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On Mod Difference Labeling of Digraphs

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Abstract

A digraph $D = (V, E)$ is a *mod difference digraph* if there exist a positive integer m and a labeling $L : V \to \{1, 2, \ldots, m-1\}$ such that $(x, y) \in E$ if and only if $L(y) - L(x) \equiv L(w)(mod \;\; m)$ for some $w \in V$. In this paper we prove that complete symmetric digraphs, unipaths and unicycles are mod difference digraphs.

Keywords: mod difference digraph, mod difference labeling.

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1. Introduction

The concepts of sum and difference graphs were first introduced by Harary [2] in 1990. A simple undirected graph G is called a *sum graph* if there exists a labeling f of the vertices of G into distinct positive integers such that any two distinct vertices u and v of G are adjacent if and only if there is a vertex w with label $f(w) = f(u) + f(v)$. *Difference graphs* are defined in a similar way except that $f(u) + f(v)$ is replaced by $|f(u) - f(v)|$. Bolland et al. [1] refined Harary's concept considering modulo addition to define mod sum labeling of a graph. The concept could be extended to digraphs also. Since the addition is commutative and positive difference of two labels is symmetric, we use modulo difference as the common operation. In this paper we define mod difference labeling of digraphs and study some classes of digraph which admit mod difference labeling. For notations and terminologies in graph theory and number theory we refer to [3], [5] and [4].

Definition 1.1. *A simple digraph $D = (V, E)$ is called a mod difference digraph if there exists a positive integer m and a labeling L of V into distinct elements of $\{1, 2, \ldots, m-1\}$ such that for the vertices u and v, $(u, v) \in E$ if and only if there is a vertex w such that $L(v) - L(u) \equiv L(w) \pmod{m}$. The labeling L is called a mod difference labeling. The minimum number of isolates required to make a digraph a mod difference digraph is called the mod difference number of D and is denoted by $\rho_d(D)$.*

Example 1.2. *A mod difference labeling of a digraph on 8 vertices is given in Figure 1.*

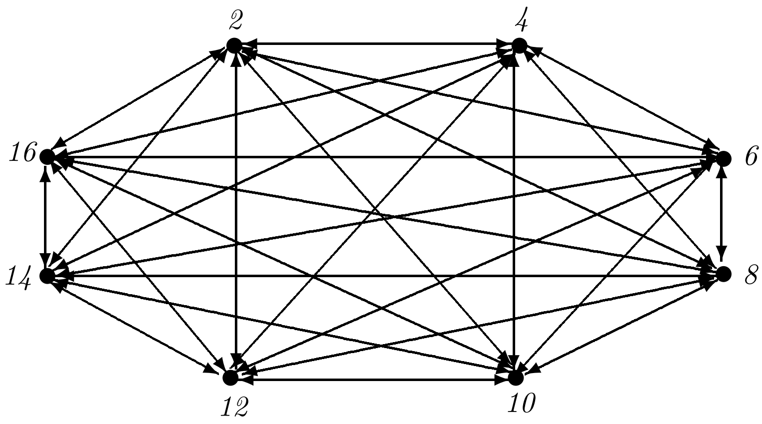

Figure 1. A mod difference labeled digraph with $m = 18$.

2. Matrix representation of mod difference graphs

A mod difference digraph with n vertices can be represented by a matrix A of order $n \times n$ as defined below.

$$a_{ij} = \begin{cases} label \;\; assigned \;\; to \;\; vertex \;\; v_i \;\; if \;\; i = j, \\ (a_{jj} - a_{ii})(mod \;\; m) \;\; whenever \;\; (v_i, v_j) \in E(D) \\ 0 \;\; otherwise. \end{cases}$$

Example 2.1. *A mod difference labeling of a digraph on 4 vertices with $m = 5$ and the corresponding matrix are given in Figure 2.*

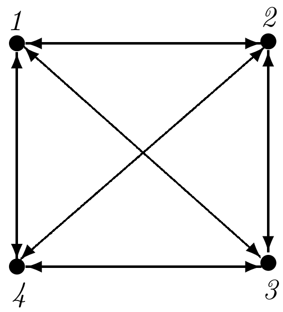

Figure 2. A mod difference labeled digraph

The corresponding matrix is, $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 2 & 1 & 2 \\ 3 & 4 & 3 & 1 \\ 2 & 3 & 4 & 4 \end{pmatrix}$

Mod difference labeling gives a method to store the digraph in the form of a single array of positive integers. The array includes the distinct elements of the matrix A which are non-zero.

3. Some classes of mod difference graphs

Theorem 3.1. *Any complete symmetric digraph, $\overleftrightarrow{K_n}$ is a mod difference digraph.*

Proof. We label the vertices $v_1, v_2, \ldots, v_n$ of $\overleftrightarrow{K_n}$ using the labeling function L given by $L(v_i) = ia$, where a is any positive integer. Let $m = (n+1)a$. Clearly $L(v_i) - L(v_j) \equiv L(v_k) \pmod{m}$, where

$$k = \begin{cases} (i-j) & \text{if } j > j, \\ (n+1) + (i-j) & \text{if } i < j. \end{cases}$$

Hence L is a mod difference labeling of $\overleftrightarrow{K_n}$. □

Example 3.2. *A mod difference labeling of $\overleftrightarrow{K_6}$ with $a = 3$ is given in Figure 3.*

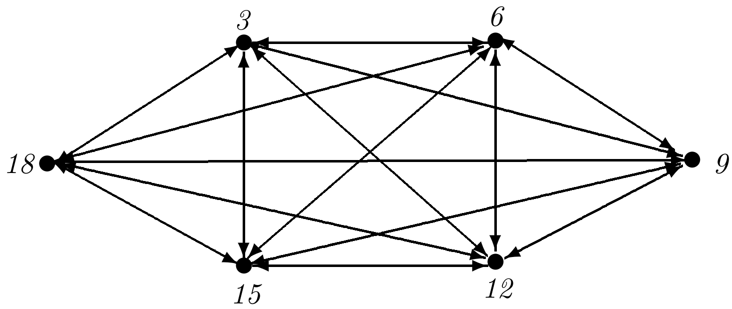

Figure 3

The following theorem show that any mod differnece labeling of $\overleftrightarrow{K_n}$ is of the form given in Theorem 3.1.

Theorem 3.3. *If the labeling L given by $L(v_i) = a_i, (i = 1, 2, \ldots, n)$, is a mod difference labeling of $\overleftrightarrow{K_n}$, then there exists an integer a such that $a_i = ia$ for all i and $m = (n+1)a$.*

Proof. Without loss of generality we assume that $a_1 < a_2 < \cdots < a_n$. Since L is a mod difference labeling, $a_2 - a_1 = a_l$ for some l. Since $a_1 < a_2$, it follows that $a_l < a_2$ and hence $a_l = a_1$. Thus $a_2 = 2a_1$. Now $a_3 - a_1 < a_3$ and hence $a_3 - a_1 = a_1$ or a_2. If $a_3 - a_1 = a_1$, then $a_2 = a_3 = 2a_1$, which is a contradiction. Thus $a_3 - a_1 = a_2$ and hence $a_3 = 3a_1$.

We now assume that $a_i = ia_1$ for $1 \leq i \leq k$. Now $a_{k+1} - a_1 < a_{k+1}$ and hence $a_{k+1} - a_1 = a_j$ for some $j, 1 \leq j \leq k$. If $j < k$, then it follows that $a_{k+1} = (j+1)a_1 = a_{j+1}$, which is a contradiction. Thus $a_{k+1} - a_1 = a_k = ka_1$, so that $a_{k+1} = (k+1)a_1$. Hence by induction $a_i = ia$, for all i. Now $a_1 - a_n \equiv a_j \pmod{m}$ for some j and hence $m = (n-1+j)a$ for some j. If $j = 1$, then $m = na$, so that $a_n = 0$ and if $j \geq 3$ then $(a_{i+1}, a_i) \notin E(D)$ which is a contradiction. Thus $j = 2$ and $m = (n+1)a$. □

Theorem 3.4. *Any unidirectional cycle $\overrightarrow{C_n}$ is a mod difference digraph.*

Proof. Let $\overrightarrow{C_n}= (v_1, v_2, \ldots, v_n, v_1)$ and define $L(v_i) = 2^{i-1}a$, where a is any positive integer.

We prove that L is a mod difference labeling of $\overrightarrow{C_n}$ with $m = (2^n - 1)a$.

For $1 \leq i \leq n-1$

$$\begin{aligned} L(v_{i+1}) - L(v_i) &= (2^i - 1)a - (2^{i-1} - 1)a \\ &= (2^i - 2^{i-1})a \\ &= 2^{i-1}a \\ &= L(v_i). \end{aligned}$$

Also $L(v_1) - L(v_n) = a - 2^{n-1}a \equiv 2^{n-1}a \pmod{n}$.

Conversely, let $L(v_i) - L(v_j) \equiv L(v_k) \pmod{m}$ for some k. Then

$$(2^{i-1}a) - (2^{j-1}a) \equiv (2^{k-1}a) \pmod{m}. \tag{1}$$

We consider two cases.

Case 1. $i > j$.

Then it follows from (1) that $(2^{i-1}a) - (2^{j-1}a) = (2^{k-1}a)$. Hence $2^{i-j} = 1 + 2^{k-j}$, so that $k = j$ and $i = j + 1$.

Case 2. $i < j$.

Then it follows from (1) that $2(^{k-1}a) = (2^{i-1}a) - (2^{j-1}a) + (2^n - 1)a = 2^{k-1}$. Hence $2^{k-1} + 1 = 2^{i-1} - 2^{j-1} + 2^n$. Let $i > 1$. Since $i > j$ and $n \geq 3$, it follows that $i = 1$ and the above equation reduces to $2^{n-j+1} = 1 + 2^{k-j}$. Hence $k = j$ and $j = n$. Thus $i = 1$ and $j = n$. Hence L is a mod sum labeling of $\overrightarrow{C_{2n}}$. $\square$

Example 3.5. *A mod difference labeling of $\overrightarrow{C_8}$ with $a = 1$ is given in Figure 4.*

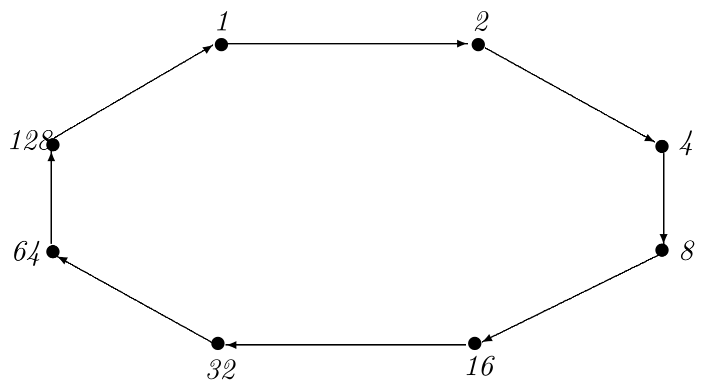

Figure 4. A mod difference labeling of unicycle with $a = 1, m = 255$

Theorem 3.6. *If the labeling L given by $L(v_i) = a_i, (i = 1, 2, \ldots, n)$, is a mod difference labeling of $\overrightarrow{C_n}$, then there exists a integer a such that $a_i = 2^{i-1}a$, $1 \leq i \leq n$ and $m = (2^n - 1)a$.*

Proof. Let the labeling L given by $L(v_i) = a_i$, be a mod difference labelig of $\overrightarrow{C_n}$. We claim that $a_1 < a_2 < \cdots < a_n$. Suppose $a_{i+1} < a_i$ for some i. Since (v_i, v_{i+1}) is an arc in $\overrightarrow{C_n}$, $a_{i+1} - a_i \equiv a_k \pmod{m}$ for some k. Hence $a_k = m + a_{i+1} - a_i$ and $a_{i+1} - a_k \equiv a_i \pmod{m}$. Also since $m - a_i > 0$, we have $a_k > a_{i+1}$.

Since L is a mod difference labeling of $\overrightarrow{C_n}$, (v_k, v_{i+1}) is an arc in $\overrightarrow{C_n}$, so that $k = i$. Thus $a_i < a_{i+1}$, which is a contradiction. Hence $a_1 < a_2 < \cdots < a_n$. We now claim that $a_i = 2^{i-1}a_1$ and $m = (2^n - 1)a_1$. Since L is a mod difference labeling, $a_2 - a_1 = a_1$ and hence $a_2 = 2a_1$. Now $a_3 - a_2 = a_1$ or a_2. If $a_3 - a_2 = a_1$, then $a_3 - a_1 = a_2$, which is not possible since (v_1, v_3) is not an arc in $\overrightarrow{C_n}$. Hence $a_3 - a_2 = a_2$, so that $a_3 = 2^2a_1$. By a similar argument we have $a_i - a_{i-1} = a_{i-1}$, so that $a_i = 2^{i-1}a_1$ for all i.

Now since (v_n, v_1) is an arc in $\overrightarrow{C_n}$, $a_1 - a_n \equiv a_j \pmod{m}$ and hence $j = n$. Hence $(1 - 2^{n-1})a_1 \equiv 2^{n-1}a_1 \pmod{m}$, so that $m = (2^n - 1)a_1$. □

Theorem 3.7. *Any unidirectional path $\overrightarrow{P_n}$ (unipath) is a mod difference digraph.*

Proof. We label the vertices $v_1, v_2, \ldots, v_n$ of $\overrightarrow{P_n}$ using the labeling function L given by $L(v_i) = 2^{i-1}a$, where a is any positive integer. Proceeding as in Theorem 3.3, it can be proved that L is a mod difference labeling of $\overrightarrow{P_n}$ with $m = (2^n + 1)a$. □

Example 3.8. *A mod difference labeling $\overrightarrow{P_6}$ with $a = 3$ is given in Figure 5*

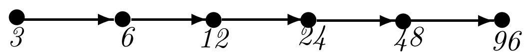

Figure 5. A mod difference labeling of unipath with $a = 3, m = 193$

Theorem 3.9. *If the labeling L given by $L(v_i) = a_i, (i = 1, 2, \ldots, n)$, is a mod difference labeling of $\overrightarrow{P_n}$, then there exists an integer a such that $a_i = 2^{i-1}a, 1 \leq i \leq n$ and $m = (2^n + 1)a$.*

Proof. Similar to that of Theorem 3.4. □

References

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