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Edge irregular total labeling of certain family of graphs

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Abstract

An edge irregular total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any different edges xy and $x'y'$ their weights $\varphi(x) + \varphi(xy) + \varphi(y)$ and $\varphi(x') + \varphi(x'y') + \varphi(y')$ are distinct. The total edge irregularity strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

We have determined the exact value of the total edge irregularity strength of the categorical product of a cycle and a path.

Keywords: irregularity strength, total edge irregularity strength, edge irregular total labeling

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1. Introduction

Throughout this paper, G is a graph, and V and E are the sets of vertices and edges of G, with cardinalities $|V|$ and $|E|$, respectively.

A labeling λ of a finite graph is a function that attaches integers (usually positive integers) to the edges and (or) the vertices of the graph, and different algebraic constraints on λ correspond to different types of labeling. There are many ways to label graphs (see Gallian survey [9]). If the domain of the labeling λ is the set of vertices or the set of edges, the labeling is called respectively *vertex labeling* or *edge labeling*. If the domain is the set of vertices and edges then we call the labeling *total labeling*. Thus, for an edge k-labeling $\sigma : E(G) \to \{1, 2, \dots, k\}$ the associated vertex-weight of a vertex $x \in V(G)$ is

$$w_\sigma(x) = \sum_{y \in N(x)} \sigma(xy),$$

where $N(x)$ is the set of neighbors of x, and for a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated edge-weight is

$$wt_{\varphi}(xy) = \varphi(x) + \varphi(xy) + \varphi(y).$$

In [6], Chartrand et al. introduced edge k-labeling of a graph G such that $w(x) \neq w(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [7], [8] and [15]. For recent results see the papers [1], [2], [4], [10] and [12].

Motivated by these papers and by a book of Wallis [20], Bača et al. in [3] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labelings.

For a graph $G = (V, E)$ we define a labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ to be an *edge irregular total k-labeling* of the graph G if for every two different edges xy and $x'y'$ of G one has $wt_{\varphi}(xy) \neq wt_{\varphi}(x'y')$. The *total edge irregularity strength*, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

In [3] we can find that $tes(G) \geq \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}$, where $\Delta(G)$ is the maximum degree of G, and also there are determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.

Recently Ivančo and Jendroľ [11] proved that for any tree T the $tes(T) = \max\left\{\left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil\right\}$. Moreover, they posed a conjecture that for arbitrary graph G different from K_5 and maximum degree $\Delta(G)$, the $tes(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}$.

The Ivančo and Jendroľ's conjecture has been verified for complete graphs and complete bipartite graphs in [13] and [14], for the Cartesian product of two paths in [16] and for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [5].

Motivated by the papers [7], [16] and [19] we study the total edge irregularity strength of the categorical product of a cycle and a path. Our result adds further support to the conjecture of Ivančo and Jendroľ.

2. Categorical Product of Graphs

The categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if u, v are adjacent in G and u', v' are adjacent in H (see e.g. [17] or [18]). If we consider graph G as the cycle C_n with $V(C_n) = \{x_i : 1 \leq i \leq n\}$, $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n\}$ with indices taken modulo n, and graph H as the path P_m with $V(P_m) = \{y_j : 1 \leq j \leq m\}$, $E(P_m) = \{y_j y_{j+1} : 1 \leq j \leq m-1\}$ then $V(C_n \times P_m) = \{(x_i, y_j) : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ is the vertex set of $C_n \times P_m$ and $E(C_n \times P_m) = \{(x_i, y_j)(x_{i+1}, y_{j+1}) : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\} \cup \{(x_i, y_{j+1})(x_{i+1}, y_j) : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\} \cup \{(x_1, y_j)(x_n, y_{j+1}) : 1 \leq j \leq m-1\} \cup \{(x_1, y_{j+1})(x_n, y_j) : 1 \leq j \leq m-1\}$ is the edge set of $C_n \times P_m$. So, $C_n \times P_m$ is the graph of order nm and size $2n(m-1)$. In this paper, we deal with the categorical product $C_n \times P_m$ for m and n even, $m \geq 2$, $n \geq 4$.

In [3] it is proved that $tes(G) \geq \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\}$. As the maximum degree $\Delta(C_n \times P_m) = 4$ then this implies that $tes(C_n \times P_m) \geq \left\lceil \frac{2n(m-1)+2}{3} \right\rceil$. To show that $\left\lceil \frac{2n(m-1)+2}{3} \right\rceil$ is an upper bound for the $tes(C_n \times P_m)$, we describe an edge irregular total $\left\lceil \frac{2n(m-1)+2}{3} \right\rceil$-labeling for $C_n \times P_m$.

3. Two Lemmas

Let $n \geq 4$, $m \geq 2$ be even integers. For integers a and b, let $[a, b]$ be an interval of integers x, $a \leq x \leq b$. We split the edge set of $C_n \times P_m$ into mutually disjoint subsets A_i and B_i, where $A_i = \{(x_i, y_j)(x_{i+1}, y_{j+1}) : j \in [1, m-1] \text{ is odd}\} \cup \{(x_i, y_{j+1})(x_{i+1}, y_j) : j \in [2, m-2] \text{ is even}\}$, $B_i = \{(x_i, y_{j+1})(x_{i+1}, y_j) : j \in [1, m-1] \text{ is odd}\} \cup \{(x_i, y_j)(x_{i+1}, y_{j+1}) : j \in [2, m-2] \text{ is even}\}$ for $i \in [1, n-1]$, and $A_n = \{(x_n, y_{j+1})(x_1, y_j) : j \in [1, m-1] \text{ is odd}\} \cup \{(x_n, y_j)(x_1, y_{j+1}) : j \in [2, m-2] \text{ is even}\}$, $B_n = \{(x_n, y_j)(x_1, y_{j+1}) : j \in [1, m-1] \text{ is odd}\} \cup \{(x_n, y_{j+1})(x_1, y_j) : j \in [2, m-2] \text{ is even}\}$. Clearly, $|A_i| = |B_i| = m-1$ for all $i \in [1, n]$ and $\bigcup_{i=1}^{n} \{A_i \cup B_i\} = E(C_n \times P_m)$.

The following two lemmas determine the exact value of the total edge irregularity strength for two particular cases.

Lemma 3.1. *Let $m \geq 2$ be an even integer. Then $tes(C_4 \times P_m) = \left\lceil \frac{8m-6}{3} \right\rceil$.*

Proof. Let $r = \left\lceil \frac{8m-6}{3} \right\rceil$. For $m \geq 2$ even, we construct the function φ_1 as follows:

$$\varphi_1((x_i,y_j)) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{if } i=1 \text{ and } j \in [1,m] \\ \frac{m+j+1}{2}, & \text{if } i=2 \text{ and } j \in [1,m-1] \text{ is odd} \\ \frac{j}{2}, & \text{if } i=2 \text{ and } j \in [2,m] \text{ is even} \\ r-m+\frac{j+1}{2}, & \text{if } i=3 \text{ and } j \in [1,m-1] \text{ is odd} \\ r+\frac{j-m}{2}, & \text{if } i=3 \text{ and } j \in [2,m] \text{ is even} \\ r-\frac{m}{2}+\lceil \frac{j}{2} \rceil, & \text{if } i=4 \text{ and } j \in [1,m] \end{cases}$$

$$\varphi_1(z) = \begin{cases} 1, & \text{if } z \in A_1 \\ \frac{m}{2}, & \text{if } z \in B_1 \\ 3m-1-r, & \text{if } z \in B_2 \\ 5m-4-r, & \text{if } z \in A_2 \\ \frac{15m}{2}-5-2r, & \text{if } z \in A_3 \\ 8m-6-2r, & \text{if } z \in B_3 \\ \frac{9m}{2}-3-r, & \text{if } z \in A_4 \\ \frac{7m}{2}-2-r, & \text{if } z \in B_4. \end{cases}$$

Observe that the edge-weights of the edges from $E(C_4 \times P_m)$, under the function φ_1, constitute the sets

$W_1 = \{wt_{\varphi_1}(e) : e \in A_1\} = \{3, 4, \ldots, m, m+1\}$,
$W_2 = \{wt_{\varphi_1}(e) : e \in B_1\} = \{m+2, m+3, \ldots, 2m-1, 2m\}$,
$W_3 = \{wt_{\varphi_1}(e) : e \in B_2\} = \{2m+1, 2m+2, \ldots, 3m-2, 3m-1\}$,
$W_4 = \{wt_{\varphi_1}(e) : e \in B_4\} = \{3m, 3m+1, \ldots, 4m-3, 4m-2\}$,
$W_5 = \{wt_{\varphi_1}(e) : e \in A_4\} = \{4m-1, 4m, \ldots, 5m-4, 5m-3\}$,
$W_6 = \{wt_{\varphi_1}(e) : e \in A_2\} = \{5m-2, 5m-1, \ldots, 6m-5, 6m-4\}$,
$W_7 = \{wt_{\varphi_1}(e) : e \in A_3\} = \{6m-3, 6m-2, \ldots, 7m-6, 7m-5\}$,
$W_8 = \{wt_{\varphi_1}(e) : e \in B_3\} = \{7m-4, 7m-3, \ldots, 8m-7, 8m-6\}$.

It is easy to see that the function φ_1 is a map from $V(C_4 \times P_m) \cup E(C_4 \times P_m)$ into $\{1, 2, \ldots, r\}$ and the edge-weights of the edges of $C_4 \times P_m$ constitute the set of consecutive integers $\bigcup_{i=1}^{8} W_i = \{3, 4, \ldots, 8m-7, 8m-6\}$.

This concludes the proof. □

Lemma 3.2. *Let $m \geq 2$ be an even integer. Then $tes(C_6 \times P_m) = \left\lceil \frac{12m-10}{3} \right\rceil$.*

Proof. Let $t = \left\lceil \frac{12m-10}{3} \right\rceil$. For $m \geq 2$ even, define the function φ_2 in the following way:

$$\varphi_2((x_i, y_j)) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{if } i = 1 \text{ and } j \in [1, m] \\ \frac{m+j+1}{2}, & \text{if } i = 2 \text{ and } j \in [1, m-1] \text{ is odd} \\ \frac{j}{2}, & \text{if } i = 2 \text{ and } j \in [2, m] \text{ is even} \\ t - \lfloor \frac{6-i}{2} \rfloor m + \frac{j-m}{2}, & \text{if } i \in [3, 6] \text{ and } j \in [2, m] \text{ is even} \\ t - \lfloor \frac{6-i-1}{2} \rfloor m - m + \lceil \frac{j}{2} \rceil, & \text{if } i \in [3, 5] \text{ and } j \in [1, m-1] \text{ is odd} \\ t - \frac{m-j-1}{2}, & \text{if } i = 6 \text{ and } j \in [1, m-1] \text{ is odd.} \end{cases}$$

$$\varphi_2(z) = \begin{cases} 1, & \text{if } z \in A_1 \\ 4m - 1 - t, & \text{if } z \in B_2 \\ \frac{13m}{2} + 1 + i(m-2) - 2t, & \text{if } z \in A_i \text{ for } i = 3 \text{ and } i = 5 \\ \frac{21m}{2} - 7 - 2t, & \text{if } z \in B_4 \\ \frac{9m}{2} - 3 - t, & \text{if } z \in A_6 \\ \frac{m}{2}, & \text{if } z \in B_1 \\ 4m - 2 - t, & \text{if } z \in A_2 \\ \frac{19m}{2} - 6 - 2t, & \text{if } z \in B_3 \\ \frac{21m}{2} - 8 - 2t, & \text{if } z \in A_4 \\ 12m - 10 - 2t, & \text{if } z \in B_5 \\ \frac{11m}{2} - 4 - t, & \text{if } z \in B_6. \end{cases}$$

It is simple to verify that under the function φ_2 the edge-weights of $C_6 \times P_m$ admit the consecutive integers from 3 to $12(m-1)+2$ and that φ_2 is the required edge irregular total t-labeling. □

4. Main Result

The main result of this paper is the following.

Theorem 4.3. *Let $m \geq 2$, $n \geq 4$ be even integers and $C_n \times P_m$ be the categorical product of the cycle C_n and the path P_m. Then*

$$tes(C_n \times P_m) = \left\lceil \frac{2n(m-1)+2}{3} \right\rceil.$$

Proof. Let $m \geq 2$, $n \geq 4$ be even integers and let $k = \left\lceil \frac{2n(m-1)+2}{3} \right\rceil$. For $m \geq 2$ and $n \in \{4, 6\}$ the assertion follows from previous lemmas. Now, for $m \geq 2$ and $n \geq 8$, we define the function ψ as follows:

$$\psi((x_i,y_j)) = \begin{cases} \lceil \frac{j}{2} \rceil, & \text{if } i \in [1,2] \text{ and } j \in [1,m] \\ (i-2)\frac{m}{2} + \lceil \frac{j}{2} \rceil, & \text{if } i \in [3, \frac{n}{2}-2] \text{ and } j \in [1,m] \\ \frac{m(n-6)}{4} + \frac{j+1}{2}, & \text{if } i = \frac{n}{2}-1 \text{ for } n \equiv 0 \ (mod\ 4) \\ & \text{and } j \in [1,m-1] \text{ is odd} \\ \frac{m(n-6)}{4} + \frac{j}{2}, & \text{if } i = \frac{n}{2}-1 \text{ for } n \equiv 2 \ (mod\ 4) \\ & \text{and } j \in [2,m] \text{ is even} \\ k - \lfloor \frac{n-i}{2} \rfloor m - \frac{m-j}{2}, & \text{if } i \in [\frac{n}{2}-1, n] \text{ for } n \equiv 0 \ (mod\ 4) \\ & \text{and } j \in [2,m] \text{ is even} \\ & \text{or } i \in [\frac{n}{2}, n] \text{ for } n \equiv 2 \ (mod\ 4) \\ & \text{and } j \in [2,m] \text{ is even} \\ k - \lfloor \frac{n-i-1}{2} \rfloor m - m + \frac{j+1}{2}, & \text{if } i \in [\frac{n}{2}, n-1] \text{ for } n \equiv 0 \ (mod\ 4) \\ & \text{and } j \in [1,m-1] \text{ is odd} \\ & \text{or } i \in [\frac{n}{2}-1, n-1] \text{ for } n \equiv 2 \ (mod\ 4) \\ & \text{and } j \in [1,m-1] \text{ is odd} \\ k - \frac{m-j-1}{2}, & \text{if } i = n \text{ and } j \in [1,m-1] \text{ is odd.} \end{cases}$$

One can easily check that under the vertex labeling ψ the edges

(i) from the set A_1 and B_1 receive the weights from the interval $[2,m]$,

(ii) from the set A_i and B_i, for $2 \leq i \leq \frac{n}{2}-3$, admit the weights from the interval $[\frac{m}{2}(2i-3)+2, \frac{m}{2}(2i-1)]$,

(iii) from the set $A_{\frac{n}{2}-2}$, for $n \equiv 2 \pmod 4$ (respectively, $B_{\frac{n}{2}-2}$ for $n \equiv 0 \pmod 4$) admit the weights from the interval $[\frac{m}{2}(n-7)+2, \frac{m}{2}(n-5)]$,

(iv) from the set $B_{\frac{n}{2}-2}$, for $n \equiv 2 \pmod 4$ (respectively, $A_{\frac{n}{2}-2}$ for $n \equiv 0 \pmod 4$) receive the weights from the interval $[k+2-\frac{5m}{2}, k-\frac{3m}{2}]$ (respectively, $[k+2-\frac{m}{4}(n-2), k-\frac{m}{4}(n-6)]$),

(v) from the set $B_{\frac{n}{2}-1}$, for $n \equiv 2 \pmod 4$ (respectively, $A_{\frac{n}{2}-1}$ for $n \equiv 0 \pmod 4$) admit the weights from the interval $[k+2-2m, k-m]$,

(vi) from the set A_n and B_n admit the weights from the interval $[k+2-\frac{m}{2}, k+\frac{m}{2}]$,

(vii) from the set A_i and A_{i+1} admit the weights from the interval $[2k+2+\frac{m}{2}(2i-2n+1), 2k+\frac{m}{2}(2i-2n+3)]$ for $\frac{n}{2}-1 \leq i < n-1$ even, if $n \equiv 2 \pmod 4$, and for $\frac{n}{2} \leq i < n-1$ even, if $n \equiv 0 \pmod 4$,

(viii) from the set B_i and B_{i+1} receive the weights from the interval $[2k+2+\frac{m}{2}(2i-2n+1), 2k+\frac{m}{2}(2i-2n+3)]$ for $\frac{n}{2}-1 \leq i < n-1$ odd, if $n \equiv 0 \pmod 4$, and for $\frac{n}{2} \leq i < n-1$ odd, if $n \equiv 2 \pmod 4$,

(ix) from the set B_{n-1} receive the weights from the interval $[2k-m+2, 2k]$.

Now, we complete the edge labels in each family from *(i)* to *(ix)* and create a total labeling of the graph $C_n \times P_m$.

(i) Each edge from the set A_1 we label by the label 1 and we receive the edge-weights from the interval $[3, m+1]$. Each edge from the set B_1 we label by the label m and we obtain the edge-weights from the interval $[m+2, 2m]$.

(ii) The edges from the set A_i, for $2 \leq i \leq \frac{n}{2}-3$, we label by $i(m-2)+3-\frac{m}{2}$ and we obtain the edge-weights from the interval $[(m-1)(2i-2)+3, (m-1)(2i-1)+2]$. Each edge from the set B_i, for $2 \leq i \leq \frac{n}{2}-3$, we label by $\frac{m}{2}+2+i(m-2)$ and we receive the edge-weights from the interval $[(m-1)(2i-1)+3, 2i(m-1)+2]$.

(iii) We label the edges from the set $A_{\frac{n}{2}-2}$ for $n \equiv 2 \pmod 4$ (respectively, $B_{\frac{n}{2}-2}$ for $n \equiv 0 \pmod 4$) by the label $(n-5)(\frac{m}{2}-1)+2$ and we obtain the edge-weights from the interval $[(m-1)(n-6)+3, (m-1)(n-5)+2]$.

(iv) The edges from the set $B_{\frac{n}{2}-2}$ for $n \equiv 2 \pmod 4$ (respectively, $A_{\frac{n}{2}-2}$ for $n \equiv 0 \pmod 4$) we label by the label $\frac{m}{2}(2n-5)-n-k+6$ (respectively, by the label $\frac{m}{2}(\frac{5n}{2}-11)-n-k+6$). In both cases we receive the edge-weights from the interval $[(m-1)(n-5)+3, (m-1)(n-4)+2]$.

(v) We label each edge from the set $B_{\frac{n}{2}-1}$ for $n \equiv 2 \pmod 4$ (respectively, $A_{\frac{n}{2}-1}$ for $n \equiv 0 \pmod 4$) by the label $m(n-2)-n-k+5$. We receive the edge-weights from the interval $[(m-1)(n-4)+3, (m-1)(n-3)+2]$.

(vi) The edges from the set B_n (respectively, A_n) we label by the label $\frac{m}{2}(2n-5)-n-k+4$ (respectively, by the label $\frac{m}{2}(2n-3)-n-k+3$) and the edge-weights create the interval $[(m-1)(n-3)+3, (m-1)(n-2)+2]$ (respectively, $[(m-1)(n-2)+3, (m-1)(n-1)+2]$).

(vii) For $\frac{n}{2}-1 \leq i < n-1$ even, if $n \equiv 2 \pmod 4$, and for $\frac{n}{2} \leq i < n-1$ even, if $n \equiv 0 \pmod 4$, we label the edges from the set A_i (respectively, A_{i+1}) by the label $mn+i(m-2)+\frac{m}{2}-2k$ (respectively, by the label $mn+i(m-2)+\frac{3m}{2}-2k-1$) and the edge-weights create the interval $[(m-1)(2i+1)+3, (m-1)(2i+2)+2]$ (respectively, $[(m-1)(2i+2)+3, (m-1)(2i+3)+2]$).

(viii) For $\frac{n}{2}-1 \leq i < n-1$ odd, if $n \equiv 0 \pmod 4$, and for $\frac{n}{2} \leq i < n-1$ odd, if $n \equiv 2 \pmod 4$, we label the edges from the set B_i (respectively, B_{i+1}) by the label $mn+i(m-2)+\frac{m}{2}-2k$ (respectively, by the label $mn+i(m-2)+\frac{3m}{2}-2k-1$) and the edge-weights form the interval $[(m-1)(2i+1)+3, (m-1)(2i+2)+2]$ (respectively, $[(m-1)(2i+2)+3, (m-1)(2i+3)+2]$).

(ix) The last $m-1$ edges from the set B_{n-1} we label by the label $2n(m-1)+2-2k$ and the last $m-1$ edge-weights create the interval $[(2n-1)(m-1)+3, 2n(m-1)+2]$.

It is a routine matter to verify that all vertex and edge labels are at most k and the edge-weights are different for all pairs of distinct edges. In fact, our total labeling has been chosen in such a way that the edge-weights of the edges from the set $\bigcup_{i=1}^{n} \{A_i \cup B_i\}$ form a consecutive sequence of integers from 3 to $2n(m-1)+2$.

Thus, the resulting total labeling is desired edge irregular k-labeling. $\square$

5. Conclusion

In this paper is determined the exact value of the total edge irregularity strength of the categorical product $C_n \times P_m$ for $m \geq 2$, $n \geq 4$ even. We have tried to find an edge irregular total $\left\lceil \frac{2n(m-1)+2}{3} \right\rceil$-labeling for $C_n \times P_m$ if at least one of m, n is odd but so far without success. So, we conclude the paper with the following open problem.

Problem 5.1. *For the categorical product $C_n \times P_m$, $m \geq 2$, $n \geq 3$, determine a total edge irregularity strength if at least one of m, n is odd.*

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