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# A note on even disjoint union of paths

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**Abstract**

A $(p, q)$-graph $G$ is edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that $f(u) + f(v) + f(uv) = k$ is a constant, called the valence of $f$, for any edge $uv$ of $G$. Moreover, $G$ is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$.

There is an interesting question to know the super edge-magicness of the even disjoint union of paths. In this paper we use an operation on digraphs that is in some sense a generalization of the Kronecker product of matrices and has a relation with (super) edge-magic graphs. In light of an operation on digraphs we solve partially the question.

**Keywords:** Magic valuations, edge-magic labelings, edge-magic graph.

**2000 Mathematics Subject Classification:** 05C78

## 1. Introduction

For most of the graph theory terminology and notation utilized here, the authors refer the reader to Chartrand and Lesniak [2]; however, to make this paper reasonably self-contained, we mention that for a graph $G$, we denote the vertex set and edge set of $G$ by $V(G)$ and $E(G)$, respectively. The seminal paper in edge-magic labelings was published by Kotzig and Rosa [8], who called these labelings *magic valuations*. These labelings were later rediscovered by Ringel and Lladó [9], who called them *edge-magic labelings*.

More formally, for a $(p, q)$-graph $G$, a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ is an *edge-magic labeling* of $G$ if $f(u) + f(v) + f(uv)$ is a constant $k$, called the valence of $f$, for any edge $uv \in E(G)$. If such a labeling exists, then $G$ is said to be an *edge-magic graph.* In [3] Enomoto et al. defined an edge-magic labeling $f$ of graph $G$ to be a *super edge-magic labeling* of $G$ if $f$ has the additional property that the vertex labels are the integers $\{1, 2, \ldots, p\}$, the smallest possible labels. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. The next result by Figueroa-Centeno et al. [5] allows us to see *super edge-magic labelings* as *vertex labelings.*

**Lemma 1.** [5] *A graph $G$ of order $p$ and size $q$ is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ is a set of $q$ consecutive integers.*

For more information about edge-magic and super edge-magic graphs the reader is referred to [5], [6], [7] and [11].

Figueroa-Centeno et al. [4] defined a new operation on digraphs as follows. Let $\vec{D}$ be a digraph and let $\Gamma = \{\vec{F_1}, \vec{F_2}, \ldots, \vec{F_s}\}$ be a family of digraphs that meet the following conditions:

1. All digraphs $\vec{F_i}$, $i \in \{1, 2, \ldots, s\}$, have the same vertex set $V$, i.e. $V(\vec{F_1}) = V(\vec{F_2}) = \cdots = V(\vec{F_s}) = V$.
2. $out(v) = in(v) = 1$ for every $v \in V(\vec{F_i})$, $i \in \{1, 2, \ldots, s\}$.
3. $\vec{F_i}$ may contain loops. In other words, each $\vec{F_i}$ is either a cycle or union of cycles of the same order such that each component has been oriented cyclically.
4. For all $i \in \{1, 2, \ldots, s\}$, if there exists a bijective function $f_i : V(\vec{F_i}) \to \{1, 2, \ldots, |V(\vec{F_i})|\}$ then every vertex $v \in V(\vec{F_i})$ is represented by value $f_i(v)$.

Consider any function $h : E(\vec{D}) \to \Gamma$. Then the product $\vec{D} \bigotimes_h \Gamma$ is a digraph with vertex set $V(\vec{D}) \times V$ and $((a, b), (c, d)) \in E(\vec{D} \bigotimes_h \Gamma) \iff (a, c) \in E(\vec{D}) \bigwedge (b, d) \in E(h(a, c))$.

Notice that the adjacency matrix of $\vec{D} \bigotimes_h \Gamma$, denoted by $A(\vec{D} \bigotimes_h \Gamma)$, is obtained by multiplying every 0 entry of $A(\vec{D})$, where $A(\vec{D})$ denotes the adjacency matrix of $\vec{D}$, by the $|V| \times |V|$ null square matrix, and every 1 entry of $A(\vec{D})$ by $A(h(a, c))$, where $A(h(a, c))$ denotes the adjacency matrix of $h(a, c)$. Notice that when $h$ is constant, we have the classical Kronecker matrix product.

In this paper we use this operation on digraphs which has a relationship to (super) edge-magic graphs. In light of the operation on digraphs we study super edge-magic properties for the disjoint union of graphs, mainly even disjoint union of paths.

## 2. Using the operation on digraphs

From now on, let $und(\vec{D})$ denote the underlying graph of a digraph $\vec{D}$ and $S_m = \{\vec{F_1}, \vec{F_2}, \ldots, \vec{F_s}\}$ denote the set of all super edge-magic 1-regular labeled digraphs of odd order $m$. Let $\#(m)$ be the cardinality of the set $S_m$, i.e., $|S_m| = \#(m)$. If $f_i$ is a super edge-magic labeling of the $und(\vec{F_i})$ then every vertex $v \in V(\vec{F_i})$ is represented by value $f_i(v)$.

For example $S_5 = \{\vec{F_1}, \vec{F_2}, \vec{F_3}, \vec{F_4}, \vec{F_5}, \vec{F_6}\}$, where each $\vec{F_i}$, $i = 1, 2, \ldots, 6$, is the digraph of order 5 such that each component is oriented cyclically and each vertex $v \in V(\vec{F_i})$, $i = 1, 2, \ldots, 6$, is represented by value $f_i(v)$, where $f_i$ is a super edge-magic labeling of the $und(\vec{F_i})$ (see Figure 1, ..., Figure 6).

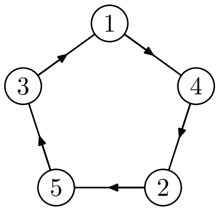


Figure 1: Digraph $\vec{F_1}$

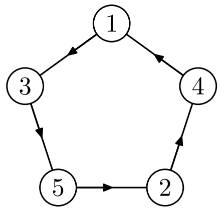


Figure 2: Digraph $\vec{F_2}$

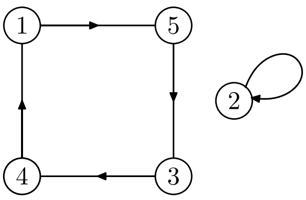


Figure 3: Digraph $\vec{F_3}$

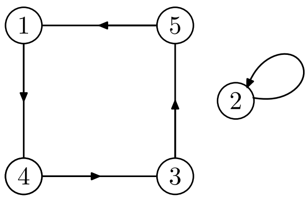


Figure 4: Digraph $\vec{F_4}$

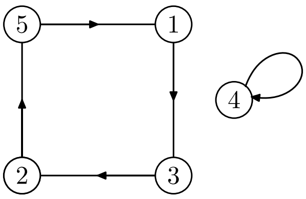


Figure 5: Digraph $\vec{F_5}$

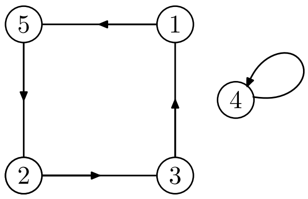


Figure 6: Digraph $\vec{F_6}$

Figueroa-Centeno et al. used the operation on digraphs and proved the following result.

**Theorem 1.** [4] *Let $F$ be a (super) edge-magic acyclic graph and $\vec{F}$ be a digraph of $F$. Consider any function $h : E(\vec{F}) \to S_m$. Then the graph $und(\vec{F} \bigotimes_h S_m) = mF$ admits a (super) edge-magic labeling. Furthermore, if $F$ has order $p$ and $t$ components, then $mF$ admits at least $[\#(m)]^{(p-t)}$ non-isomorphic (super) edge-magic labelings.*

Many researchers have studied the following problem: If a graph $G$ is (super) edge-magic, is the disjoint union of multiple copies of the graph $G$ (super) edge-magic as well?

This problem is very interesting if graph $G$ is a path. Wallis et al. [10] proved that all paths $P_n$ have edge-magic labelings. For super edge-magic labelings of paths we proved the following result.

**Theorem 2.** [1] *The path $P_n$, $n \geq 2$, has a super edge-magic labeling with a valence $k$ if and only if either*

*(i) $k = \frac{5n}{2} + 1$ for $n$ even or*

*(ii) $k = \frac{5n+1}{2}$ or $k = \frac{5n+3}{2}$ for $n$ odd.*

What is known for (super) edge-magic labelings of the disjoint union of paths?

Figueroa-Centeno et al. considered this problem and proved the following theorem.

**Theorem 3.** [6] *If $G$ is a (super) edge-magic bipartite or tripartite graph, and $m$ is odd, then $mG$ is (super) edge-magic.*

The previous result answers the problem for the case when the graph $G$ is a path and $m$ is odd. For an even disjoint union of multiple copies of the path $P_n$, there is known only the following result.

**Theorem 4.** [6] *The forest $F \cong 2P_n$ is super edge-magic if and only if $n \neq 2$ or 3.*

What can we say for $m$ even, $m \geq 4$?

From Theorem 3 and Theorem 4 we have an answer for the case when $m \equiv 2 \pmod 4$, $m \geq 6$ and $n \geq 4$.

Moreover, if we consider a super edge-magic labeling of $2P_n$, $n \neq 2$ and 3, from Theorem 4, then applying Theorem 1 we obtain an exponential lower bound for the number of super edge-magic labelings and we can claim the following.

**Theorem 5.** *If $m \equiv 2 \pmod 4$, $m \geq 2$ and $n \geq 4$, then the graph $mP_n$ admits at least $\left[\# \left(\frac{m}{2}\right)\right]^{(2n-2)}$ non-isomorphic super edge-magic labelings.*

It remains to investigate whether $mP_n$ has a super edge-magic labeling for $m \equiv 0 \pmod 4$. Therefore we propose the following open problem.

**Open Problem 1.** *Let $P_n$ be a path of order $n$. For the graph $mP_n$, $m \equiv 0 \pmod 4$, $m \geq 4$ and $n \geq 4$, determine if there is a super edge-magic labeling.*

Figueroa-Centeno et al. [6] studied super edge-magic labeling for the union of a path and a star. They have shown:

**Theorem 6.** *The forest $F \cong P_n \cup K_{1,r}$ is super edge-magic for every integer $n \geq 4$ and $r \geq 1$.*

Again, using the digraph operation $\vec{F} \bigotimes_h S_m$ for $F \cong P_n \cup K_{1,r}$ by Theorem 1 and previous result from Theorem 6 we can claim:

**Theorem 7.** *If $m$ is odd, $m \geq 3$, then the graph $m(P_n \cup K_{1,r})$ admits at least $[\#(m)]^{(n+r-1)}$ non-isomorphic super edge-magic labelings for every integer $n \geq 4$ and $r \geq 1$.*

For the graph $m(P_n \cup K_{1,r})$, for $m$ even, so far we have not found any super edge-magic labeling. For further investigation it leads us to suggest the following.

**Open Problem 2.** *For the graph $m(P_n \cup K_{1,r})$, $m \geq 2$ even, $n \geq 4$ and $r \geq 1$, determine if there is a super edge-magic labeling.*

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