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Total labellings, irregularity strengths and colourings

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Abstract

Let $G = (V, E)$ be a simple graphs with the vertex set V and edge set E. For a positive integer k a total k-labelling is a map $\varphi : V \cup E \to \{1, 2, \ldots, k\}$. The weight of a vertex x under the total labelling φ is $\mathrm{wt}(x) = \varphi(x) + \sum_{y \in N(x)} \varphi(xy)$ and the weight of an edge xy under φ is $\mathrm{wt}(xy) = \varphi(x) + \varphi(xy) + \varphi(y)$.

We are interested in determining the smallest positive integer k such that there is a total k-labelling φ fulfilling one of the following possibilities

(i) $\mathrm{wt}(e) \neq \mathrm{wt}(f)$ for every two district edges of E, or

(ii) $\mathrm{wt}(e) \neq \mathrm{wt}(f)$ for every two adjacent edges of E, or

(iii) $\mathrm{wt}(x) \neq \mathrm{wt}(y)$ for every two district vertices of V, or

(iv) $\mathrm{wt}(x) \neq \mathrm{wt}(y)$ for every two adjacent vertices of V.

We survey recent results and open questions for any of the above four cases.

Keywords: total edge irregularity strength, total vertex irregularity strength, edge colouring total labelling, vertex colouring total labelling, irregularity strength.

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1. Introduction

All graphs considered in this paper are simple and finite.

A *labelling* of a graph is a map that carries graph elements to the number (usually to the positive or non–negative integers). The most common choices of domain are the set of all vertices (*vertex* labellings), the edge set alone (*edge* labellings), or the set of all vertices and edges (*total* labellings). Other domains are possible. The most complete recent survey of graph labellings is that of Gallian [10].

In many cases it is interesting to consider the sum of all labels associated with a graph elements. This will be called the weight of the elements. As in the study of magic total labellings, see e.g. a recent book of Wallis [21], the *weight of a vertex* x under a total labelling σ of elements of a graph $G = (V, E)$ is

$$\mathrm{wt}(x) = \sigma(x) + \sum_{y \in N(x)} \sigma(xy) \tag{1}$$

and the *weight of an edge* $e = xy$ is

$$\mathrm{wt}(e) = \sigma(x) + \sigma(xy) + \sigma(y)\,. \tag{2}$$

Note that here and in the sequel for given graph G and its vertex v, $N_G(v)$, $\deg_G(v)$, $V(G)$, $E(G)$, $\sigma(G)$ and $\Delta(G)$ (or simply $N(v)$, $\deg(v)$, V, E, δ and Δ if G is known from the context) denote the set of *neighbours* and the *degree* of v in G, the *set of vertices*, the *set of edges*, the *minimum degree* and the *maximum degree* of G, respectively. Further, we put $|V(G)| = p$, $|E(G)| = q$, and $\{1, 2, \ldots, k\} = [k]$.

In this paper we give a survey on four problems that have appeared recently in connection with total labellings of graphs.

2. Motivations

Motivation for the research mentioned throughout this paper has been an intensive work on the irregularity strength of graphs. For a positive integer k, an (edge) k-labelling of G is a function $\varphi : E \to [k]$. We call $w(e) = \varphi(e)$ the *weight* of an edge $e \in E$. The weight of a vertex $v \in V$ is defined to be $w(v) = \sum_{u \in N(v)} \varphi(uv)$. The greatest value of φ is called the *strength* of φ. An (edge) k-labelling is *irregular* if the obtained weights of all vertices are distinct. The smallest strength of an irregular k-labelling is called the *irregularity strength* of G and is denoted by $s(G)$. If it does not exist, we write $s(G) = \infty$. It is easy to see that $s(G) = +\infty$ iff G contains no K_2 and at most one K_1.

The irregularity strength was introduced by Chartrand et al. [7] and was motivated by the well known fact that a simple graph of order at least 2 contains a pair of vertices with the same degree. On the other hand, a multigraph can be *irregular*, i.e. the degrees of its vertices can all be distinct. Suppose one wants to multiply the edges of a graph G in order to create an irregular multigraph of it. Then $s(G)$ is equal to the smallest maximum multiplicity of an edge in such multigraph.

Aigner and Triesch [2] proved that $s(G) \leq p - 1$ if G is connected, $G \neq K_3$, and $s(G) \leq p + 1$ otherwise. Nierhoff [16] showed that $s(G) \leq p - 1$ for all graphs with $s(G) < +\infty$, $G \neq K_3$. The bound is tight e.g. for stars. For very nice surveys on this parameter see Lehel [14], a contribution by Frieze, Gould, Karoński and Pfender [9] or a very recent paper by Cuckler and Lazebnik [6].

In subsequent sections we introduce four parameters mainly motivated by the irregularity strength. In the first one, the total edge irregularity strength we ask for the minimum

k that there is a total k-labelling that distinguishes any to edges by their weights. The second one is a relaxation of the first one, motivated also a classical edge colouring. We ask for the minimum k that there is a total k-labelling that distinguishes neighbouring edges. The remaining two are vertex analogues of the previous ones. The last one had been motivated by the classical regular vertex colouring.

3. Edge irregular total labellings

Let for a graph $G = (V, E)$ a function $\varphi : V \cup E \to \{1, 2, \dots, k\} = [k]$ be a *total k-labelling.* A total k-labelling is an *edge irregular total k-labelling* of the graph G if for every two distinct edges e and f of G there is

$$\mathrm{wt}(e) \neq \mathrm{wt}(f)\,.$$

The minimum k for which the graph G has an edge irregular total k-labelling is called the *total edge irregularity strength* of the graph G and is denoted by $\mathrm{tes}(G)$. This parameter was introduced by Bača, Jendrol', Miller and Ryan [3].

In their introductory paper [3] Bača et al. proved

Theorem 1. [3] *Let $G = (V, E)$ be a graph with non–empty edge set E. Then*

$$\left\lceil \frac{|E| + 2}{3} \right\rceil \leq \mathrm{tes}(G) \leq |E|\,.$$

Proof. To get the upper bound let us label each vertex of G with label 1 and the edges consecutively with labels $1, 2, \dots, |E|$. It is easy to see that $\mathrm{wt}(e) \neq \mathrm{wt}(f)$ for any two distinct edges e and f of G.

To obtain the lower bound notice that the heaviest edge e has the weight $\mathrm{wt}(e) \geq |E|+2$. This weight is the sum of three labels so at least one label is at least $(|E| + 2)/3$. □

Already in the paper [3] it is shown that the upper bound is too far from being optimal. It is easy to see that if a graph H is a subgraph if a graph G then $\mathrm{tes}(H) \leq tes(G)$. Therefore it is very important to determine $\mathrm{tes}(K_n)$ of complete graph K_n.

Bača et al. conjectured the following

Conjecture 1. [3] $\mathrm{tes}(K_p) = \lceil (p^2 - p + 4)/6 \rceil$ *for any $p \neq 5$, $p \geq 2$.*

They proved that $\mathrm{tes}(K_5) = 5$. This conjecture has been verified recently by Jendrol', Miškuf and Soták [12].

In fact the conjectured value is equal to the lower bound given by Theorem 1. The other easy lower bound is

$$\mathrm{tes}(G) \geq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil.$$

Bača et al. [3] determined the value tes(G) for G being a path, a wheel, a friendship graph and a star. Miškuf and Jendroľ [15] found the exact values of tes(G) for plane grids. For generalized Petersen graph $P(n,k)$ it was proved in [8] that $\text{tes}\big(P(n,k)\big) = n+1$ for $k \neq \frac{n}{2}$ and $\lceil \frac{5n+4}{6} \rceil$ for $k = \frac{n}{2}$. Ivančo and Jendroľ [11] have proved

Theorem 2. [11] *Let T be a tree. Then*

$$\text{tes}(T) = \max\left\{ \left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil \right\}.$$

They also have supposed that the following conjecture is true

Conjecture 2. [11] *Let G be a graph of order at least 3, $G \not\cong K_5$. Then*

$$\text{tes}(G) = \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

This conjecture is proved to be true for bipartite graphs by Jendroľ, Miškuf and Soták [12]. Very nice results have recently been achieved by Brandt, Miškuf and Rautenbach [5]. They proved among other the following three theorems that confirm the validity of Conjecture 2 for wide families of graphs:

Theorem 3. [5] *Let G be a graph of order p and size q. If $\Delta \geq 2q/3$, then*

$$\text{tes}(G) = \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

Theorem 4. [5] *Let $G = (V,E)$ be a graph of order $p = |V|$, size $q = |E|$, and maximum degree Δ with $0 < \Delta < \frac{q}{1000\sqrt{8p}}$. Then*

$$\text{tes}(G) = \left\lceil \frac{|E|+2}{3} \right\rceil.$$

Theorem 5. [5] *For every integer Δ there is a constant $n(\Delta)$ such that every graph $G = (V,E)$ without isolated vertices of order $|V| = p \geq n(\Delta)$, size $q = |E|$, and maximum at most Δ satisfies*

$$\text{tes}(G) = \left\lceil \frac{|E|+2}{3} \right\rceil.$$

4. Edge–colouring total labellings

The edge irregular total k-labelling φ of a graph $G = (V,E)$ defined in the previous Section distinguishes every two distinct edges e and f of G by their weights, i.e. $\text{wt}(e) \neq \text{wt}(f)$. In this Section we consider such total k-labelling $\varphi : V \cup E \to [k]$ that $\text{wt}(e) \neq \text{wt}(f)$

for every two adjacent edges e and f of G (note that two edges e and f are said to be adjacent if they share a vertex). Following [4] we call such a total k-labelling the *edge–colouring total k-labelling.* The smallest integer k for which there exists an edge colouring total k-labelling is denoted by $\chi'_t(G)$. The parameter $\chi'_t(G)$ has natural upper and lower bounds in terms of the maximum degree Δ of G.

Theorem 6. *Let G be a graph of maximum degree Δ. Then*

$$\left\lceil \frac{\Delta+1}{2} \right\rceil \leq \chi'_t(G) \leq \Delta + 1 \,.$$

Proof. The upper bound is obvious from Vizing's Theorem [20], since a proper edge colouring together with a constant labelling of vertices defines an edge–colouring total labelling of G. To get the lower bound observe that the largest weight among the weights of edges incident with a vertex v of maximum degree Δ is at least $\Delta + 1 + \varphi(v)$, where $\varphi(v)$ is the label of v. Hence there must be

$$2\chi'_t(G) + \varphi(v) \geq \Delta + 1 + \varphi(v)$$

and we are done. □

Recently Brandt, Budajová, Rautenbach and Stiebitz [4] improved the upper bound as follows

Theorem 7. [4] *If G is a graph of maximum degree Δ, then*

$$\chi'_t(G) \leq \left\lfloor \frac{1}{2}\left(\Delta + \left\lfloor \sqrt{2\Delta(1+\ln(2\Delta^2 - 2\Delta + 2))} \right\rfloor\right) \right\rfloor + 1 = \frac{\Delta}{2} + \mathcal{O}(\sqrt{\Delta \log \Delta}\,) \,.$$

Brandt et al. [4] proposed the following

Problem 1. [4] *Is there a constant K with*

$$\chi'_t(G) \leq \frac{\Delta+1}{2} + K$$

for all graphs G of maximum degree Δ ?

The answer is yes for forests, namely the following is true

Theorem 8. [4] *If F is a forest of maximum degree Δ, then*

$$\chi'_t(F) = \left\lceil \frac{\Delta+1}{2} \right\rceil \,.$$

For complete graphs K_p, $p \geq 1$, it is known

Theorem 9. [4] *If $p \not\equiv 2(mod\ 4)$, then $\chi_t'(K_p) = \lceil \frac{p}{2} \rceil$ and if $p = 2(mod\ 4)$,* then $\frac{p}{2} \le \chi_t'(G) \le \frac{p}{2} + 1$.

5. Vertex irregularity strength

For a graph $G = (V, E)$ let $\varphi : V \cup E \to [k]$ be a total k-labelling. Following Bača et al. [3] we define a total k-labelling σ to be a *vertex irregular total k-labelling* of the graph G if for every two distinct vertices u and v of G there is

$$\mathrm{wt}(u) \neq \mathrm{wt}(v)\,.$$

The minimum k for which the graph G has a vertex irregular total k-labelling is called the *total vertex irregularity strength* of the graph G, $\mathrm{tvs}(G)$. This invariant was introduced recently by Bača et al. in [3] where they showed few bounds on this parameter and determined it for some families of graphs.

Theorem 10. [3] *Let $G = (V, E)$ be a graph. Then*

(i) $\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \le \mathrm{tvs}(G) \le p + \Delta - 2\delta + 1$;

(ii) $\mathrm{tvs}(G) \le p - 1 - \left\lfloor \frac{p-2}{\Delta+1} \right\rfloor$ *if G has no isolated vertices or edges;*

(iii) $\mathrm{tvs}(G) \le s(G)$, *where $s(G)$ in the irregularity strength of G.*

There are several families of graph G for which $\mathrm{tvs}(G)$ is known. For example $\mathrm{tvs}(K_p) = 2$ for every complete graph on $p \ge 2$ vertices [3], $\mathrm{tvs}(P(n, k)) = \lceil \frac{n}{2} \rceil + 1$, if $k \neq \frac{n}{2}$ and $P(n, \frac{n}{2}) = \frac{n}{2} + 1$ for the generalized Petersen graph $P(n, k)$ [8].

In [22] there are achieved the exact values for $K_{m,n}$ for some pairs of m, n. In [8] there are proved the exact values of $\mathrm{tvs}(G)$ for some families of graphs including also ladders, Möbius ladders, Knödel graphs, flower snarks and related graphs.

Recently Przybyło [17], [18] proved the following two theorems

Theorem 11. [18] *Let G be a graph of order p with no isolated vertices.*

(i) If $\Delta \le \left\lfloor \left(\frac{p}{\ln p}\right)^{\frac{1}{2}} \right\rfloor$, *then* $\mathrm{tvs}(G) \le 2p\left(\frac{1}{\delta} + \frac{1}{\Delta}\right)$;

(ii) if $\left\lfloor \left(\frac{p}{\ln p}\right)^{\frac{1}{4}} \right\rfloor + 1 \le \Delta \le \left\lfloor p^{\frac{1}{2}} \right\rfloor$, *then* $\mathrm{tvs}(G) \le 18\,\frac{p}{\Delta}$;

(iii) if $\Delta \ge \left\lfloor p^{\frac{1}{2}} \right\rfloor + 1$, $\delta \ge \lceil 10 \log p \rceil$ *then* $\mathrm{tvs}(G) \le 96(\log p)^{\frac{p}{\delta}}$.

Theorem 12. [17] *Let G be a d-regular graph of order p with no isolated vertices.*

(i) *If* $d \leq \left\lfloor \left(\frac{p}{\ln p}\right)^{\frac{1}{4}} \right\rfloor$, *then* $\mathrm{tvs}(G) \leq 2\,\frac{p}{d} + 1$ *;*

(ii) *if* $\left\lfloor \left(\frac{p}{\ln p}\right)^{\frac{1}{4}} \right\rfloor + 1 \leq d \leq \left\lfloor p^{\frac{1}{2}} \right\rfloor$, *then* $\mathrm{tvs}(G) \leq 12\,\frac{p}{d} + 1$ *;*

(iii) *if* $d \geq \left\lfloor p^{\frac{1}{2}} \right\rfloor + 1$, *then* $\mathrm{tvs}(G) \leq 48\,(\log p)^{\frac{p}{d}+1}$.

6. Vertex–colouring total labellings

A total k-labelling $\varphi : V \cup E \to [k]$ of a graph $G = (V, E)$ is defined to be the *neighbour–distinguishing total k-labelling* (or the vertex–colouring total k-labelling, see [4]) if for every edge $e = uv \in E(G)$

$$\mathrm{wt}(u) \neq \mathrm{wt}(v)\,.$$

The smallest integer k for which there exists a neighbour–distinguishing total k-labelling is denoted by $\chi_t(G)$. This parameter was introduced by Przybyło and Woźniak in 2007, see [19], who proved first results on this parameter and posed some open problems. We survey some of them here.

Surprisingly easily one may prove the following

Theorem 13. [19] *If G is a bipartite graph, then*

$$\chi_t(G) \leq 2\,.$$

Proof. Let us first arbitrarily assign weights 1 or 2 to the edges of G. Then assign 1 or 2 to the vertices so that resulting weights of the vertices in one colour class are even and odd in the other one. □

Theorem 14. [19] *If G be a graph of with chromatic number $\chi_0(G)$, then*

$$\chi_t(G) \leq \left\lfloor \frac{\chi_0(G)}{2} \right\rfloor + 1\,.$$

Already in Bača et al. [3] there is proved (in other terminology) that

Theorem 15. [3] $\chi_t(K_p) = 2$ *for every* $p \geq 2$.

Theorem 16. [19] *If G is a 4-regular graph, then* $\chi_t(G) = 2$.

Surprising but very elegant is the following

Theorem 17. [19] *For every graph G there is* $\chi_t(G) \leq 11$.

Przybyło and Woźniak asked if the labels 1 and 2 are enough for every graph. More precisely, they posed.

Conjecture 3. [19] *Let G be a simple graph. Then* $\chi_t(G) \leq 2$.

Similar parameter, but in the case of an *edge* (not total) labelling was introduced and studied in [13] by Karónski, Luczak and Thomason. They asked if each connected graph of order at least 3 permits a *neighbour–distinguishing edge* 3*-labelling*, and showed that this statement holds for the graph G with $\chi_0(G) \leq 3$. It was proved by Addario–Berry, Dalal and Reed [1] that each connected graph of order at least 3 permits a neighbour–distinguishing edge 16-labelling.

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