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Some Open Problems on Graph Labelings

S. Arumugam
Core Group Research Facility (CGRF)
National Center for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626190.
e-mail: *s.arumugam.klu@gmail.com*
G.S. Bloom
Computer Science Department, City College of New York and
Computer Science Program, Graduate Center of the City University of New York
New York, USA.
e-mail:*bloom@cs.ccny.cuny.edu*
Mirka Miller
Graph Theory and Applications (GTA) Research Group
School of Electrical Engineering and Computer Science
The University of Newcastle, Australia and
Department of Mathematics, University of West Bohemia,
Pilsen, Czech Republic.
e-mail: *mirka.miller@newcastle.edu.au*
and
Joe Ryan
Graph Theory and Applications (GTA) Research Group
School of Electrical Engineering and Computer Science
The University of Newcastle, Australia.
e-mail: *joe.ryan@newcastle.edu.au*

Abstract

In this note we present some of the open problems on various aspects of graph labelings which were posed by the participants during IWOGL 2009 and which have not been included in any of the other papers appearing in this volume.

Keywords: mod sum labeling, distance magic labeling, completely graceful graph, greedy permutation, graceful number, harmonious number.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [5]. For basic definitions of various types of graph labelings we refer to dynamic survey of Gallian [8]

2. Mod Sum Labeling of Graphs (Posed by Mirka Miller)

A graph $G(V,E)$ is called a *sum graph* if there is an injective labeling, called *sum labeling*, L from V to a set of distinct positive integers S such that $xy \in E$ if and only if there is a vertex w in V such that $L(w) = L(x) + L(y) \in S$. Every graph can be made into a sum graph by adding some isolated vertices, if necessary. The smallest number of isolated vertices that need to be added to a graph H to obtain a sum graph is called the *sum number* of H; it is denoted by $\sigma(H)$.

Mod sum graphs are defined in the same way except that the sum is taken modulo some positive integer m. The *mod sum number* $\rho(G)$ is defined as the smallest number of isolates that need to be added to the graph G, over all possible choices of m to obtain a mod sum graph. For more details, see the references [4], [8], [9], [11], [21], [22], [23] and [24].

The table below gives the current state of knowledge of mod sum numbers.

Graph Class	Restrictions	Mod Sum Number
Trees, T_n	$n \geq 3$	0
Cycles, C_n	$n \geq 4$	0
$K_{p+1,q}$	$q \geq 1$ and $p \geq r_q + r_{q-1} - 1$	0
$K_{2,p}$	$p \geq 1$	0
$K_{n_1,n_2,\ldots,n_m}$	If there exists n_i and n_j such that $n_i < n_j < 2n_i$	> 0
W_4		0
W_n	$n \geq 6$, n even	2
W_n	$n \geq 5$, n odd	n
$H_{2,n}$	$n \geq 0$	0
$H_{m,n}$	$n > m \geq 3$	> 0
K_2		1
K_3		1
K_n	$n \geq 4$	n

Clearly, the mod sum number of a graph G cannot be more than the sum number of G, $\rho(G) \leq \sigma(G)$ and it could be reasonably expected that by choosing a suitable m, the mod sum number should be always smaller than the sum number. However, this is not true: there do exist graphs for which $\rho(G) = \sigma(G)$, for example, the odd wheels.

Problem 2.1. *Find other families of graphs for which $\rho(G) = \sigma(G)$. More generally, characterize all such graphs.*

3. Distance Magic Labeling (Posed by Dalibor Froncek)

A *distance magic labeling* or 1-*vertex-magic vertex labeling* of a graph $G(V,E)$ with $|V| = n$ is a mapping $\mu : V \to \{1, 2, \ldots, n\}$ with the property that for every vertex $x \in V$ the weight of x, denoted $w(x)$, which is equal to the sum of labels of all neighbors of x, is always equal to a constant m, called the magic constant of μ.

Theorem 3.2. [12] *Let an r-regular graph G on n vertices have a distance magic labeling. Then r is even.*

Theorem 3.3. [7] *An r-regular distance magic graph G on n vertices (where n is even) exists if and only if $2 \leq r \leq n-2$, r is even and either $n \equiv r \equiv 0(mod\ 4)$ or $n \equiv r+2(mod\ 4)$.*

Theorem 3.4. [6] *Let $q > 1$ be an odd number, $s \geq 1, r = 2^s q$ and $n = tq$ for some odd $t, t \geq 2^s + 1$. Then an r-regular distance magic graph G on n vertices (where n is odd) exists.*

Problem 3.5. *Find r-regular distance magic graph on $n = tq$ vertices*

1. *for $r = 2^s q$, where $s \geq 1$ and $t \leq 2^s$.*
2. *for $r = 2^s$.*

4. Completely Graceful Graphs (Posed by G.S. Bloom)

A *bijective labeling* $\varphi(V)$ from the vertices of a simple graph $G(V,E)$ to a subset of N is called a *difference labeling* of G, when an induced labeling of the edges is defined as $\varphi(uv) = |\varphi(u) - \varphi(v)|$ for every edge uv. If $\varphi(V)$ results in every edge difference being repeated exactly k times, we say that the labelling is *k-equitable* [2,3]. When a difference labeling is k-equitable for some positive k and when $\varphi(V) \subset \{0, 1, 2, \ldots, |E|\}$, we use the term *optimally k-equitable* to express this.

For example, Figure 1 shows that P_9 is optimally k-equitable for $k = 1, 2, 4$, and 8.

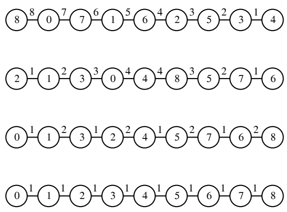

Figure 1

If $G \neq$ tree is optimally k-equitable for *all* k in the set $\{k\colon k$ is a proper divisor of $|E|\}$, then we say that G is *completely equitable*. If $G =$ tree, we also require G to be optimally $|E|$-equitable for G to be completely equitable. In Figure 1 we have shown that P_9 is completely and optimally equitable. In Figure 2 we show that C_{12} is k-equitable for $k \in \{1, 2, 3, 4, 6\}$, and consequently C_{12} is also completely and optimally equitable.

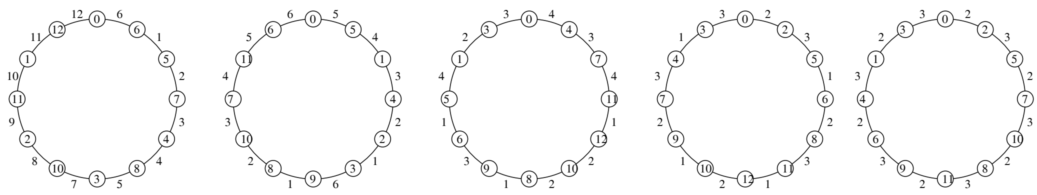

Figure 2

In [1] we discuss optimally k-equitable graphs with edge label multiset $= \{1^k, 2^k, 3^k, \ldots, (|E|/k)^k\}$. Graphs with these labelings are called *k-fold graceful*. If a graph is k-fold graceful for all divisors k of $|E|$, they are *completely graceful*. In [3] we show, among other results, that P_n is completely graceful, and that K_n, $K_{1,n}$, and the Petersen graphs are not. Figure 1 gives an example of a set of labelings that show P_9 is completely graceful. Note that the labelings in Figure 2 show that C_{12} is graceful (1-graceful), 2-graceful, 3-graceful, and 4-graceful. Nevertheless, C_{12} is not completely graceful, since we can prove that it is not 6-graceful.

Problem 4.1. *What graphs are completely graceful or at least completely and optimally equitable?*

5. A Greedy Permutation of the Natural Numbers (Posed by P.J. Slater)

For the set $N = \{1, 2, 3, \ldots\}$ of positive integers, a permutation $P = (a_1, a_2, a_3, \ldots)$ of N (that is, a sequence of positive integers such that every positive integer appears exactly once) has a difference sequence $D = (d_1, d_2, d_3, \ldots)$ where $d_k = |a_{k+1} - a_k|$. Answering a question of Roger Entringer, in [17] it was shown that there is such a permutation P whose difference sequence D is also a permutation. (In [13] a question of Paul Erdos was answered affirmatively, showing that permutation P of N can be chosen so that its difference sequence and all succeeding difference sequences are also permutations.)

Permutation P with D also a permutation can be thought of as a graceful labeling of the infinite path $P(\infty)$. See [14, 15, 16, 18] for results on labelings of infinite graphs. For example, we have the following result.

Theorem 5.1. [16]

1. *All countably infinite trees are k-graceful for each $k \geq 1$.*
2. *Any countably infinite tree with an infinite set of independent edges is bijectively-k-graceful for each $k \geq 1$.*
3. *A countably infinite tree with no infinite independent edge set is bijectively-k-graceful if and only if the number of vertices of infinite degree is one and $k = 1$.*

To construct a permutation P^* of N in a greedy manner let $a_1 = 1$ and $a_2 = 2$. (Note that $d_1 = 1$.) Assuming that $a_1, a_2, \ldots, a_t$ have been defined, define a_{t+1} to be the smallest positive integer such that $a_{t+1} \neq a_i$ for $1 \leq i \leq t$ and $d_t = |a_{t+1} - a_t| \neq d_k$ for $1 \leq k \leq t-1$. By definition, each positive integer appears at most once in P^* and the associated difference sequence D^*. We have the following.

$P^* = (1, 2, 4, 7, 3, 8, 14, 5, 12, 20, 6, 16, 27, 9, 21, 34, 10, 25, 41, 24, \ldots)$

$D^* = (1, 2, 3, 4, 5, 6, 9, 7, 8, 14, 10, 11, 18, 12, 13, 24, 15, 16, 17, \ldots)$

Theorem 5.2. [17] *The sequence P^* is a permutation of N.*

Conjecture 5.3. *The sequence D^* is a permutation of N.*

There are apparently many interesting properties and questions about the permutation P^*. For example, what is the growth rate of P^*? One curious question is the following.

Problem 5.4. *Is there a value $i \geq 2$ such that $a_i < a_{i+1} < a_{i+2} < a_{i+3}$?*

6. Labelings of Super-Subdivision of a Graph (Posed by G. Sethuraman)

Let G be a graph with t edges. A graph H is called a *super-subdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some $m_i, 1 \leq i \leq t$ (m_i may vary for each edge) in such a way that the ends of e_i are merged with the two vertices of the partite set of $K_{2,m}$ of cardinality two after removing the edge e_i from G. If each m_i is chosen arbitrarily then the super-subdivision is called *arbitrary super-subdivision*.

In [20] Sethuraman and Selvaraju have given an algorithm to construct a special type of super-subdivision graphs from every nontrivial connected graph G and the family of super-subdivision graphs obtained by the algorithm is denoted by $SS(G)$. The following result shows that each member of $SS(G)$ of any nontrivial connected graph G admits α-valuation.

Theorem 6.1. [20] *For any non-trivial connected graph G, each super-subdivision graph in $SS(G)$ admits α-valuation.*

It is interesting to note that each member of $SS(G)$ admits other labelings like cordial or n-elegant too. But graphs that admit arbitrary super-subdivisions with α-valuation are very few, for example, paths and stars admit arbitrary super-subdivision with α-valuation [19]. It appears that more interesting results can be obtained on super-subdivision graphs. In this direction we ask the following questions to initiate further study on super-subdivision graphs.

Problem 6.2. *Characterize connected graphs which admit arbitrary super-subdivisions with α- valuation.*

Problem 6.3. *Are there graphs apart from $K_{2,m}$ which can be used for edge replacement in defining the super-subdivision that will admit graceful labeling or α-valuation?*

Problem 6.4. *What are the other labelings that the graphs in $SS(G)$ would admit for a given connected graph G?*

7. Graceful Number of a Graph (Posed by G. Sethuraman)

It is well known that every graph G is an induced subgraph of a graceful graph. *Graceful number* of a graph G, denoted $g(G)$, is defined to be $min\{|E(G_g)|-|E(G)| : G_g$ is a graceful graph containing G as its vertex induced subgraph$\}$. Thus, graceful number of a graceful graph is 0 and for any non-graceful graph the graceful number is at least 1. Similarly one can define *Harmonious number* $h(G)$ of a graph G, with respect to harmonious labeling.

Problem 7.1. *For a given graph G, determine $g(G)$.*

Problem 7.2. *Find tight upper bounds for $g(G)$.*

Problem 7.3. *For a given graph G, determine $h(G)$.*

Problem 7.4. *Find tight upper bounds for $h(G)$.*

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