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On 5-equitability of one point union of shells

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Abstract

A labeling of a graph is G a function f from the vertex set $V(G)$ to the set of integers $\{0,\dots,k-1\}$. Such a labeling induces a labeling f on the edge set $E(G)$ by defining $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$. Cahit calls this labeling k-equitable if f assigns the labels $\{0,\dots,k-1\}$ equitably to the vertices as well as edges.

Multiple shells are one point unions of many shells. In this paper it is proved that multiple shells are 5-equitable.

Keywords: multiple shells, k-equitable labeling, cordial labeling.

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1. Introduction

Definition 1.1. *A k-labeling of a graph G is a map $f : V(G) \longrightarrow \{0,1,\cdots,k-1\}$. A k-labeling defines a map, also called f from $E(G)$ to $\{0,1,\cdots,k-1\}$ given by $f(xy) = |f(x) - f(y)|$. For a k-labeling f, by $v_f(i)$ we mean the number of vertices with label i and by $e_f(i)$ we mean the number of edges with label i. A k-labeling is said to be k-equitable if $|\ v_f(i) - v_f(j)\ | \leq 1$ and $|\ e_f(i) - e_f(j)\ | \leq 1$ for all i,j.*

In 1987, this concept was introduced and studied by Cahit [3, 4, 5] as a weaker version of the concept of graceful and harmonious graphs. The graphs which are 2-equitable are called cordial.

We take $k = 5$ and to simplify the data we will write $v_f(0,1,2,3,4)$ and $e_f(0,1,2,3,4)$ for the 5-tuple $(v_f(0),\dots,v_f(4))$ and $(e_f(0),\cdots,e_f(4))$ respectively.

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By a *shell* of size n we mean the graph obtained by taking a cycle of length n and adding exactly $n-3$ chords which are concurrent at a vertex in the initial cycle. This vertex is called the *apex* of the shell.

Definition 1.2. *By a multiple shell $MS(n_1, \cdots, n_k)$, we mean the graph obtained by taking the one point union of k shells of sizes $n_1, \cdots, n_k$, where all the apexes are identified.*

Multiple shells were shown to be cordial by Andar et al. [1] and 3-equitable by Bapat and Limaye [2].

For a multiple shell G we have $\mid V(G) \mid = \sum_1^k n_i - k + 1$ and $\mid E(G) \mid = \sum_1^k (2n_i - 3) = 2\sum_1^k n_i - 3k$. Hence if we have 5 shells such that $\sum n_i$ is a multiple of 5 then the number of vertices is of the form $5N+1$ and the number of edges is a multiple of 5. A shell with $5t+j$ vertices is said to be of Type j.

A multiple shell is said to be homogeneous of Type j if it is one point union of shells, all of which are of Type j. For a shell S_n we denote the vertex set as $\{u, v_1, v_2, \cdots, v_{n-1}\}$.

We consider shells S_n where $n \geq 11$.

2. Homogeneous multiple shells of Type 1

We first define several labelings as follows:

1. $O_{A,1}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially repeating $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{A,1}}(0,1,\cdots,4) = (t+1,t,t,t,t)$ and $e_{O_{A,1}}(0,1,\cdots,4) = (2t-1,2t,2t,2t,2t)$.

2. $O_{B,1}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,0,4,3,3,0,2,2,1,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{B,1}}(0,1,\cdots,4) = (t+1,t,t,t,t)$ and $e_{O_{B,1}}(0,1,\cdots,4) = (2t+1,2t,2t-1,2t-1,2t)$.

3. $O_{C,1}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2,4,0,3,3,0,4,2,1,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{C,1}}(0,1,\cdots,4) = (t+1,t,t,t,t)$ and $e_{O_{C,1}}(0,1,\cdots,4) = (2t,2t-1,2t,2t,2t)$.

4. $O_{D,1}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1,2,0,3,3,0,4,4,2,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{D,1}}(0,1,\cdots,4) = (t+1,t,t,t,t)$ and $e_{O_{D,1}}(0,1,\cdots,4) = (2t,2t,2t,2t,2t-1)$.

The following table gives the consolidated data:

Labeling f	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$	Equitable or not
$O_{A,1}$	$(t+1,t,t,t,t)$	$(2t-1,2t,2t,2t,2t)$	Equitable
$O_{B,1}$	$(t+1,t,t,t,t)$	$(2t+1,2t,2t-1,2t-1,2t)$	Non-equitable
$O_{C,1}$	$(t+1,t,t,t,t)$	$(2t,2t-1,2t,2t,2t)$	Equitable
$O_{D,1}$	$(t+1,t,t,t,t)$	$(2t,2t,2t,2t,2t-1)$	Equitable

We use this data to prove the following:

Proposition 2.1. *A homogeneous multiple shell of Type 1 obtained by taking one point union of k shells of Type 1, $2 \le k \le 5$ is 5-equitable.*

Proof. Let S_i be shell of size $n_i = 5t_i + 1$. We assign the labelings f_i, to this shell, $2 \le i \le 5$ and call the resulting labeling f. We note that the total number of edges is $\sum(2n_i - 3) = 10\sum t_i - k$. The following table gives the data of these labelings: In this table T stands for $\sum t_i$.

f_1	f_2	f_3	f_4	f_5	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$O_{A,1}$	$O_{B,1}$	–	–	–	$(T+1,T,T,T,T,)$	$(2T,2T,2T-1,2T-1,2T)$
$O_{A,1}$	$O_{B,1}$	$O_{C,1}$	–	–	$(T+1,T,T,T,T,)$	$(2T,2T-1,2T-1,2T-1,2T)$
$O_{A,1}$	$O_{B,1}$	$O_{C,1}$	$O_{D,1}$	–	$(T+1,T,T,T,T)$	$(2T,2T-1,2T-1,2T-1,2T-1)$
$O_{A,1}$	$O_{A,1}$	$O_{B,1}$	$O_{C,1}$	$O_{D,1}$	$(T+1,T,T,T,T)$	$(2T-1,2T-1,2T-1,2T-1,2T-1)$

This shows that all the homogeneous multiple shells of Type 1, where at most 5 shells are involved are 5-equitable. □

Theorem 2.2. *All homogeneous shells of Type 1 are 5-equitable.*

Proof. Let G be a graph obtained by taking one point union of k shells $S_1, S_2, \cdots, S_k$ of Type 1 with $5t_i + 1$ vertices $1 \le i \le k$. Let $k = 5m + r, 0 \le r \le 4$. Every time when we finish the labeling process, we call the resulting labeling f.

First form s number of 5-tuple shells using $\{S_{5z+1}, S_{5z+2}, S_{5z+3}, S_{5z+4}, S_{5z+5}\}, 0 \le z \le m-1$. Assign the labeling constructed in the previous Proposition to each of them. For each of these labelings the vertices other than the apex as well as all the edges are equitably labeled. Moreover the apex has been assigned the label 0. Hence if $r = 0$ it completes the labeling and $v_f(0,1,2,3,4) = (T+1,T,T,T,T)$ and $e_f(0,1,2,3,4) = (2T-m, 2T-m, 2T-m, 2T-m-1, 2T-m)$ where $T = \sum_{1}^{5s} t_i$.

If $r = 1$ one shell remains to be labeled. Assign the labeling $O_{A,1}$ to it. The resulting labeling has $v_f(0,1,2,3,4) = (T+1,T,T,T,T), e_f(0,1,2,3,4) = (2T-m-1, 2T-m, 2T-m, 2T-m, 2T-m)$.

If $r = 2$ two shells remain to be labeled. Assign the labeling given for double shell in the previous proposition. The resulting labeling has $v_f(0,1,2,3,4) = (T+1,T,T,T,T), e_f(0,1,2,3,4) = (2T-m, 2T-m, 2T-m-1, 2T-m-1, 2T-m)$.

If $r = 3$ three shells remain to be labeled. Assign the labeling given for triple shell in the previous proposition. The resulting labeling has $v_f(0,1,2,3,4) = (T+1,T,T,T,T), e_f(0,1,2,3,4) = (2T-m, 2T-m-1, 2T-m-1, 2T-m-1, 2T-m)$.

If $r = 4$ four shells remain to be labeled. Assign the labeling given for 4-tuple shell in the previous proposition. The resulting labeling has $v_f(0,1,2,3,4) = (T+1,T,T,T,T), e_f(0,1,2,3,4) = (2T-m, 2T-m-1, 2T-m-1, 2T-m-1, 2T-m-1)$.

The following table consolidates the vertex numbers and edge numbers for these labelings.

k	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$5m$	$(T+1,T,T,T,T))$	$(2T-m, 2T-m, 2T-m, 2T-m, 2T-m)$
$5m+1$	$(T+1,T,T,T,T)$	$(2T-m-1, 2T-m, 2T-m, 2T-m, 2T-m)$
$5m+2$	$(T+1,T,T,T,T)$	$(2T-m, 2T-m, 2T-m-1, 2T-m-1, 2T-m)$
$5m+3$	$(T+1,T,T,T,T)$	$(2T-m, 2T-m-1, 2T-m-1, 2T-m-1, 2T-m)$
$5m+4$	$(T+1,T,T,T,T)$	$(2T-m, 2T-m-1, 2T-m-1, 2T-m-1, 2T-m-1)$

□

3. Homogeneous shells of Type 2

We first define several labelings as follows:

1. $O_{A,2}$: This is defined as follows: Assign the label 0 to the apex u. and 1 to v_1. Label $v_2, v_3, \cdots$ sequentially repeating $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{A,2}}(0,1,\cdots,4) = (t+1,t+1,t,t,t)$ and $e_{O_{A,1}}(0,1,\cdots,4) = (2t, 2t+1, 2t, 2t, 2t)$.

2. $O_{B,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,2,2,4,1,3,4,0,3,0,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{B,2}}(0,1,\cdots,4) = (t+1,t,t,t,t+1)$ and $e_{O_{B,2}}(0,1,\cdots,4) = (2t-1, 2t, 2t+1, 2t+1, 2t)$.

3. $O_{C,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,4,0,2,2,4,1,3,3,0,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{C,2}}(0,1,\cdots,4) = (t+1,t,t,t,t+1)$ and $e_{O_{C,2}}(0,1,\cdots,4) = (2t+1, 2t-1, 2t+1, 2t, 2t)$.

4. $O_{D,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $3,4,1,3,0,4,4,0,2,2,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{D,2}}(0,1,\cdots,4) = (t+1,t,t,t,t+1)$ and $e_{O_{D,2}}(0,1,\cdots,4) = (2t, 2t, 2t, 2t, 2t+1)$.

5. This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2,2,2,1,3,3,3,4,0,4,1$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{E,2}}(0,1,\cdots,4) = (t,t,t+1,t+1,t)$ and $e_{O_{E,2}}(0,1,\cdots,4) = (2t+1, 2t, 2t, 2t, 2t)$.

6. $O_{F,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $3, 3, 0, 1, 4, 0, 4, 2, 2, 3, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{F,2}}(0, 1, \cdots, 4) = (t+1, t, t, t+1, t)$ and $e_{O_{F,2}}(0, 1, \cdots, 4) = (2t, 2t, 2t, 2t+1, 2t)$.

7. $O_{G,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 4, 0, 4, 2, 2, 3, 3, 0, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{G,2}}(0, 1, \cdots, 4) = (t+1, t, t+1, t, t)$ and $e_{O_{G,2}}(0, 1, \cdots, 4) = (2t, 2t, 2t+1, 2t, 2t)$.

8. $O_{H,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 4, 0, 4, 3, 3, 0, 2, 2, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{H,2}}(0, 1, \cdots, 4) = (t+1, t, t+1, t, t)$ and $e_{O_{H,2}}(0, 1, \cdots, 4) = (2t+1, 2t, 2t, 2t, 2t)$.

9. $O_{I,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $0, 1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{I,2}}(0, 1, \cdots, 4) = (t+2, t, t, t, t)$ and $e_{O_{I,2}}(0, 1, \cdots, 4) = (2t, 2t+1, 2t, 2t, 2t)$.

10. $O_{J,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $0, 1, 3, 3, 3, 2, 2, 4, 0, 4, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{J,2}}(0, 1, \cdots, 4) = (t+1, t, t, t+1, t)$ and $e_{O_{J,2}}(0, 1, \cdots, 4) = (2t+1, 2t, 2t, 2t, 2t)$.

11. $O_{K,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4, 0, 4, 1, 3, 3, 3, 2, 2, 0, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{K,2}}(0, 1, \cdots, 4) = (t+1, t, t, t+1, t)$ and $e_{O_{K,2}}(0, 1, \cdots, 4) = (2t+1, 2t, 2t, 2t, 2t)$.

12. $O_{L,2}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4, 0, 4, 1, 3, 0, 3, 2, 2, 0, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{L,2}}(0, 1, \cdots, 4) = (t+2, t, t, t, t)$ and $e_{O_{L,2}}(0, 1, \cdots, 4) = (2t, 2t, 2t, 2t+1, 2t)$.

The following table gives the consolidated data:

Labeling f	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$	Equitable or not
$O_{A,2}$	$(t+1, t+1, t, t, t)$	$(2t, 2t+1, 2t, 2t, 2t)$	Equitable
$O_{B,2}$	$(t+1, t, t, t, t+1)$	$(2t-1, 2t, 2t+1, 2t+1, 2t)$	Non-equitable
$O_{C,2}$	$(t+1, t, t, t, t+1)$	$(2t+1, 2t-1, 2t+1, 2t, 2t)$	Non-equitable
$O_{D,2}$	$(t+1, t, t, t, t+1)$	$(2t, 2t, 2t, 2t, 2t+1)$	Equitable
$O_{E,2}$	$(t, t, t+1, t+1, t)$	$(2t+1, 2t, 2t, 2t, 2t)$	Equitable
$O_{F,2}$	$(t+1, t, t, t+1, t)$	$(2t, 2t, 2t, 2t+1, 2t)$	Equitable
$O_{G,2}$	$(t+1, t, t+1, t, t)$	$(2t, 2t, 2t+1, 2t, 2t)$	Equitable
$O_{H,2}$	$(t+1, t, t+1, t, t)$	$(2t+1, 2t, 2t, 2t, 2t)$	Equitable
$O_{I,2}$	$(t+2, t, t, t, t)$	$(2t, 2t+1, 2t, 2t, 2t)$	Non-equitable
$O_{J,2}$	$(t+1, t, t, t+1, t)$	$(2t+1, 2t, 2t, 2t, 2t)$	Equitable
$O_{K,2}$	$(t+1, t, t, t+1, t)$	$(2t+1, 2t, 2t, 2t, 2t)$	Equitable
$O_{L,2}$	$(t+2, t, t, t, t)$	$(2t, 2t, 2t, 2t+1, 2t)$	Non-Equitable

We use this data to prove the following:

Proposition 3.1. *A homogeneous multiple shell of Type 2 obtained by taking one point union of k shells of Type 1, $2 \leq k \leq 5$ is 5-equitable.*

Proof. Let S_i be shell of size $n_i = 5t_i + 1$. We assign the labelings f_i to this shell, $2 \leq i \leq k$ and call the resulting labeling f. The total number of edges in the resulting multiple shell is $\sum(2n_i - 3) = \sum(10t_i + 1) = 10T + k$.

The following table gives the data of these labelings: In this table T stands for $\sum t_i$.

f_1	f_2	f_3	f_4	f_5	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$O_{A,2}$	$O_{C,2}$	$-$	$-$	$-$	$(T+1, T+1, T, T, T+1,)$	$(2T+1, 2T, 2T+1, 2T, 2T)$
$O_{D,2}$	$O_{G,2}$	$O_{J,2}$	$-$	$-$	$(T+1, T, T+1, T+1, T+1,)$	$(2T+1, 2T, 2T+1,$ $2T, 2T+1)$
$O_{A,2}$	$O_{G,2}$	$O_{D,2}$	$O_{J,2}$	$-$	$(T+1, T+1, T+1, T+1, T+1$	$(2T+1, 2T+1,$ $2T+1, 2T, 2T+1)$
$O_{A,2}$	$O_{G,2}$	$O_{D,2}$	$O_{J,2}$	$O_{L,2}$	$(T+2, T+1, T+1, T+1, T+1)$	$(2T+1, 2T+1, 2T+1,$ $2T+1, 2T+1)$

This incompletes the proof. □

Theorem 3.2. *All homogeneous shells of Type 2 are 5-equitable.*

Proof. Similar to that of Theorem 2.2. □

4. Homogeneous shells of Type 3

We first define several labelings as follows:

1. $O_{A,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $3, 1, 1, 2, 4, 0, 3, 3, 0, 4, 2, 1.$ once followed by $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ One can easily see that $v_{O_{A,3}}(0, 1, \cdots, 4) =, (t+1, t+1, t, t+1, t)$ and $e_{O_{A,3}}(0, 1, \cdots, 4) = (2t, 2t+1, 2t+1, 2t+1, 2t)$.
2. $O_{B,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $3, 3, 0, 4, 4, 1, 0, 4, 2, 2, 3, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{B,3}}(0, 1, \cdots, 4) = (t+1, t, t, t+1, t+1)$ and $e_{O_{B,3}}(0, 1, \cdots, 4) = (2t+1, 2t, 2t, 2t+1, 2t+1)$.
3. $O_{C,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 2, 1, 4, 4, 0, 3, 3, 0, 4, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{C,3}}(0, 1, \cdots, 4) = (t+1, t, t+1, t, t+1)$ and $e_{O_{C,1}}(0, 1, \cdots, 4) = (2t+1, 2t, 2t, 2t+1, 2t+1)$.

4. $O_{D,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $3, 3, 2, 2, 2, 0, 4, 4, 1, 4, 0, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{D,3}}(0, 1, \cdots, 4) = (t{+}1, t, t{+}1, t, t{+}1)$ and $e_{O_{D,3}}(0, 1, \cdots, 4) = (2t + 2, 2t, 2t, 2t, 2t + 1)$.

5. $O_{E,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 2, 2, 3, 3, 3, 1, 4, 0, 4, 0, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{E,3}}(0, 1, \cdots, 4) = (t{+}1, t, t{+}1, t{+}1, t)$ and $e_{O_{E,3}}(0, 1, \cdots, 4) = (2t + 2, 2t, 2t, 2t, 2t + 1)$.

6. $O_{F,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 4, 0, 4, 0, 3, 3, 1, 2, 0, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{F,3}}(0, 1, \cdots, 4) = (t + 2, t + 1, t, t, t)$ and $e_{O_{F,3}}(0, 1, \cdots, 4) = (2t, 2t + 1, 2t + 1, 2t, 2t + 1)$.

7. $O_{G,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 4, 0, 4, 4, 2, 2, 3, 3, 0, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{G,3}}(0, 1, \cdots, 4) = (t{+}1, t, t{+}1, t, t{+}1)$ and $e_{O_{G,3}}(0, 1, \cdots, 4) = (2t + 1, 2t, 2t + 1, 2t, 2t + 1)$.

8. $O_{H,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $0, 3, 2, 4, 0, 3, 3, 1, 0, 4, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{H,3}}(0, 1, \cdots, 4) = (t + 2, t, t, t + 1, t)$ and $e_{O_{H,3}}(0, 1, \cdots, 4) = (2t, 2t + 1, 2t + 1, 2t + 1, 2t)$.

9. $O_{I,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $0, 3, 2, 4, 0, 4, 1, 3, 0, 2, 1, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{I,3}}(0, 1, \cdots, 4) = (t + 2, t + 1, t, t, t)$ and $e_{O_{I,3}}(0, 1, \cdots, 4) = (2t, 2t + 1, 2t + 1, 2t + 1, 2t)$.

10. $O_{J,3}$: This is defined as follows: assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $0, 2, 1, 4, 4, 0, 3, 3, 0, 4, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{J,3}}(0, 1, \cdots, 4) = (t + 2, t, t, t, t + 1)$ and $e_{O_{J,3}}(0, 1, \cdots, 4) = (2t + 1, 2t, 2t, 2t + 1, 2t + 1)$.

The following table gives the consolidated data:

Labeling f	$v_f(0, 1, 2, 3, 4)$	$e_f(0, 1, 2, 3, 4)$	Equitable or not
$O_{A,3}$	$(t + 1, t + 1, t, t + 1, t)$	$(2t, 2t + 1, 2t + 1, 2t + 1, 2t)$	Equitable
$O_{B,3}$	$(t + 1, t, t, t + 1, t + 1)$	$(2t + 1, 2t, 2t, 2t + 1, 2t + 1)$	Equitable
$O_{C,3}$	$(t + 1, t, t + 1, t, t + 1)$	$(2t + 1, 2t, 2t, 2t + 1, 2t + 1)$	Equitable
$O_{D,3}$	$(t + 1, t, t + 1, t, t + 1)$	$(2t + 2, 2t, 2t, 2t, 2t + 1)$	Non-equitable
$O_{E,3}$	$(t + 1, t, t + 1, t + 1, t)$	$(2t + 2, 2t, 2t, 2t, 2t + 1)$	Non-equitable
$O_{F,3}$	$(t + 2, t + 1, t, t, t)$	$(2t, 2t + 1, 2t + 1, 2t, 2t + 1)$	Non-equitable
$O_{G,3}$	$(t + 1, t, t + 1, t, t + 1)$	$(2t + 1, 2t, 2t + 1, 2t, 2t + 1)$	Equitable
$O_{H,3}$	$(t + 2, t, t, t + 1, t)$	$(2t, 2t + 1, 2t + 1, 2t + 1, 2t)$	Non-equitable
$O_{I,3}$	$(t + 2, t + 1, t, t, t)$	$(2t, 2t + 1, 2t + 1, 2t + 1, 2t)$	Non-equitable
$O_{J,3}$	$(t + 2, t, t, t, t + 1)$	$(2t + 1, 2t, 2t, 2t + 1, 2t + 1)$	Non-equitable

We use this data to prove the following:

Proposition 4.1. *A homogeneous multiple shell of Type 3 obtained by taking one point union of k shells of Type 1,* $2 \leq k \leq 5$ *is* 5*-equitable.*

Proof. Let S_i be shell of size $5t_i + 1$. We assign the labelings $f_1, \cdots$ to these shells and call the resulting labeling f. The following table gives the data of these labelings: In this table T stands for $\sum t_i$.

f_1	f_2	f_3	f_4	f_5	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$O_{A,3}$	$O_{C,3}$	$-$	$-$	$-$	$(T+1,T+1,T+1,T+1,T+1,)$	$(2T+1,2T+1,2T+1,$ $2T+2,2T+1)$
$O_{A,3}$	$O_{D,3}$	$O_{I,3}$	$-$	$-$	$(T+2,T+2,T+1,T+1,T+1,)$	$(2T+2,2T+2,2T+2,$ $2T+2,2T+1)$
$O_{A,3}$	$O_{C,3}$	$O_{E,3}$	$O_{F,3}$	$-$	$(T+2,T+2,T+2,T+2,T+1)$	$(2T+3,2T+2,2T+2,$ $2T+2,2T+3)$
$O_{A,3}$	$O_{C,3}$	$O_{D,3}$	$O_{F,3}$	$O_{H,3}$	$(T+3,T+2,T+2,T+2,T+2)$	$(2T+3,2T+3,2T+3,$ $2T+3,2T+3)$

This shows that all the homogeneous multiple shells of Type 3, where at most 5 shells are involved, are 5-equitable. □

This can be used as before to prove the following:

Theorem 4.2. *All homogeneous multiple shells of Type 3 are* 5*-equitable.*

5. Homogeneous shells of Type 4

We first define several labelings as follows:

1. $O_{A,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,4,0,3,3,1,2,2$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{A,4}}(0,1,\cdots,4) = (t+1,t,t+1,t+1,t+1)$ and $e_{O_{A,4}}(0,1,\cdots,4) = (2t+1,2t+1,2t+1,2t+1,2t+1)$.
2. $O_{B,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1,1,3,3,0,4,4,2$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{B,4}}(0,1,\cdots,4) = (t+1,t+1,t,t+1,t+1)$ and $e_{O_{B,4}}(0,1,\cdots,4) = (2t+1,2t+1,2t+1,2t+1,2t+1)$.
3. $O_{C,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2,2,1,1,4,4,0,3$ once followed by repeatedly $1,2,4,0,3,3,0,4,2,1$. One can easily see that $v_{O_{C,4}}(0,1,\cdots,4) = (t+1,t+1,t+1,t,t+1)$ and $e_{O_{C,4}}(0,1,\cdots,4) = (2t+1,2t+1,2t+1,2t+1,2t+1)$.

4. $O_{D,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 2, 1, 1, 4, 0, 0, 3$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{D,4}}(0, 1, \cdots, 4) = (t+2, t+1, t+1, t, t)$ and $e_{O_{D,4}}(0, 1, \cdots, 4) = (2t + 2, 2t + 1, 2t + 1, 2t + 1, 2t)$.

5. $O_{E,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 2, 1, 1, 3, 0, 4, 0$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{E,4}}(0, 1, \cdots, 4) = (t+2, t+1, t+1, t, t)$ and $e_{O_{E,4}}(0, 1, \cdots, 4) = (2t + 1, 2t + 2, 2t + 1, 2t, 2t + 1)$.

6. $O_{F,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 1, 3, 3, 0, 4, 0, 2$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{F,4}}(0, 1, \cdots, 4) = (t+2, t+1, t, t+1, t)$ and $e_{O_{F,4}}(0, 1, \cdots, 4) = (2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$.

7. $O_{G,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 2, 2, 4, 4, 0, 3, 0$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{G,4}}(0, 1, \cdots, 4) = (t+2, t, t+1, t, t+1)$ and $e_{O_{G,4}}(0, 1, \cdots, 4) = (2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$.

8. $O_{H,4}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 3, 3, 0, 4, 0, 2, 1$ once followed by repeatedly $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$. One can easily see that $v_{O_{H,4}}(0, 1, \cdots, 4) = (t+2, t, t+1, t+1, t)$ and $e_{O_{H,4}}(0, 1, \cdots, 4) = (2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$.

The following table gives the consolidated data:

Labeling f	$v_f(0, 1, 2, 3, 4)$	$e_f(0, 1, 2, 3, 4)$	Equitable or not
$O_{A,4}$	$(t + 1, t, t + 1, t + 1, t + 1)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Equitable
$O_{B,4}$	$(t + 1, t + 1, t, t + 1, t + 1)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Equitable
$O_{C,4}$	$(t + 1, t + 1, t + 1, t, t + 1)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Equitable
$O_{D,4}$	$(t + 2, t + 1, t + 1, t, t)$	$(2t + 2, 2t + 1, 2t + 1, 2t + 1, 2t)$	Non-equitable
$O_{E,4}$	$(t + 2, t + 1, t + 1, t, t)$	$(2t + 1, 2t + 2, 2t + 1, 2t, 2t + 1)$	Non-equitable
$O_{F,4}$	$(t + 2, t + 1, t, t + 1, t)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Non-equitable
$O_{G,4}$	$(t + 2, t, t + 1, t, t + 1)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Non-equitable
$O_{H,4}$	$(t + 2, t, t + 1, t + 1, t)$	$(2t + 1, 2t + 1, 2t + 1, 2t + 1, 2t + 1)$	Non-equitable

We use this data to prove the following:

Proposition 5.3. *A homogeneous multiple shell of Type 4 obtained by taking one point union of k shells of Type 4,* $2 \leq k \leq 5$ *is* 5*-equitable.*

Proof. Let S_i be shell of size $5t_i + 4$. We assign the labelings f_i to this shell, $2 \leq i \leq 5$ and call the resulting labeling f. The following table gives the data of these labelings: As before, in this table T stands for $\sum t_i$.

f_1	f_2	f_3	f_4	f_5	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$O_{A,4}$	$O_{F,4}$	–	–	–	$(T+2,T+1,T+1,T+2,T+1)$	$(2T+2,2T+2,2T+2,$ $2T+2,2T+2)$
$O_{A,4}$	$O_{C,4}$	$O_{F,4}$	–	–	$(T+2,T+2,T+2,T+2,T+2,)$	$(2T+3,2T+3,2T+3,$ $2T+3,2T+3)$
$O_{A,4}$	$O_{B,4}$	$O_{F,4}$	$O_{G,4}$	–	$(T+3,T+2,T+2,T+3,T+3)$	$(2T+4,2T+4,2T+4,$ $2T+4,2T+4)$
$O_{A,4}$	$O_{C,4}$	$O_{F,4}$	$O_{F,4}$	$O_{G,4}$	$(T+4,T+3,T+3,T+3,T+3)$	$(2T+5,2T+5,2T+5,$ $2T+5,2T+5)$

This shows that all the homogeneous multiple shells of Type 4, where at most 5 shells are involved, are 5-equitable. □

This can be used to prove the following:

Theorem 5.4. *All homogeneous multiple shells of Type 4 are* 5*-equitable.*

6. Homogeneous shells of Type 5

We first define several labelings as follows:

1. $O_{A,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,4,0,3,3,1,2,2,1$ once followed repeatedly by $1,2,4,0,3,3,0,4,2,1$ One can easily see that $v_{O_{A,5}}(0,1,\cdots,4) = (t+1,t+1,t+1,t+1,t+1)$ and $e_{O_{A,5}}(0,1,\cdots,4) = (2t+2,2t+2,2t+1,2t+1,2t+1)$.
2. $O_{B,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2,2,1,4,0,0,4,3,3$ once followed repeatedly by $1,2,4,0,3,3,0,4,2,1$ One can easily see that $v_{O_{B,5}}(0,1,\cdots,4) = (t+2,t,t+1,t+1,t+1)$ and $e_{O_{B,5}}(0,1,\cdots,4) = (2t+2,2t+1,2t+1,2t+1,2t+2)$.
3. $O_{C,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1,4,0,4,2,2,3,3,1$ once followed repeatedly by $1,2,4,0,3,3,0,4,2,1$ One can easily see that $v_{O_{C,5}}(0,1,\cdots,4) = (t+1,t+1,t+1,t+1,t+1)$ and $e_{O_{C,5}}(0,1,\cdots,4) = (2t+1,2t+1,2t+2,2t+1,2t+2)$.
4. $O_{D,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $4,1,4,0,2,2,3,3,1$ once followed repeatedly by $1,2,4,0,3,3,0,4,2,1$ One can easily see that $v_{O_{D,5}}(0,1,\cdots,4) = (t+1,t+1,t+1,t+1,t+1)$ and $e_{O_{D,5}}(0,1,\cdots,4) = (2t+1,2t+1,2t+2,2t+2,2t+1)$.
5. $O_{E,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2,2,3,1,1,4,0,0,4$ once followed repeatedly by $1,2,4,0,3,3,0,4,2,1$ One can easily see that $v_{O_{E,5}}(0,1,\cdots,4) = (t+2,t+1,t+1,t,t+1)$ and $e_{O_{E,5}}(0,1,\cdots,4) = (2t+2,2t+1,2t+1,2t+1,2t+2)$.

6. $O_{F,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 3, 0, 4, 1, 1, 4, 2, 0$ once followed repeatedly by $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ One can easily see that $v_{O_{F,5}}(0, 1, \cdots, 4) = (t+2, t+1, t+1, t, t+1)$ and $e_{O_{F,5}}(0, 1, \cdots, 4) = (2t, 2t+2, 2t+2, 2t+2, 2t+1)$.

7. $O_{G,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $1, 1, 4, 0, 0, 4, 2, 3, 3$ once followed repeatedly by $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ One can easily see that $v_{O_{G,5}}(0, 1, \cdots, 4) = (t+2, t+1, t, t+1, t+1)$ and $e_{O_{G,5}}(0, 1, \cdots, 4) = (2t+2, 2t+1, 2t+1, 2t+1, 2t+2)$.

8. $O_{H,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 3, 0, 0, 4, 1, 1, 4, 2$ once followed repeatedly by $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ One can easily see that $v_{O_{H,5}}(0, 1, \cdots, 4) = (t+2, t+1, t+1, t, t+1)$ and $e_{O_{H,5}}(0, 1, \cdots, 4) = (2t+1, 2t+2, 2t+1, 2t+2, 2t+1)$.

9. $O_{I,5}$: This is defined as follows: Assign the label 0 to the apex u. Then assign the labels to $v_1, v_2, \cdots$ sequentially $2, 2, 1, 3, 3, 0, 4, 0, 1$ once followed repeatedly by $1, 2, 4, 0, 3, 3, 0, 4, 2, 1$ One can easily see that $v_{O_{I,5}}(0, 1, \cdots, 4) = (t+2, t+1, t+1, t+1, t)$ and $e_{O_{I,5}}(0, 1, \cdots, 4) = (2t+2, 2t+2, 2t+1, 2t+1, 2t+1)$.

The following table gives the consolidated data:

Labeling f	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$	Equitable or not
$O_{A,5}$	$(t+1, t+1, t+1, t+1, t+1)$	$(2t+2, 2t+2, 2t+1, 2t+1, 2t+1)$	Equitable
$O_{B,5}$	$(t+2, t, t+1, t+1, t+1)$	$(2t+2, 2t+1, 2t+1, 2t+1, 2t+2)$	Non-equitable
$O_{C,5}$	$(t+1, t+1, t+1, t+1, t+1)$	$(2t+1, 2t+1, 2t+2, 2t+1, 2t+2)$	Equitable
$O_{D,5}$	$(t+1, t+1, t+1, t+1, t+1)$	$(2t+1, 2t+1, 2t+2, 2t+2, 2t+1)$	Equitable
$O_{E,5}$	$(t+2, t+1, t+1, t, t+1)$	$(2t+2, 2t+1, 2t+1, 2t+1, 2t+2)$	Non-equitable
$O_{F,5}$	$(t+2, t+1, t+1, t, t+1)$	$(2t, 2t+2, 2t+2, 2t+2, 2t+1)$	Non-equitable
$O_{G,5}$	$(t+2, t+1, t, t+1, t+1)$	$(2t+2, 2t+1, 2t+1, 2t+1, 2t+2)$	Non-equitable
$O_{H,5}$	$(t+2, t+1, t+1, t, t+1)$	$(2t+1, 2t+2, 2t+1, 2t+2, 2t+1)$	Non-equitable
$O_{I,5}$	$(t+2, t+1, t+1, t+1, t)$	$(2t+2, 2t+2, 2t+1, 2t+1, 2t+1)$	Non-equitable

We use this data to prove the following:

Proposition 6.1. *A homogeneous multiple shell of Type 5 obtained by taking one point union of k shells of Type 1,* $2 \le k \le 5$ *is* 5*-equitable.*

Proof. Let S_i be shell of size $5t_i + 1$. We assign the labelings f_1 to these shells and call the resulting labeling f. The following table gives the data of these labelings: In this table T stands for $\sum t_i$.

f_1	f_2	f_3	f_4	f_5	$v_f(0,1,2,3,4)$	$e_f(0,1,2,3,4)$
$O_{C,5}$	$O_{H,5}$	$-$	$-$	$-$	$(T+2,T+2,T+2,T+1,T+2)$	$(2T+2,2T+3,2T+3,$ $2T+3,2T+3)$
$O_{B,5}$	$O_{D,5}$	$O_{I,5}$	$-$	$-$	$(T+3,T+2,T+3,T+3,T+2,)$	$(2T+5,2T+4,2T+4,$ $2T+4,2T+4)$
$O_{B,5}$	$O_{D,5}$	$O_{F,5}$	$O_{I,5}$	$-$	$(T+4,T+3,T+4,T+3,T+3)$	$(2T+5,2T+6,2T+6,$ $2T+6,2T+5)$
$O_{B,5}$	$O_{D,5}$	$O_{F,5}$	$O_{G,5}$	$O_{I,5}$	$(T+5,T+4,T+4,T+4,T+4)$	$(2T+7,2T+7,2T+7,$ $2T+7,2T+7)$

This shows that all the homogeneous multiple shells of Type 5, where at most 5 shells are involved, are 5-equitable. □

Again using this one can prove the following:

Theorem 6.2. *All homogeneous multiple shells of Type 5 are* 5*-equitable.*

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