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Recent advances in Rosa-type labelings of graphs

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Abstract

This short survey complements a recent comprehensive survey by Saad El-Zanati and Charles Vanden Eynden [6]. We present here some new generalizations of ρ- and α-labelings which were not known when their survey was written.

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1. Introduction

Decomposition of complete and complete bipartite graphs into mutually isomorphic subgraphs is a topic on the boundary between graph theory and design theory, which relies heavily on graph labelings. For a comprehensive survey, we refer the reader to the recent paper by S. El-Zanati and C. Vanden Eynden [6]. To make this paper self-contained, we first give a brief overview of known labeling methods used in decompositions of complete graphs. Then we present some newer methods which appeared since [6] was written.

Definition 1.1. *Let G be a graph with n vertices (isolated vertices are allowed). We say that the complete graph K_n has a G-decomposition if there are subgraphs $G_1, G_2, \dots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i.*

The decomposition is cyclic if $s = n$ and there exists an ordering $(x_1, x_2, \dots, x_n)$ of the vertices of K_n and isomorphisms $\phi_i : G \to G_i, i = 1, 2, \dots, n$ such that $\phi_i(x_j) = x_{i+j}$ for every $j = 1, 2, \dots, n$, where the subscripts are taken modulo n.

In 1975, R. Wilson [17] presented the following fundamental result on G-decompositions of K_n.

Theorem 1.2. *Let G be a graph with m edges. Then there exists a constant n_0 such that for every $n \geq n_0$ satisfying $m|n(n-1)/2$ and the necessary degree conditions the complete graph K_n has a G-decomposition.*

In light of this result, we are left with the quest for G-decompositions of complete graphs whose orders are "small" relative to the number of edges of G. Results of this type are often achieved by employing graph labeling techniques.

In 1967 A. Rosa [16] introduced some important types of vertex labelings. Because we will need a definition of a labeling which is more general than the classical one, we state it here.

Definition 1.3. *A labeling of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of elements of an Abelian group $\mathcal{A}$ of order $2n+1$.*

The most general among the original Rosa's labelings is the rosy labeling (called ρ-valuation by AR).

Definition 1.4. *Let G be a graph with n edges and ρ be an injection from $V(G)$ into a subset S of elements of the additive group Z_{2n+1}. The length of an edge e with endvertices x and y is defined as $\ell(e) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. The injection ρ is called rosy labeling (also ρ-labeling or ρ-valuation) if the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$.*

The importance of the rosy labeling can be seen from the following result due to Rosa [16].

Theorem 1.5. *Let G be a graph with n edges. Then the complete graph K_{2n+1} has a cyclic G-decomposition if and only if G admits a rosy labeling.*

A proof of this theorem is based on the fact that a labeled K_{2n+1} has exactly $2n+1$ edges of length i for every $i = 1, 2, \ldots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is then constructed by taking a labeled copy of G, say G_0, and then adding an element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \ldots, 2n$.

Besides the rosy labeling, A. Rosa introduced some other labelings, which are special cases of the rosy labeling. Although rosy labeling is the most general (and in some sense most useful), it is the graceful labeling which is best known.

Definition 1.6. *Let G be a graph with n edges and β be a rosy labeling of G. If the set of labels of vertices of G is a subset of the set $\{1, 2, \ldots, n\}$, then β is called a graceful labeling (or β-labeling or β-valuation).*

Obviously, because every graceful labeling is a rosy labeling, its existence for a given G implies the existence of a cyclic G-decomposition of K_{2n+1}. The rosy labeling is less restrictive and therefore often easier to find. It is worth noting that in spite of that there has been significantly more effort in the investigation of graceful graphs than in rosy graphs. (We say that a graph *is* graceful or rosy when it has a graceful or rosy labeling.) The popularity of graceful graphs is affected, besides the attractive name, by the Kotzig-Ringel-Rosa conjecture, which predicts that *all trees are graceful*. In fact, G. Ringel [15] only conjectured that for a given tree T on $n+1$ vertices, there is a T-decomposition of K_{2n+1}, while A. Kotzig later conjectured that there is a *cyclic* decomposition—see [16]. A. Rosa's discovery of β-labeling translated the problem into the language of graph labelings and Golomb's introduction of the name "graceful" [12] gave it its catchy name. This conjecture is naturally challenging, and apparently very hard. There are too many papers constructing various classes of graceful trees to be listed here. We refer the interested reader to an exhaustive survey of graph labelings due to Gallian [11]. In spite of concentrated efforts of many authors, the conjecture is far from being solved. The most significant "closed" families of graceful trees are caterpillars [16], trees with at most four vertices of degree 1 [16], trees of diameter at most 4 [18] and at most 5 [13], and all trees with at most 27 vertices [1].

2. α-labeling and generalizations

While a rosy or graceful labeling of a graph G with n edges in general guarantees the existence of a G-decomposition of the graph K_{2n+1} only, a more restrictive α-labeling introduced by A. Rosa guarantees G-decompositions of an infinite class of complete graphs.

Definition 2.1. *Let G be a graph with n edges and α be a graceful labeling of G. If there exists a constant α_0 such that for every two vertices x, y of G with $\alpha(x) < \alpha(y)$ it holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$, then α is called an α-labeling (or α-valuation).*

It is obvious that a graph with an α-labeling must be bipartite. The following result was proved by A. Rosa.

Theorem 2.2. *If a bipartite graph G with n edges has an α-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

It is well known that there are graphs that admit a rosy or graceful labeling but not an α-labeling. The smallest example is the 3-*comet* consisting of three paths P_3 with their endvertices glued together in one vertex of degree 3.

Nice generalization of Theorem 2.2 was proved by S. El-Zanati, C. Vanden Eynden, and N. Punnim [7]. They relaxed the properties of the α-labeling to obtain a ρ^+-labeling. The difference between these labelings is that while in an α-labeling of a bipartite graph with the partite sets X, Y we require *all* vertices in X to have the labels smaller than *every* vertex in Y, in a ρ^+-labeling we only require that all *neighbors* of each given vertex $y \in Y$ have their labels smaller than $\rho^+(y)$. Moreover, we can use labels from the set $\{0, 1, \ldots, 2n\}$ while in an α-labeling only from the set $\{0, 1, \ldots, n\}$.

Definition 2.3. *Let G be a bipartite graph G with n vertices and a bipartition X, Y. A rosy labeling ρ^+ is called a ρ^+-labeling if for every edge $xy \in E(G)$ with $x \in X, y \in Y$ it holds that $\rho^+(x) < \rho^+(y)$.*

The following theorem was proved in [7].

Theorem 2.4. *If a bipartite graph G with n edges has a ρ^+-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

Another generalization of the α-labeling was obtained by the author [9].

Definition 2.5. *Let G be a bipartite graph G with n edges. Than G has an α_2-labeling if*

1. *α_2 is a rosy labeling with the label set L*
2. *$L = L_0 \cup L_1 \cup L_2$ and $L_i \cap L_j = \emptyset$ for $i \neq j$*
3. *There exist integers λ_1, λ_2 such that $0 \leq l_0 \leq \lambda_1 < l_1 \leq \lambda_2 < l_2$ for all labels $l_i \in L_i, i = 0, 1, 2$*
4. *If xy is an edge of G and $\alpha_2(x) < \alpha_2(y)$, then $\alpha_2(x) \in L_i$ and $\alpha_2(y) \in L_{i+1}$ for $i \in \{0, 1\}$ and $\alpha_2(y) - \alpha_2(x) \leq n$.*

Notice that when restricted to a pair of sets L_i and L_{i+1}, the labeling is "alpha-like" in the sense that that the length of an edge is always equal to the difference between the higher and lower label (in that order).

The existence of an α_2-labeling of G guarantees a G-decomposition in the same way as an α-labeling.

Theorem 2.6. *Let G be a graph on n edges that admits an α_2-labeling. Then for any positive integer k there exists a G-decomposition of the complete graph K_{2nk+1}.*

There is no labeling known so far for general graphs with n edges that would allow G-decompositions of K_{2nk+1} when $k > 1$. However, when G contains an edge e such that $G - e$ is bipartite, such a labeling exists, as proved by A. Blinco, S. El-Zanati, and C. Vanden Eynden [2].

Definition 2.7. *Let G be a tripartite graph G with tripartition X, Y, Z. Then G is almost bipartite if $Z = \{z\}$ and there is a unique vertex $\bar{y} \in Y$ such that $\bar{y}z \in E(G)$.*

For almost bipartite graphs, a γ-labeling can be defined as follows.

Definition 2.8. *Let G be an almost bipartite graph G with n edges as defined above. A rosy labeling γ is called a γ-labeling if for every edge $xw \in E(G)$ with $x \in X, w \in Y \cup Z$ it holds that $\gamma(x) < \gamma(w)$ and $\gamma(z) - \gamma(\bar{y}) = n$, where $\bar{y}$ and z are the vertices described in Definition 2.7.*

A γ-labeling also mimics the properties of an α-labeling, as follows from a theorem proved by Blinco, El-Zanati, and Vanden Eynden [2].

Theorem 2.9. *Let G be an almost bipartite graph G with n edges that admits a γ-labeling. Then for any positive integer k there exists a G-decomposition of the complete graph K_{2nk+1}.*

3. Labeling with group products

There are classes of graphs for which finding a G-decomposition through a rosy labeling is too complicated. This is in particular true for some G-decompositions of complete bipartite graphs into regular graphs. In that case, more general labelings become useful. Because this survey is focused on decompositions of complete graphs, we restrict our attention to that case only. For recent applications to complete bipartite graph decompositions, we refer the reader to [4] and [5].

To simplify our notation, we will often identify vertices with their respective labels. We will say "a vertex χ" rather than "a vertex x with $\rho(x) = \chi$".

While the rosy labeling is based on the additive group Z_{2n+1}, the product rosy labeling uses for labels elements of products of odd additive groups, $Z_{2n_1+1} \times Z_{2n_2+1} \times \cdots \times Z_{2n_k+1}$. We denote this group by $\mathcal{Z}$.

We denote by $\chi = (x_1, x_2, \ldots, x_k)$ a vertex $\chi \in \mathcal{Z}$ and by $[\chi\psi]$ an edge joining vertices χ and ψ.

Definition 3.1. *Let $k \geq 1, n_i \geq 1$ for $i = 1, 2, \ldots, k$ and $2p + 1 = (2n_1 + 1)(2n_2 + 1) \ldots (2n_k + 1)$. Let G be a graph with p edges and the vertex set $V \subseteq \mathcal{Z} = Z_{2n_1+1} \times Z_{2n_2+1} \times \cdots \times Z_{2n_k+1}$. Let $e = [(x_1, x_2, \ldots, x_k)(y_1, y_2, \ldots, y_k)]$ be an edge of G. We define the dimension of e as $\dim(e) = (a_1, a_2, \ldots, a_k)$, where $a_k = \min(x_k - y_k, y_k - x_k)$. Moreover if for $i \in \{1, 2, \ldots, k-1\}$ we have $x_i - y_i \neq 0$ while $x_{i+1} - y_{i+1} = x_{i+2} - y_{i+2} = \cdots = x_k - y_k = 0$, then*

$a_{i+1} = a_{i+2} = \cdots = a_k = 0,$

$a_i = \min(x_i - y_i, y_i - x_i) \leq n_i$, *and for $j = 1, 2, \ldots, i-1$ we set*

$a_j = x_j - y_j$ *if* $x_i = y_i + a_i \pmod{2n_i + 1}$, *or*

$a_j = y_j - x_j$ *if* $y_i = x_i + a_i \pmod{2n_i + 1}$.

We say that G has a product rosy labeling $\rho^\times$ if the set of dimensions of all edges of G is equal to

$$\bigcup_{i=1}^{k} \{(l_1, l_2, \ldots, l_{i-1}, l_i, 0, 0, \ldots, 0) \mid 0 \leq l_j \leq 2n_j \text{ for } j = 1, 2, \ldots, i-1,\ 1 \leq l_i \leq n_i\}$$

(Formally, $\rho^\times$ can be viewed as the mapping from the unlabeled graph G into $\mathcal{Z}$.)

Denote by $\mathcal{Z}_i = Z_{2n_{i+1}+1} \times Z_{2n_{i+2}+1} \times \cdots \times Z_{2n_k+1}$. Notice that when for some edge $e = [\chi\psi]$ we have $\dim(e) = (a_1, 0, 0, \ldots, 0)$, then $x_j = y_j$ for $j = 2, 3, \ldots, k$ and the vertices $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$ belong to the same coset of $\mathcal{Z}/\mathcal{Z}_1$ and $x_1 \neq y_1$. Therefore, in this case a_1 cannot be equal to zero. At the same time, because here $a_1 = \min(x_1 - y_1, y_1 - x_1)$, it is obvious that $a_1 \le n_1$.

On the other hand, when the vertices $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$ belong to different cosets of $\mathcal{Z}/\mathcal{Z}_1$, we always subtract both entries in the same order. It means that either $\dim(e) = (x_1 - y_1, x_2 - y_2, \dots, x_k - y_k)$, when for some $i > 1$ we have $0 < x_i - y_i < y_i - x_i$ and $x_l = y_l$ for $l = i+1, i+2, \dots, k$, or $\dim(e) = (y_1 - x_1, y_2 - x_2, \dots, y_k - x_k)$ when $0 < y_i - x_i < x_i - y_i$ and $x_l = y_l$ for $l = i+1, i+2, \dots, k$. In this case the difference $x_1 - y_1$ or $y_1 - x_1$ can be any element of Z_{2n_1+1}.

Similarly when $a_j \neq 0$ for some $j > 0$ and $a_{j+1} = a_{j+2} = \cdots = a_k = 0$, then $x_l = y_l$ for $l = j+1, j+2, \dots, k$ and the vertices $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$ belong to the same coset of $\mathcal{Z}/\mathcal{Z}_j$ and $x_j \neq y_j$. Again, in this case $a_j = \min(x_j - y_j, y_j - x_j)$, and $0 < a_j \le n_j$.

When the vertices $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$ belong to different cosets of $\mathcal{Z}/\mathcal{Z}_j$, we always subtract both entries in the same order. It means that either $\dim(e) = (x_1 - y_1, x_2 - y_2, \dots, x_k - y_k)$ (when for some $i > j$ we have $x_i - y_i < y_i - x_i$ and $x_l = y_l$ for $l = i+1, i+2, \dots, k$), or $\dim(e) = (y_1 - x_1, y_2 - x_2, \dots, y_k - x_k)$ (when for some $i > j$ we have $y_i - x_i < x_i - y_i$ and $x_l = y_l$ for $l = i+1, i+2, \dots, k$). In this case again the difference $x_j - y_j$ or $y_j - x_j$ for $1 \le j < i$ can be any element of Z_{2n_j+1}.

We also remark that if we allow $k = 1$, then the labeling is equivalent to a rosy labeling on Z_{2n_1+1}. For obvious reasons, we will further assume that every n_i is greater than zero.

We observed above that the decompositions based on rosy labelings are cyclic. Although it is not true for decompositions arising from product rosy labelings in general, one important property of rosy decompositions is carried over to the product rosy decompositions.

Definition 3.2. *Let $\mathcal{H} = \{H_0, H_1, \dots, H_{2p}\}$ be an H-decomposition of K_{2p+1} defined by isomorphisms $\varphi_i(H_0) = H_i$ for $i = 0, 1, \dots, 2p$. If for a vertex $x \in H$ the mapping $\Phi(x) : \{\varphi_i(x) : i = 0, 1, \dots, 2p\} \to V(K_{2p+1})$ is a bijection, we say that x is a bijective vertex in $\mathcal{H}$. If all vertices of H are bijective in $\mathcal{H}$, then we say that $\mathcal{H}$ is a bijective decomposition of K_{2p+1}.*

Definition 3.3. *Let $\mathcal{H} = \{H_0, H_1, \dots, H_{2p}\}$ be an H-decomposition of K_{2p+1} defined by isomorphisms $\varphi_i(H_0) = H_i$ for $i = 0, 1, \dots, 2p$. If for a vertex $x \in H$ we have $\{\varphi_i(x) | i = 0, 1, \dots, 2p\} = V(K_{2p+1})$, then we say that x is a bijective vertex in $\mathcal{H}$. If all vertices of H are bijective in $\mathcal{H}$, then we say that $\mathcal{H}$ is a bijective decomposition of K_{2p+1}.*

As an example of a graph that admits both bijective and non-bijective decomposition, we use a caterpillar R with four edges. There are two such caterpillars, P_5, and R with adjacent vertices x_0 and x_1 of degree 2 and 3, respectively, and three vertices of degree 1: y_0 adjacent to x_0 and y_1, y_2 adjacent to x_1. Obviously, K_9 has a bijective R-decomposition, since R is graceful, and the decomposition is in fact cyclic. Let us now denote the vertices of K_9 by pairs (i, j), where $i, j \in \{0, 1, 2\}$. Let the nine copies of R be R_{rs}, where also

$r, s \in \{0,1,2\}$, and let $R_{rs} = \psi_{rs}(R)$. Set $\psi_{rs}(x_l) = (r, s+l)$ and $\psi_{rs}(y_t) = (r+1, t)$, where the addition is performed modulo 3. One can check that this collection gives an R-decomposition of K_9. Then we have

$$\psi_{00}(y_0) = (1,0), \psi_{00}(y_1) = (1,1), \psi_{00}(y_2) = (1,2), \psi_{00}(x_0) = (0,0), \psi_{00}(x_1) = (0,1),$$
$$\psi_{01}(y_0) = (1,0), \psi_{01}(y_1) = (1,1), \psi_{01}(y_2) = (1,2), \psi_{01}(x_0) = (0,1), \psi_{01}(x_1) = (0,2),$$
$$\psi_{02}(y_0) = (1,0), \psi_{02}(y_1) = (1,1), \psi_{02}(y_2) = (1,2), \psi_{02}(x_0) = (0,2), \psi_{02}(x_1) = (0,0)$$

in which, e.g., the vertex $(1,0)$ is an image of y_0 in precisely three copies of R. Namely, in R_{00}, R_{01}, R_{02}. Therefore, the decomposition is *not* bijective.

The following theorem was proved in [8] for $k = 2$ only. Therefore, we prove it here in full generality.

Theorem 3.4. *Let $2p+1 = (2n_1+1)(2n_2+1)\dots(2n_k+1)$ with $k > 1$ and $n_i \geq 1$ and G be a graph with p edges that admits a product rosy labeling. Then there exists a G-decomposition of the complete graph K_{2p+1}. Moreover, this decomposition is bijective.*

Proof. Let $G_0 \cong G$. Define mappings $\rho'_\beta : \mathcal{Z} \to \mathcal{Z}$ for $\chi, \beta \in \mathcal{Z}$ by $\rho'_\beta(\chi) = (\chi + \beta)$. Obviously, the induced mappings $\rho^\times_\beta : V(G_0) \to V(G_\beta)$ are graph isomorphisms. Our goal is to show that the family $\mathcal{G} = \{G_\beta | \beta \in \mathcal{Z}\}$ is a G-decomposition of K_{2p+1}.

First we want to prove that the isomorphisms are edge-dimension-preserving. Suppose that $\beta = (b_1, b_2, \dots, b_k)$,

$$\dim([(x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k)]) = (a_1, a_2, \dots, a_k)$$

where, without loss of generality (WLOG), $a_j = x_j - y_j$ for $j = 1, 2, \dots, k$ and $0 < a_i \leq n_i, a_{i+1} = \dots = a_k = 0$ for some $i \in \{1, 2, \dots, k\}$. Then

$$[\rho^\times_\beta(x_1, x_2, \dots, x_k)\rho^\times_\beta(y_1, y_2, \dots, y_k)] = [(x_1 + b_1, \dots, x_k + b_k)(y_1 + b_1, \dots, y_k + b_k)]$$

and since $x_j + b_j - (y_j + b_j) = x_j - y_j$ for $j = 1, 2, \dots, k$, we have

$$\rho^\times_\beta(x_1, x_2, \dots, x_k) - \rho^\times_\beta(y_1, y_2, \dots, y_k) = (a_1, a_2, \dots, a_k).$$

Because $0 < a_i \leq n_i, a_{i+1} = \dots = a_k = 0$, we observe that $(a_1, a_2, \dots, a_k)$ satisfies our definition of an edge dimension and hence

$$\dim([\rho^\times_\beta(x_1, x_2, \dots, x_k)\rho^\times_\beta(y_1, y_2, \dots, y_k)]) = (a_1, a_2, \dots, a_k).$$

Now we show that an arbitrary edge $[(x'_1, x'_2, \dots, x'_k)(y'_1, y'_2, \dots, y'_k)]$ of K_{2p+1} with dimension $(a_1, a_2, \dots, a_k)$ belongs to at least one image of G_0. We can suppose WLOG that $a_j = x'_j - y'_j$ for $j = 1, 2, \dots, k$ and $0 < a_i \leq n_i, a_{i+1} = \dots = a_k = 0$ for some $i \in \{1, 2, \dots, k\}$. Because G_0 has a $\rho^\times$-labeling, there is exactly one edge of dimension $(a_1, a_2, \dots, a_k)$ in G_0, say $e = [(x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k)]$ where again $a_j = x_j - y_j$ for $j = 1, 2, \dots, k$.

Now let $\beta = (b_1, b_2, \ldots, b_k) = (x'_1, x'_2, \ldots, x'_k) - (x_1, x_2, \ldots, x_k)$. Because

$$(y'_1, y'_2, \ldots, y'_k) = (x'_1, x'_2, \ldots, x'_k) - (a_1, a_2, \ldots, a_k)$$

and

$$(y_1, y_2, \ldots, y_k) = (x_1, x_2, \ldots, x_k) - (a_1, a_2, \ldots, a_k),$$

we immediately get

$$(y'_1, y'_2, \ldots, y'_k) - (y_1, y_2, \ldots, y_k) = (x'_1, x'_2, \ldots, x'_k) - (x_1, x_2, \ldots, x_k) = \beta.$$

This implies that the edge $[(x'_1, x'_2, \ldots, x'_k)(y'_1, y'_2, \ldots, y'_k)]$ belongs to G_β, which we wanted to show.

Now we need to show that every edge of K_{2p+1} belongs to at most one factor of $\mathcal{G}$. Suppose the contrary and let an edge $\chi'\psi' = [(x'_1, x'_2, \ldots, x'_k)(y'_1, y'_2, \ldots, y'_k)]$ with dimension $\alpha = (a_1, a_2, \ldots, a_k)$ belong to G_β and G_γ for some $\beta \neq \gamma$. By the definition of the dimension, we have $a_j = x'_j - y'_j$ for $j = 1, 2, \ldots, k$ with $0 < a_i \leq n_i, a_{i+1} = \cdots = a_k = 0$ for some $i \in \{1, 2, \ldots, k\}$. Suppose WLOG that the pre-images of χ' and ψ' in $\rho^\times_\beta$ are $\chi = \chi' - \beta = (x_1, x_2, \ldots, x_k)$ and $\psi = \psi' - \beta = (y_1, y_2, \ldots, y_k)$, respectively. Then we must have $0 < x_i - y_i = a_i \leq n_i, a_{i+1} = a_{i+2} = \cdots = a_k = 0$ and $a_j = x_j - y_j$ for $j = 1, 2, \ldots, i-1$. This means that $\chi - \psi = \alpha$.

Because G_0 contains only one edge of dimension α, we must have $\chi' = \chi + \gamma, \psi' = \psi + \gamma$, or $\chi' = \psi + \gamma, \psi' = \chi + \gamma$. Suppose that the latter is true and let $\gamma = (c_1, c_2, \ldots, c_k)$. Because $\dim([\chi'\psi']) = \alpha = \chi' - \psi'$, we must have $\alpha = (\psi + \gamma) - (\chi + \gamma) = \psi - \chi$. But above we have shown that $\alpha = \chi - \psi$. This would imply that $\alpha = -\alpha$, which is impossible in $\mathcal{Z}$. Therefore, we must have $\chi' = \chi + \gamma, \psi' = \psi + \gamma$. Then $\chi' = \chi + \gamma = \chi + \beta$ and $\psi' = \psi + \gamma = \psi + \beta$ which implies that $\gamma = \beta$, a contradiction. Therefore, every edge of K_{2p+1} belongs to precisely one graph of $\mathcal{G}$.

The fact that all $2p + 1$ images of any vertex $\chi \in G_0$ are mutually distinct vertices of K_{2p+1} follows easily from the properties of group $\mathcal{Z}$. We have constructed the G-decomposition as the family $\mathcal{G} = \{G_\beta | \beta \in \mathcal{Z}\}$, where each graph in $\mathcal{G}$ is obtained by $\rho^\times_\beta : V(G_0) \to V(G_\beta)$ and $\rho^\times_\beta(\chi) = (\chi + \beta)$. Therefore, the set of $2p + 1$ images of any vertex χ of G is the set $\{\chi + \beta | \beta \in \mathcal{Z}\}$ which forms the whole group $\mathcal{Z}$. This completes the proof. □

4. Decompositions of K_{km}

As mentioned in Section 2, if a bipartite graph G with n edges admits an α-labeling or a ρ^+-labeling, then there exists a G-decomposition of K_{2nk+1} for every integer $k \geq 1$.

It turns out that even if we relax the assumption of the existence of an α-labeling or a ρ^+-labeling of a *bipartite* graph G by assuming only a rosy labeling, a similar (but of course weaker) result holds.

In [3] M. Buratti and A. Pasotti proved a result on difference matrices, which is in [14] re-stated as follows.

Theorem 4.1. *If a graph G with n edges and chromatic number $\chi(G)$ cyclically decomposes K_k and K_m, where $k \equiv m \equiv 1 \pmod{2n}$ and $\chi(G)$ does not exceed the smallest prime factor of m, then there exists a cyclic G-decomposition of K_{km}.*

Here the notion of cyclic decomposition is used in a more general way than stated in Definition 1.1. By "decomposing K_k cyclically" we mean that the decomposing permutation is cyclic and the length of the cycle is a divisor of k, not necessarily k itself as in Definition 1.1.

Because a bipartite graph G has $\chi(G) = 2$, the following corollary is easy to prove. It was stated in [3] in a more general form related to Theorem 4.1.

Corollary 4.2. *If a bipartite graph G with n edges has a ρ-labeling, then there exists a cyclic G-decomposition of $K_{(2n+1)^r}$ for any positive integer r.*

It was proved in [10] that if we restrict ourselves to bipartite graphs while assuming the existence of *any* G-decomposition rather than a cyclic one, we can still get a result similar to Theorem 4.1.

The result is based on an idea that can be viewed as a "two-step" decomposition. We recall that a *composition* $G[H]$ of graphs G and H (also called a *lexicographic product*) is a graph that arises from G by replacing each vertex of G by a copy of H and each edge of G by $K_{m,m}$, where m is the order of H. In particular, if $H = \overline{K}_m$, the graph consisting of m isolated vertices, then we say that we *blow up* G into $G[\overline{K}_m]$.

Then the following theorem holds.

Theorem 4.3. *If a bipartite graph G decomposes K_k, then G also decomposes the complete k-partite graph $K_{m,m,\ldots,m}$ for any $m \geq 2$.*

The following theorems can be then proved.

Theorem 4.4. *Let G be a bipartite graph which decomposes K_k and K_m. Then G also decomposes K_{km}.*

Theorem 4.5. *Let G be a bipartite graph which decomposes K_s. Then G decomposes also K_{s^r} for any $r \geq 1$.*

Notice that because the existence of a rosy labeling of a graph G with n edges implies the existence of a G-decomposition of K_{2n+1}, Corollary 4.2 follows from Theorem 4.5.

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