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Note on Super Antimagicness of Disconnected Graphs

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Abstract

A *labeling* of a graph is a mapping that carries some sets of graph elements into numbers (usually the positive integers). An (a, d)-*edge-antimagic total labeling* of a graph $G(V, E)$ with p vertices and q edges is a one-to-one mapping f from $V(G) \cup E(G)$ onto the set $\{1, 2, \dots, |V(G)| + |E(G)|\}$, such that the set of all the edge-weights, $w_f(uv) = f(u) + f(uv) + f(v), uv \in E(G)$, forms an arithmetic sequence starting from a and having a common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

In the paper we mainly study the super (a, d)-edge-antimagic total labelings of disconnected graphs.

Keywords: edge-antimagic total labeling, super edge-antimagic total labeling

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1. Introduction

For a (p, q)-graph G with p vertices and q edges, a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ is a *total labeling* of G and the associated edge-weight of an edge $uv \in E(G)$ is $w_f(uv) = f(u) + f(uv) + f(v)$.

An (a, d)-*edge-antimagic total labeling* ((a, d)-*EAT* for short) of G is a total labeling with the property that the edge-weights form an arithmetic sequence starting from a and having common difference d, where $a > 0$ and $d \geq 0$ are two integers. Definition of (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [13] as a natural extension of *magic valuation* which defined by Kotzig and Rosa in [11]. Magic valuation is also very often known as *edge-magic labeling.* Kotzig and Rosa [11] showed that all caterpillars have magic valuations and conjectured that all trees have magic valuations.

An (a, d)-EAT labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits an (a, d)-EAT labeling or a super (a, d)-EAT labeling is called an (a, d)-EAT graph or super (a, d)-EAT graph, respectively.

Let (p, q)-graph be a super (a, d)-EAT. It is easy to see that the minimum possible edge-weight is at least $p+4$ and the maximum possible edge-weight is no more than $3p+q-1$. Thus

$$a + (q-1)d \leq 3p + q - 1 \quad \text{and} \quad d \leq \frac{2p+q-5}{q-1}.$$

Therefore, for any connected (p, q)-graph where $p - 1 \leq q$, we have $d \leq 3$.

In this paper we mainly investigate the existence of super (a, d)-EAT labeling for disconnected graphs. We concentrate on the following problem: If a graph G is (super) (a, d)-EAT, is the disjoint union of m copies of the graph G, denoted by mG, (super) (a, d)-EAT as well?

2. Super (a, d)-EAT labeling for $d = 0$ and $d = 2$

Figueroa-Centeno, Ichishima and Muntaner-Batle in [8] proved the following theorem.

Theorem 1. [8] *If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.*

The next corollary immediately follows from previous theorem.

Corollary 1. *If T is a (super) edge-magic tree and m is odd, then mT is (super) edge-magic.*

Kotzig and Rosa [11] have shown that all cycles are edge-magic. Thus we have the following corollary.

Corollary 2. *If m is odd and $n > 1$, then the 2-regular graph mC_{2n} is edge-magic.*

Bača, Lin and Muntaner-Batle [1] proved that every path on n vertices has a super edge-magic labeling. From Theorem 1, it follows

Corollary 3. *If m is odd, $m \geq 3$ and $n \geq 2$, then the graph mP_n is super edge-magic.*

In [9], there are several constructions of super edge-magic labelings for disjoint union of graphs. For instance, Ivančo and Lučkaničová [10] described super edge-magic labelings for union of stars $K_{1,m} \cup K_{1,n}$. Chen [5] showed that mP_2 is super edge-magic if and only if m is odd. Bača and Barrientos [2] proved that the graph mK_n has a super edge-magic labeling if and only if either *(i)* $n = 2$ and $m \geq 3$ is odd, or *(ii)* $n = 3$ and $m \geq 3$ is odd. Dafik, Miller, Ryan and Bača [6] showed that the graph $mK_{n,n,n}$ is super edge-magic if and only if $n = 1$ and m is odd, $m \geq 3$.

Bača, Lascsáková and Semaničová [3] used the connection between α-labeling and edge-magic labeling and proved the following theorem.

Theorem 2. [3] *Let G be a graph of order n and size $n-1$, $n \geq 3$. If G admits an α-labeling, and m is odd, $m \geq 1$, then mG admits a super edge-magic labeling.*

Rosa [12] has shown that caterpillar always has an α-labeling. Thus, from Theorem 2, it follows

Corollary 4. *Let C be a caterpillar of order n, $n \geq 3$. If m is odd, $m \geq 1$, then the graph mC is super edge-magic.*

For an even disjoint union of graphs there are only a few known results. Figueroa-Centeno, Ichishima and Muntaner-Batle [8] have shown that the forest $2P_n$, $n > 1$, has a super edge-magic labeling if and only if $n \neq 2$ or 3. Using an operation, which is, in some sense, a generalization of the Kronecker product of matrices for digraphs, Bača, Lin and Muntaner-Batle in [4] proved that if $m \equiv 2 \pmod 4$, $m \geq 6$ and $n \geq 4$, then the graph mP_n admits a super edge-magic labeling.

Next, consider the following result by Figueroa-Centeno et al.

Lemma 1. [7] *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \dots, p\}$, such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f can be extended to a super edge-magic labeling of G with valence (edge-weight) $a = p + q + s$, where $s = min(S)$ and

$$S = \{a - (p + 1), a - (p + 2), \dots, a - (p + q)\}.$$

This lemma implies that all previous results on super edge-magic labelings can be extended to a super $(a - q + 1, 2)$-EAT labelings.

3. Super $(a, 1)$-EAT labeling

In this section, we will study the super $(a, 1)$-EAT labeling of disconnected graphs. The main result is the following,

Theorem 3. *Let G_i be a super $(a, 1)$-EAT graph of order p and size q, $i = 1, 2, \dots, m$. Then the disjoint union $\bigcup_{i=1}^m G_i$ is also a super $(b, 1)$-EAT graph.*

Proof. Let G_i, $i = 1, 2, \dots, m$, be a graph with p vertices and q edges. Note that G_i is not necessarily isomorphic to G_j for $i \neq j$. Suppose that each G_i, $i = 1, 2, \dots, m$, admits a super $(a, 1)$-EAT labeling f_i such that

$$\begin{aligned} f_i : \quad & V(G_i) \longrightarrow \{1, 2, \dots, p\} \\ & E(G_i) \longrightarrow \{p + 1, p + 2, \dots, p + q\} \end{aligned}$$

and

$$\{f_i(u) + f_i(v) + f_i(uv) : uv \in E(G_i)\} = \{a, a + 1, \dots, a + q - 1\}.$$

Define the labeling f for the vertices and edges of $\bigcup_{i=1}^{m} G_i$ in the following way:

$$f(x) = \begin{cases} m[f_i(x) - 1] + i & \text{if } x \in V(G_i), \\ mf_i(x) + 1 - i & \text{if } x \in E(G_i). \end{cases}$$

It is not difficult to see that the function f assigns the labels

$$\begin{array}{cccccc} 1, & m+1, & 2m+1, & \dots & (p-1)m+1 & \text{to the vertices of } G_1, \\ 2, & m+2, & 2m+2, & \dots & (p-1)m+2 & \text{to the vertices of } G_2, \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ i, & m+i, & 2m+i, & \dots & (p-1)m+i & \text{to the vertices of } G_i, \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ m, & 2m, & 3m, & \dots & pm & \text{to the vertices of } G_m. \end{array}$$

And for the edge labels of $\bigcup_{i=1}^{m} G_i$ we have

$$\begin{array}{ccccc} (p+1)m, & (p+2)m, & \dots & (p+q)m & \text{of } G_1, \\ (p+1)m-1, & (p+2)m-1, & \dots & (p+q)m-1 & \text{of } G_2, \\ \vdots & \vdots & & \vdots & \vdots \\ (p+1)m+1-i, & (p+2)m+1-i, & \dots & (p+q)m+1-i & \text{of } G_i, \\ \vdots & \vdots & & \vdots & \vdots \\ pm+1, & (p+1)m+1, & \dots & (p+q-1)m+1 & \text{of } G_m. \end{array}$$

It is easy to see that the labeling f is a bijective function which assigns the set of integers $\{1, 2, \dots, mp+mq\}$ to the vertices and edges of $\bigcup_{i=1}^{m} G_i$, thus f is a total labeling. Furthermore, f assigns the numbers $1, 2, \dots, pm$ to the vertices of $\bigcup_{i=1}^{m} G_i$, therefore, the f is a super total labeling.

For the edge-weight of $uv \in E(G_i)$ we have

$$\begin{aligned} f(u) + f(v) + f(uv) &= m[f_i(u) - 1] + i + m[f_i(v) - 1] + i + mf_i(uv) + 1 - i \\ &= m[f_i(u) + f_i(v) + f_i(uv) - 2] + 1 + i. \end{aligned}$$

It means that the edge-weights in the components are

$$\begin{array}{lcccc} G_1\colon & m(a-2)+2, & m(a-1)+2, & \dots & m(a+q-3)+2 \\ G_2\colon & m(a-2)+3, & m(a-1)+3, & \dots & m(a+q-3)+3 \\ \vdots & \vdots & \vdots & & \vdots \\ G_i\colon & m(a-2)+1+i, & m(a-1)+1+i, & \dots & m(a+q-3)+1+i \\ \vdots & \vdots & \vdots & & \vdots \\ G_m\colon & m(a-1)+1, & ma+1, & \dots & m(a+q-2)+1. \end{array}$$

The reader can easily verify that the edge-weights are distinct and consecutive

$$\{f(u)+f(v)+f(uv) : uv \in E(\bigcup_{i=1}^{m} G_i)\} = \{m(a-2)+2, m(a-2)+3, \dots, m(a+q-2)+1\}.$$

This implies that $\bigcup_{i=1}^{m} G_i$ has a super $(m(a-2)+2, 1)$-EAT labeling. □

Using the previous theorem we can get the following corollary.

Corollary 5. *Let G be a super $(a, 1)$-EAT graph. Then the disjoint union of arbitrary number of copies of G, i.e. mG, $m \geq 1$, also admits a super $(b, 1)$-EAT labeling.*

Moreover for m copies of graph G which is $(a, 1)$-EAT but not super EAT, we can also derive the following result.

Theorem 4. *Let G be an $(a, 1)$-EAT graph. Then mG, $m \geq 1$ is also a $(b, 1)$-EAT graph.*

Proof. Let G be a (p, q) graph and let f be an $(a, 1)$-EAT labeling of G

$$f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, p+q\}.$$

For every vertex v in G, we denote by symbol v_i the corresponding vertex of v in the i-th copy of G in mG. Analogously, let u_iv_i denote the corresponding edge of uv in the i-th copy of G in mG.

We define a labeling g of mG in the following way

$$g(v_i) = m[f(v)-1]+i \qquad \text{for } v \in V(G),\ i = 1, 2, \dots, m,$$
$$g(u_iv_i) = mf(uv)+1-i \qquad \text{for } uv \in E(G),\ i = 1, 2, \dots, m.$$

Let $t \in \{1, 2, \dots, p+q\}$, we consider the following two cases.

Case 1. If the number t is assigned by the labeling f to a vertex of G then the corresponding vertices in the copies in mG will receive labels

$m(t-1)+1,$	$m(t-1)+2,$	$\dots$	$m(t-1)+i,$	$\dots$	$mt.$
in G_1	in G_2	$\dots$	in G_i	$\dots$	in G_m

Case 2. If the number t is assigned by the labeling f to an edge of G then the corresponding edges in the copies in mG will have labels

$mt,$	$mt-1,$	$\dots$	$mt+1-i,$	$\dots$	$m(t-1)+1.$
in G_1	in G_2	$\dots$	in G_i	$\dots$	in G_m.

It is easy to see that the edge labels and vertex labels in mG are not overlapping and the maximum used label is $mp+mq$, thus g is a total labeling. Moreover, following the same line of reasoning as in the proof of Theorem 3, we know that the edge-weights form an arithmetic sequence with the difference 1. This produces the desired result. □

4. Super $(a,3)$-EAT labeling

In this section, we will prove the following theorem for super $(a,3)$-EAT labelings of disjoint union of graphs,

Theorem 5. *Let G_i be a super $(a,3)$-EAT graph of order p and size q, $i=1,2,\dots,m$. The disjoint union $\bigcup_{i=1}^m G_i$ is a super $(b,3)$-EAT graph.*

Proof. Let G_i, $i=1,2,\dots,m$, be a super $(a,3)$-EAT (p,q)-graph. Therefore there exists a super $(a,3)$-EAT labeling f_i for G_i such that

$$f_i: \begin{array}{l} V(G_i) \longrightarrow \{1,2,\dots,p\} \\ E(G_i) \longrightarrow \{p+1,p+2,\dots,p+q\} \end{array}$$

and

$$\{f_i(u)+f_i(v)+f_i(uv) : uv \in E(G_i)\} = \{a, a+3, \dots, a+3(q-1)\}.$$

We define the labeling f for $\bigcup_{i=1}^m G_i$ in the following way,

$$f(x) = m[f_i(x)-1]+i \quad \text{if } x \in V(G_i) \cup E(G_i).$$

It is not difficult to see that the function f assigns the labels

$$\begin{array}{cccccc}
1, & m+1, & 2m+1, & \dots & (p-1)m+1 & \text{to the vertices of } G_1, \\
2, & m+2, & 2m+2, & \dots & (p-1)m+2 & \text{to the vertices of } G_2, \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
i, & m+i, & 2m+i, & \dots & (p-1)m+i & \text{to the vertices of } G_i, \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
m, & 2m, & 3m, & \dots & pm & \text{to the vertices of } G_m.
\end{array}$$

For the edge labels we will have

$$\begin{array}{ccccc}
pm+1, & (p+1)m+1, & \dots & (p+q-1)m+1 & \text{of } G_1, \\
pm+2, & (p+1)m+2, & \dots & (p+q-1)m+2 & \text{of } G_2, \\
\vdots & \vdots & & \vdots & \vdots \\
pm+i, & (p+1)m+i, & \dots & (p+q-1)m+i & \text{of } G_i, \\
\vdots & \vdots & & \vdots & \vdots \\
(p+1)m, & (p+2)m, & \dots & (p+q)m & \text{of } G_m.
\end{array}$$

It is easy to see that f is a total labeling and the numbers $1,2,\dots,pm$ are assigned to the vertices of $\bigcup_{i=1}^m G_i$, i.e. f is a super total labeling.

For the edge-weight of $uv \in E(G_i)$ we get

$$\begin{aligned} f(u)+f(v)+f(uv) &= m[f_i(u)-1]+i+m[f_i(v)-1]+i+m[f_i(uv)-1]+i \\ &= m[f_i(u)+f_i(v)+f_i(uv)-3]+3i. \end{aligned}$$

The edge-weights in the components are

$$\begin{array}{lcccc} G_1: & m(a-3)+3, & ma+3, & \dots & m(a+3q-6)+3 \\ G_2: & m(a-3)+6, & ma+6, & \dots & m(a+3q-6)+6 \\ \vdots & \vdots & \vdots & & \vdots \\ G_i: & m(a-3)+3i, & ma+3i, & \dots & m(a+3q-6)+3i \\ \vdots & \vdots & \vdots & & \vdots \\ G_m: & ma, & m(a+3), & \dots & m(a+3q-3). \end{array}$$

So the set of the edge-weights is

$$\{f(u)+f(v)+f(uv) : uv \in E(\textstyle\bigcup_{i=1}^{m} G_i)\} = \{m(a-3)+3, m(a-3)+6, \dots, m(a+3q-3)\}.$$

This implies that $\bigcup_{i=1}^{m} G_i$ has a super $(m(a-3)+3, 3)$-EAT labeling. □

Following from above theorem, immediately we get

Corollary 6. *Let G be a super $(a,3)$-EAT graph. Then the disjoint union of arbitrary number of copies of G, i.e. mG, $m \geq 1$, also admits a super $(b,3)$-EAT labeling.*

As the technique of the verification is very similar to the proofs presented in the previous sections, we present the following result without the proof.

Theorem 6. *Let G be an $(a,3)$-EAT graph. Then mG, $m \geq 1$, is also a $(b,3)$-EAT graph.*

5. Conclusions

In this paper we deal with the following problem: If a graph G is (super) (a,d)-EAT, is the disjoint union of m copies of the graph G (super) (a,d)-EAT as well? We solved this problem for any connected graph and any m with the differences $d = 1$ and $d = 3$.

For $d = 0$ and $d = 2$, there are a few results known for some families of graphs and when m odd. However, at this point, we do not know anything in general about the existence of super (a,d)-EAT labelings of mG, if G is a super (a,d)-EAT, $m \geq 2$ and $d = 0, 2$. Therefore for the further investigation we propose:

Open Problem 1. *Let G be a super (a,d)-EAT graph, $d \in \{0,2\}$. For the graph mG determine if there is a super (a,d)-EAT labeling, for $d \in \{0,2\}$ and all $m \geq 2$.*

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