $\{1, 2, 3, \ldots, p-1\}$ occurs twice with a negative sign, and each of p and $p+1$ occurs once with a negative sign. That is,

$$\begin{aligned} &\{(p+2)+(p+2)+(p+3)+(p+3)+\cdots+(2p+1)+(2p+1)\} \\ -&\{(1)+(1)+(2)+(2)+\cdots+(p-1)+(p-1)+(p)+(p+1)\} = 2p^2+2p-1. \quad (1) \end{aligned}$$

Since both p and $p+1$ occur only once in the summation, we get

$$\{f(x_1), f(x_{2p+1})\} = \{f(a_p), f(a_{p+1})\}. \tag{2}$$

(ii) Each of the numbers in $\{p+3, p+4, \ldots, 2p+1\}$ occurs twice with a positive sign, each of $\{1, 2, 3, \ldots, p\}$ occurs twice with a negative sign, and each of $p+1$ and $p+2$ occurs once with a positive sign. That is,

$$\{(p+1)+(p+2)+(p+3)+(p+3)+(p+4)+(p+4)+\cdots+(2p+1)+(2p+1)\} \\ -\{(1)+(1)+(2)+(2)+\cdots+(p)+(p)\} = 2p^2+2p-1. \tag{3}$$

Since both p and $p+2$ occur only once in the summation, we get

$$\{f(x_1), f(x_{2p+1})\} = \{f(a_{p+1}), f(a_{p+2})\}. \tag{4}$$

Equations (1) and (3) are of the form $\{b_1+b_2+\cdots+b_{2p}\}-\{c_1+c_2+\cdots+c_{2p}\} = S = 2p^2+2p-1$, where c_{2p-1}, c_{2p} occur only once in the equation (1) and b_1, b_2 occur only once in the equation (3). Let $S_1 = \sum_{i=1}^{2p} b_i$ and $S_2 = \sum_{j=1}^{2p} c_j$. Observe that reduction of S is possible only in the following cases.

1. *S_1 decreases and S_2 unchanged:* In this case, the following possibilities arise for equation (1).

   (i) if we replace b_i, for some i, $1 \le i \le 2p$, by c_j that occur only once (i.e p or $p+1$), then the reduction in S is $b_i - c_j$. If $c_j = b_i - l$, then the reduction of S is equal to l. Therefore, if we replace $(p+2)$ by $(p+1)$, we get reduction 1 in S and if we replace $p+2$ by p or $p+3$ by $p+1$, we get reduction 2 in S.

   (ii) if we replace b_{i_1} and b_{i_2}, $b_{i_1} \ne b_{i_2}$, $1 \le i_1, i_2 \le 2p$, by c_{2p-1} and c_{2p} (i.e p and $p+1$) respectively, then the reduction is $(b_{i_1}-c_{2p-1})+(b_{i_2}-c_{2p})$. Here reduction in S is at least 3.

   Possibilities for (3) are as follows.

   (iii) if we replace b_i, for some i, $3 \le i \le 2p$, by b_j (i.e $p+1$ or $p+2$) that occurs only once, then the reduction in S is $b_i - b_j$. Therefore, if we replace $p+3$ by $p+2$, we get reduction in S is 1 and if we replace $p+3$ by $p+1$ or $p+4$ by $p+2$, we get reduction in S is 2.

   (iv) if we interchange b_1 with b_{i_1}, and b_2 with b_{i_2} where $b_{i_1} \ne b_{i_2}$, $3 \le i_1, i_2 \le 2p$, then the reduction in S is $(b_{i_1} - b_1) + (b_{i_2} - b_2)$. Here reduction in S is at least 3.

(v) if we replace $2b_i$, for some i, $3 \leq i \leq 2p$, (since b_i occurs twice) by $2b_j$, b_j occur only once (i.e $(p+1)$ or $(p+2)$) and replace b_j by b_i, then the reduction in S is $2(b_i - b_j) - (b_i - b_j) = b_i - b_j$. Therefore, if $b_i = p+3$ and $b_j = p+2$, we get reduction in S is 1. If $b_i = p+3$ and $b_j = p+1$ or $b_i = p+4$ and $b_j = p+2$, we get reduction in S is 2.

2. S_1 *unchanged and* S_2 *increases:* In this case, the following possibilities arise for equation (1).

(i) if we replace c_i, for some i, $1 \leq i \leq 2p-2$, by c_j (i.e p or $p+1$) that occur only once, then the reduction in S is $c_j - c_i$. Therefore, if we replace $p-1$ by p, we get reduction in S is 1 and if we replace $p-2$ by p or $p-1$ by $p+1$, we get reduction in S is 2.

(ii) if we interchange c_{2p-1} with c_{i_1}, and c_{2p} with c_{i_2} where $c_{i_1} \neq c_{i_2}$, $1 \leq i_1, i_2 \leq 2p-2$, then the reduction in S is $(c_{i_1} - c_{2p-1}) + (c_{i_2} - c_{2p})$. Here reduction in S is at least 3.

(iii) if we replace $2c_i$, for some i, $1 \leq i \leq 2p-2$, by $2c_j$, c_j occur only once (i.e (p) or $(p+1)$) and replace c_j by c_i. Then the reduction in S is $c_j - c_i$. Therefore, if we replace $p-1$ by p, we get reduction in S is 1 and if we replace $p-2$ by p or $p-1$ by $p+1$, we get reduction in S is 2.

Possibilities for equation (3) are as follows.

(iv) if we replace c_i, for some i, $1 \leq i \leq 2p$, by b_j that occur only once (i.e $p+1$ or $p+2$), then the reduction in S is $b_j - c_i$. If $c_i = b_j - l$, then the reduction in S is equal to l. Therefore, if we replace (p) by $(p+1)$, we get reduction in S is 1 and if we replace p by $p+2$ or $p-1$ by $p+1$, we get reduction in S is 2.

(v) if we replace c_{i_1} and c_{i_2}, $c_{i_1} \neq c_{i_2}$, $1 \leq i_1, i_2 \leq 2p$, by b_1 and b_2 (i.e $p+1$ and $p+2$) respectively, then the reduction in S is $(b_1 - c_{i_1}) + (b_2 - c_{i_2})$. Here reduction in S is at least 3.

3. S_1 *decreases by* S_1' *and* S_2 *decreases by* S_2', $S_1' > S_2'$*:* This case is possible for equation (1), if we replace $p+1$ by p and replace b_i, for some i, $3 \leq i \leq 2p$, by $p+1$. Then the reduction in S is $b_i - (p+1) - 1$. Therefore, if $b_i = p+3$, we get reduction in S is 1 and if $b_i = p+4$, we get reduction in S is 2. This case is not possible in equation (3) because S_2 cannot decrease further.

4. S_1 *increases by* S_1' *and* S_2 *increases by* S_2', $S_2' > S_1'$*:* This case is not possible in equation (1). In equation (3), this case is possible if we replace $p+1$ by $p+2$ and replace c_j for some i, $1 \leq j \leq 2p-2$, by $p+1$. Then the reduction in S is $p+1-c_j-1$. Therefore, if $c_j = p-1$, we get reduction in S is 1 and if $c_j = p-2$, we get reduction in S is 2.

5. S_1 *decreases and* S_2 *increases:* Here reduction in S is at least 2. In both the equations (1) and (3), this case is possible if we interchange b_i and c_j, $1 \leq i, j \leq 2p$. Then the

reduction is $(b_i - c_j) + (b_i - c_j) = 2(b_i - c_j)$. If $c_j = b_i - l$, then the reduction in S is equal to $2l$. Therefore in equation (1), if we interchange $p+1$ and $p+2$ and in equation (3), if we interchange p and $p+1$, we get reduction in S is 2. If we interchange more than one pair, then the reduction is more than 2.

Here we are interested for the case where reduction in S is equal to 1. Notice that in case 1(i) S reduces by one and since p and $p+2$ occur only once, we get $\{f(a_p), f(a_{p+2})\} = \{0, f(x_n)\}$. Similarly from the cases 1(iii), 1(v), 2(i), 2(iii), 2(iv), 3 and 4, we get $\{f(a_{p-1}), f(a_{p+1})\}$ or $\{f(a_{p+1}), f(a_{p+3})\} = \{0, f(x_n)\}$. $\square$

Theorem 3.2. *Let f be any radio k-coloring of a path P_n: $a_1\ a_2\ \dots a_n$, $n = 2p+1$ and $x_1, x_2, \dots, x_n$ be an ordering of vertices of P_n such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$ with $\sum_{i=2}^{n} d(x_i, x_{i-1}) = 2p^2 + 2p - 3$. Then any of $\{f(a_{p+1}), f(a_{p+2})\}$, $\{f(a_p), f(a_{p+3})\}$, $\{f(a_{p+2}), f(a_{p+3})\}$, $\{f(a_{p+1}), f(a_{p+4})\}$, $\{f(a_{p-2}), f(a_{p+1})\}$, $\{f(a_{p-1}), f(a_p)\}$, $\{f(a_{p-1}), f(a_{p+2})\}$ or $\{f(a_p), f(a_{p+1})\}$ is equal to $\{0, f(x_n)\}$.*

Proof. By looking at the cases in the proof of Theorem 3.1 where reduction in S is equal to 2, we get the result. $\square$

Notation 3.3. *For any radio k-coloring f of a path P_n and an ordering $x_1, x_2, \dots, x_n$ of vertices of P_n with $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$, we define $\epsilon_i = (f(x_i) - f(x_{i-1})) - (1 + k - d(x_i, x_{i-1}))$, $2 \leq i \leq n$. It is clear from the definition of a radio k-coloring that $\epsilon_i \geq 0$, $2 \leq i \leq n$.*

Theorem 3.4. *For any odd integer $n \geq 11$, $n = 2p+1$, $rc_{n-4}(P_n) \geq 2p^2 - 6p + 9$.*

Proof. Let P_n: $a_1\ a_2\ a_3 \dots a_{n-1}\ a_n$. Let f be a minimal radio $(n-4)$-coloring of P_n. Let $x_1, x_2, \dots, x_n$ be an ordering of vertices of P_n such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$. Then

$$\begin{aligned} f(x_n) - f(x_1) &= \sum_{i=2}^{n} (f(x_i) - f(x_{i-1})) \\ &= \sum_{i=2}^{n} (1 + (n-4) - d(x_i, x_{i-1}) + \epsilon_i) \\ &= (n-1)(1 + (n-4)) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i. \end{aligned}$$

$$f(x_n) = (n-1)(1 + (n-4)) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \epsilon_i + 1. \tag{5}$$

Suppose that $f(x_n) = 2p^2-6p+8$. Consider the coloring f of $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}$. Now f is a radio (n)-coloring of $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}$ and the number of vertices in $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}$ is $2(p-2)+1$. So, by Theorem 1.1, the span of f on $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}$ is greater than or equal to $(2(p-2)+1-1)(2(p-2)+1) - \frac{1}{2}(2(p-2)+1-1)^2+2 = 2p^2-6p+6$. That is the span of f on $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}\}$ is either $2p^2-6p+6$ or $2p^2-6p+7$ or $2p^2-6p+8$. If we substitute $n = 2(p-2)+1$, $\sum_{i=2}^{n} d(x_i, x_{i-1}) = 2p^2+2p-1$ and $f(x_n) = 2p^2-6p+8$ in (5), we get $\sum_{i=2}^{n} \epsilon_i = 2$. So the coloring f of $P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}$ has the distance summation $2(p-1)^2+2(p-1)-1$ or $2(p-1)^2+2(p-1)-2$ or $2(p-1)^2+2(p-1)-3$. From equations (2), (4) and Theorems 3.1, 3.2, we have any of

$$\begin{aligned}
&\{f(a_{p-2}), f(a_{p-1})\}, \{f(a_{p-1}), f(a_p)\},\ \{f(a_{p-3}), f(a_{p-1})\},\ \{f(a_{p-1}), f(a_{p+1})\},\\
&\{f(a_{p-2}, f(a_p))\},\ \{f(a_{p-1}), f(a_p)\},\ \{f(a_{p-2}), f(a_{p+1})\},\ \{f(a_p), f(a_{p+1})\},\\
&\{f(a_{p-1}), f(a_{p+2})\},\ \{f(a_{p-4}), f(a_{p-1})\},\ \{f(a_{p-3}), f(a_{p-2})\},\ \{f(a_{p-3}), f(a_p)\} \text{ or}\\
&\{f(a_{p-2}), f(a_{p-1})\} \text{ is equal to} \{0,\ \text{span of } f \text{ on } P_n \setminus \{a_{2p-2}, a_{2p-1}, a_{2p}, a_{2p+1}\}\} \qquad (6)
\end{aligned}$$

Similarly as above if we consider the coloring f of $P_n \setminus \{a_1, a_2, a_3, a_4\}$, we get

$$\begin{aligned}
&\{f(a_{p+2}), f(a_{p+3})\},\ \{f(a_{p+3}), f(a_{p+4})\},\ \{f(a_{p+1}), f(a_{p+3})\},\ \{f(a_{p+3}), f(a_{p+5})\},\\
&\{f(a_{p+2}, f(a_{p+4}))\},\ \{f(a_{p+3}), f(a_{p+4})\},\ \{f(a_{p+2}), f(a_{p+5})\},\ \{f(a_{p+4}), f(a_{p+5})\},\\
&\{f(a_{p+3}), f(a_{p+6})\},\ \{f(a_p), f(a_{p+3})\},\ \{f(a_{p+1}), f(a_{p+2})\},\ \{f(a_{p+1}), f(a_{p+4})\} \text{ or}\\
&\{f(a_{p+2}), f(a_{p+3})\}, \text{ is equal to } \{0, \text{span of } f \text{ on } P_n \setminus \{a_1, a_2, a_3, a_4\}\} \qquad (7)
\end{aligned}$$

Since there is no common set of colors (positive integers) on the left hand side of the equations (6) and (7) and $p \geq 5$, we will get a contradiction because the colors with difference 1 should be of at least $k = 2p-3$ distance apart or the colors with difference 2 should be of at least $k-1 = 2p-4$ distance apart or a color c can be repeated only if it is at least $k+1 = 2p-2$ distance apart from the previous c color. Therefore $rc_{n-4}(P_n) \geq 2p^2-6p+9$ when $n = 2p+1$. □

From Theorem 2.1 and Theorem 3.4, we summarize the main result of the paper.

Theorem 3.5. *For any odd integer* $n \geq 11$, $n = 2p+1$, $rc_{n-4}(P_n) = 2p^2-6p+9$.

As an application of Theorem 3.5, in the following, we obtained an improved lower bound of $rc_{n-4}(P_{n+i})$, $i \geq 1$ and for any odd integer $n \geq 11$.

Theorem 3.6. *For any two integers* $i \geq 1$ *and* $n \geq 11$, $rc_{n-4}(P_{n+i}) \geq 2p^2-6p+9$, *if* $n = 2p+1$.

Proof. For integers m and n, $m < n$, we get $rc_k(P_n) \geq rc_k(P_m)$ (because by deleting $(n-m)$ number of vertices from any end of $f(P_n)$, where f is a radio k-coloring of P_n, one gets a radio a radio k-coloring of P_m). Now from Theorem 3.5, we have $rc_{n-4}(P_n) = 2p^2 - 6p + 9$, if $n \geq 11$, $n = 2p + 1$. Therefore for $i \geq 1$ and $n \geq 11$, $rc_{n-4}(P_{n+i}) \geq rc_{n-4}(P_n) = 2p^2 - 6p + 9$, if $n = 2p + 1$. $\square$

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