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Secondary and Internal Distances of Sets in Graphs

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Abstract

For any given type of a set of vertices in a connected graph $G = (V, E)$, we seek to determine the smallest integers $(x, y : z)$ such that all minimal (or maximal) sets S of the given type, where $|V| > |S| \geq 2$, have the property that every vertex $v \in V - S$ is within distance at most x to a vertex $u \in S$ (shortest distance), and within distance at most y to a second vertex $w \in S$ (second shortest distance). We also seek to determine the smallest integer z such that every vertex $u \in S$ is within distance at most z to a closest neighbor $w \in S$ (the internal distance). A dominating set $S \subseteq V$ in a graph G is a set having the property that every vertex $v \in V - S$ is within distance 1 to some vertex in S, or equivalently, whose shortest distance $x = 1$. In this paper we determine the secondary distances y and internal distances z for 31 types of sets in graphs, whose shortest distance $x = 1$.

Keywords: Open neighborhood, closed neighborhood, secondary distance.

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1. Introduction

Let $G = (V, E)$ be a connected graph. The *open neighborhood of a vertex* $v \in V$ is the set $N(v) = \{u | uv \in E\}$ of vertices u adjacent to v in G, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is defined as $deg(v) = |N(v)|$. The *open neighborhood of a set* $S \subset V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of* S is the set $N[S] = \bigcup_{v \in S} N[v]$.

The *subgraph of a graph* $G = (V, E)$ *induced by a set* $S \subseteq V$ is the subgraph $G[S] = (S, E \cap (S \times S))$, whose vertex set is S, and whose edge set consists of all edges in E joining two vertices in S.

Definition 1. *A set S of vertices is a* dominating set *if $N[S] = V$, that is every vertex $v \in V$ is either in S, or is in $V - S$ and is adjacent to at least one vertex $u \in S$.*

The *distance* $d(u, v)$ between two vertices in a connected graph G equals the minimum number of edges in a path from vertex u to vertex v.

Definition 2. *A set S of vertices is a* distance-k dominating set *if every vertex $v \in V - S$ is at most distance k to at least one vertex in S.*

We will say that the *distance* $d(u, S)$ between a vertex $u \in V$ and a set $S \subset V$ equals the minimum value of $d(u, v)$ over all vertices $v \in S$. It follows, therefore, that for every vertex $u \in S$, $d(u, S) = 0$.

In a similar way, we define the *dominating distance* $dd(S)$ *of a set* S to equal

$$dd(S) = max\{d(v, S)|v \in V - S\}.$$

It follows therefore, that if $dd(S) = k$, then every vertex $v \in V - S$ is distance at most k to at least one vertex in S.

Let $S = \{u_1, u_2, \ldots, u_k\}$ be any set of vertices in a graph G and let $v \in V - S$ be any vertex in $V - S$. Let us assume that the vertices in S have been ordered so that $d(v, u_1) \leq d(v, u_2) \leq \ldots \leq d(v, u_k)$. It follows that $d(v, S) = d(v, u_1)$. We define the *secondary distance* $sd(v, S)$ of a vertex $v \in V - S$ to S to equal $d(v, u_2)$, that is, $sd(v, S) = d(v, u_2)$. Similarly, we define the *secondary distance* $sd(S)$ of a set S to equal the maximum value of $sd(v, S)$ over all vertices $v \in V - S$.

Thus, if $dd(S) = x$ and $sd(S) = y$ then every vertex in $V - S$ is at most distance x to a nearest vertex in S and at most distance y to a second nearest vertex in S.

We will also be interested in determining the smallest value z such that every vertex $u \in S$ is at most distance z to another other vertex in S. We call this the *internal distance* $id(S)$ of S.

Therefore, we will say that a set S is an $(x, y : z)$ set if its dominating distance is at most x ($dd(S) \leq x$), its secondary distance is at most y ($sd(S) \leq y$), and its internal distance is at most z ($id(S) \leq z$). Notice that if a set S is an $(x, y : z)$ set, then it is also an $(x', y' : z')$ set for any $x' > x$, $y' > y$, or $z' > z$.

Using our previous terminology, it follows that a set S is a dominating set if and only if its dominating distance equals one, $dd(S) = 1$, and is a distance-k dominating set if and only if $dd(S) \leq k$.

In this paper we determine the secondary distances y and internal distances z for 31 types of sets S whose dominating distance $dd(S) = 1$. This paper is a sequel to one by Hedetniemi, Hedetniemi, Rall and Knisely [39] that introduced the concept of secondary domination in graphs.

2. Types of (1,4:3) Dominating Sets

In this section we define a variety of dominating sets, each of which is a $(1, 4 : 3)$ set. Thus, for each type of dominating set S, where $|V| > |S| \geq 2$, every vertex $v \in V - S$ adjacent to at least one vertex in S and is within distance at most 4 to a second vertex in S, while every vertex $u \in S$ is (guaranteed to be) within distance at most 3 to another vertex in S.

2.1. Standard domination

Our first theorem was originally proved by Hedetniemi, Hedetniemi, Rall and Knisely in [39]. We provide the same proof here and use it as a model of all subsequent proofs in this paper; the proof that every dominating set S has internal distance $id(S) = 3$ is new here.

Theorem 3. *Every dominating set S in a connected graph $G = (V, E)$ is a (1,4:3)-set.*

Proof. Let $S \subset V$ be any dominating set in a connected graph $G = (V, E)$. We will first show that $id(S) \leq 3$, that is, for any vertex $u \in S$, there exists another vertex $w \in S$ such that $d(u, w) \leq 3$. Assume that there exists a vertex $u \in S$ for which $d(u, S - \{u\}) > 3$. Let u, x, y be the first three vertices on a shortest path from vertex u to another vertex w in S, where $x, y \in V - S$. Notice that vertex u cannot be adjacent to vertex y, else this is not a shortest path from u to another vertex in S. But since S is a dominating set, vertex y must be adjacent to at least one vertex, say $z \in S$. It follows therefore that $d(u, z) \leq 3$, contradicting our assumption that $d(u, S - \{u\}) > 3$. Therefore, $id(S) \leq 3$.

Since S is a dominating set, every vertex $v \in V - S$ must be adjacent to at least one vertex, say $u \in S$. But since $id(S) \leq 3$, it follows that vertex u is at most distance three to another vertex in S. Therefore, vertex $v \in V - S$ must be at most distance four to a second vertex in S, and we have that $sd(S) \leq 4$. □

It is important to observe that the (1,4:3) bounds in this theorem can be achieved, and thus the values of $y = 4$ and $z = 3$ cannot be decreased. The set $S = \{2, 5\}$ in Figure 1 can be seen to be a (1,4:3) set. We will have occasion to refer to this example many times. Therefore, we will refer to this as the P_6:2,5 example.

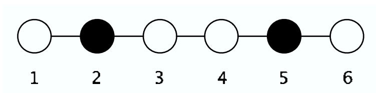

Figure 1: A (1,4:3) dominating set

Theorem 3 together with the P_6:2,5 example can be used to show that each of the following seven types of dominating sets are (1,4:3) sets, since the set $S = \{2, 5\}$ is a dominating set of each type: independent dominating, k-dependent dominating, acyclic dominating, bipartite dominating, odd (and externally odd) dominating, internally strong dominating, and open irredundant dominating.

2.2 Independent domination

Definition 4. *A set S of vertices is an* independent dominating set *if $N[S] = V$ and no two vertices in S are adjacent.*

The following is a well known result due to Berge [1].

Proposition 5. *Every maximal independent set S in a graph G is an independent dominating set.*

Note that the set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ independent dominating set.

2.3. k-dependent domination

Definition 6. *A set $S \subset V$ is called* k-dependent *if every vertex $u \in S$ is adjacent to at most k vertices in S, that is, deg(u) in the induced subgraph $G[S]$ is at most k.*

Notice that from the definition of k-dependence it follows that any k-dependent set S is also j-dependent for all $j \geq k$.

The concept of k-dependence was first introduced in 1985 by Fink and Jacobson [23]; see also Favaron [19, 20].

Definition 7. *A set S of vertices is a* k-dependent dominating set *if $N[S] = V$ and S is k-dependent.*

Note that the set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a 0-dependent dominating set. Therefore, it provides an example of a (1,4:3) k-dependent dominating set for all $k \geq 0$.

For a paper on k-dependent domination see Favaron, Hedetniemi, Hedetniemi and Rall [22].

2.4. Acyclic domination

Definition 8. *A set S in a graph G is called* acyclic *if the induced subgraph $G[S]$ does not contain any cycles.*

Definition 9. *A set S of vertices in a graph G is an* acyclic dominating set *if $N[S] = V$ and the set S is acyclic.*

Acyclic domination was introduced by Hedetniemi, Hedetniemi and Rall [38].

The set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ acyclic dominating set.

2.5. Bipartite domination

Definition 10. *A set S of vertices is called* bipartite *if the induced subgraph $G[S]$ is bipartite, that is, contains no cycles of odd length.*

Definition 11. *A set S of vertices is a* bipartite dominating set *if $N[S] = V$ and the set S is bipartite.*

Bipartite domination has been studied by Ko and Shepherd [43].

The set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ bipartite dominating set.

2.6. Odd and externally odd domination

Definition 12. *A set S of vertices is an* odd dominating set *if for every vertex $v \in V$, $|N[v] \cap S|$ is odd, that is, every closed neighborhood $N[v]$ in G contains an odd number of vertices in S.*

We use the term *closed* odd domination to reflect the fact that the definition involves the use of closed neighborhoods $N[v]$. We also use the term *total* odd domination to reflect the fact that all vertices $v \in V$ are required to satisfy the odd intersection with S condition. However, the term odd domination has been used in the literature to mean closed and total, without saying as such.

For papers on odd domination see Caro et al [7, 8].

We say that a set S is an *(externally) odd dominating set* if for every vertex $v \in V - S$, $|N(v) \cap S|$ is odd. Thus, we only require that vertices in $V - S$ are adjacent to an odd number of vertices in S.

The set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ odd and externally odd dominating set.

2.7. Internally strong domination

The following type of domination is newly defined here.

A graph $G = (V, E)$ is called *k-regular* if every vertex $v \in V$ has degree k, that is, $|N(v)| = k$.

A vertex $v \in V$ in a graph G is called *strong* if for every vertex $u \in N(v)$, $deg(v) \geq deg(u)$. Given a set S, we say that a vertex $v \in S$ is *internally strong (with respect to S)* if $deg_S(v) \geq deg_S(u)$ for every vertex u adjacent to v in S.

Definition 13. *A set S of vertices is an* internally strong dominating set *if $N[S] = V$ and every vertex $u \in S$ is internally strong with respect to S.*

Notice that if S is an internally strong dominating set, then every connected component of the induced graph $G[S]$ is a regular subgraph, where two connected components can be regular of different degrees.

The set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ internally strong dominating set.

2.8. Open irredundant domination

We say that a vertex $u \in S$ has an *external private neighbor (with respect to S)* if u is adjacent to at least one vertex $v \in V - S$ and no other vertex $w \in S$ is adjacent to v.

Definition 14. *A set S of vertices is an* open irredundant dominating set *if $N[S] = V$ and every vertex $u \in S$ has an external private neighbor (with respect to S).*

Open irredundant dominating sets have also been called *private dominating sets* by Hedetniemi, Hedetniemi and Jacobs [36]; see also Haynes and Henning [32]. It was first observed by Bollobás and Cockayne [2], that every non-trivial, connected graph G has at least one open irredundant minimum dominating set.

The set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is a $(1, 4 : 3)$ open irredundant dominating set.

2.9. Restrained domination

Definition 15. *A set S of vertices is called* restrained *if every vertex $v \in V - S$ is adjacent to at least one other vertex $w \neq v \in V - S$.*

Definition 16. *A set S of vertices is called a* restrained dominating set *if $N[S] = V$ and S is a restrained set.*

Restrained domination was introduced by Domke, Hattingh, Hedetniemi, Laskar and Markus [13]. For other papers on restrained domination see also [14, 15, 40, 52, 9, 12].

Since all dominating sets, including restrained dominating sets, are (1,4:3) sets, we have the following result.

Theorem 17. *Every restrained dominating set S in a graph G is a (1,4:3) set.*

Notice that the set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is not a restrained dominating set. Thus we need another example. A graph with a restrained dominating set $S = \{3, 6\}$ achieving the (1,4:3) bounds is shown in Figure 2.

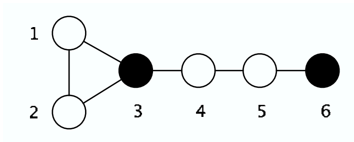

Figure 2: A (1,4:3) restrained dominating set

2.10. 1-moveable domination

Definition 18. *A set S of vertices in a graph G is a* 1-moveable dominating set *if $N[S] = V$ and for every vertex $u \in S$ there exists a vertex $v \in V - S$ such that the set obtained by deleting u from S and adding v is also a dominating set, that is, the set $S' = S - \{u\} \cup \{v\}$ is a dominating set.*

The new concept of a 1-moveable set is due to Gera and Horton [26]. In 1-moveable domination, the focus is on placing one guard on each vertex in a dominating set S, such that every guard in S can move to at least one adjacent vertex in $V - S$ in such a way that the resulting placement of guards still forms a dominating set.

Since all dominating sets, including 1-moveable dominating sets, are (1,4:3) sets, we have the following result.

Theorem 19. *Every 1-moveable dominating set S in a graph G is a (1,4:3) set.*

Since the set $S = \{2, 5\}$ in the $P_6 : 2, 5$ example is not a 1-moveable dominating set, we need to construct a 1-moveable dominating set that is a (1,4:3) set. The set $S = \{7, 8\}$ in Figure 3 is a (1,4:3) 1-moveable dominating set, since the guard at 7 can be moved to vertex 2, and the guard at 8 can be moved to vertex 5, and in either case a dominating set of guards will still be in place.

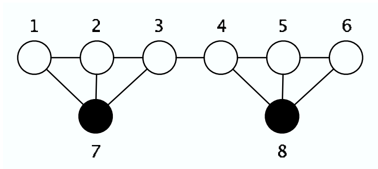

Figure 3: A (1,4:3) 1-moveable dominating set

3. (1,3:3) Dominating Sets

In this section we present four types of dominating sets that are (1,3:3) sets.

3.1. 2-maximal independent sets

An independent set S of vertices is called *maximal independent* if no proper superset of S is also an independent set. Equivalently, S is maximal independent if it is not possible to remove 0 vertices and add 1 vertex in $V - S$ to S to create a larger independent set. Viewed this way we could say that a maximal independent set is a 1-*maximal independent set.*

We say that a 1-maximal independent set S is a 2-*maximal independent set* if it is not possible to remove one vertex from S and then add two other vertices to create a larger independent set. To illustrate this, consider the previous example P_6:2,5. It is easy to see that the set $S = \{2, 5\}$ in the path P_6 is a 1-maximal independent set, but it is not a 2-maximal independent set because we can remove vertex 2 and then add vertices 1 and 3 to create a larger independent set.

The concepts of k-minimal and k-maximal invariants of graphs was first introduced and studied by Bollobás, Cockayne and Mynhardt in [3]; see also p. 95 of Haynes, Hedetniemi and Slater [31].

We have previously shown that every 1-maximal independent set is a (1,4:3) set. But for 2-maximal independent sets we have the following different result.

Theorem 20. *Every* 2*-maximal independent set S in a graph G is a (1,3:3) set.*

Proof. Let S be an arbitrary 2-maximal independent set for which $sd(S) > 3$. This means that there exists a vertex $v \in V - S$ that is adjacent to at least one vertex $u \in S$, and whose next closest vertex $w \in S$ is at least distance four from v. Let the vertices on a shortest path to w be labeled in order, $v, x, y, z, ..., w$, where we are given that vertices y and z are in $V - S$, and there may be other vertices between z and w on this shortest path to w that are also not vertices in S. There are two cases for vertex x: either $x = u$, or $x \notin S$.

Case 1. $x \notin S$.

Since S is 2-maximal independent, it is, by definition, also 1-maximal, and therefore it is a dominating set. Thus, vertices x and y must both be adjacent to a vertex in S. If either of these two vertices are adjacent to a vertex in S other than vertex u, then the secondary distance of v is at most three, contradicting our assumption that $sd(v, S) > 3$. Thus, vertices v, x and y must be dominated only by vertex $u \in S$. But since $v, x, y, z, ..., w$ is a shortest path from v to w, we know that vertex v is not adjacent to vertex y. Therefore, it follows that the set $S' = S - \{u\} \cup \{v, y\}$ is also an independent set, contradicting our assumption that S is a 2-maximal independent set.

Case 2. $x = u$.

In this case, we know that vertices v and y are only adjacent to vertex $u \in S$, and we know that vertices v and y are not adjacent, since this is a shortest secondary path to a vertex in S. Thus, as in Case 1, the set $S' = S - \{u\} \cup \{v, y\}$ is also an independent set, contradicting our assumption that S is a 2-maximal independent set. □

The fact that the internal distance $id(S) = 3$ follows from the proof of Theorem 3, that shows for any dominating set S, $id(S) = 3$.

It remains to show that the (1,3;3) bounds can be achieved for 2-maximal independent sets. The set $S = \{2, 5\}$ in Figure 4 can be seen to be a (1,3:3) 2-maximal independent set.

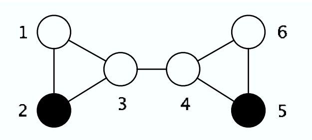

Figure 4: A (1,3:3) 2-maximal independent set

3.2. Mobile domination

Let $S \subset V$ be an arbitrary set of vertices in a graph G. We say that a set $S' \subset V$ is a *shift* of S if there exists a 1-to-1 function

$$sh : S \rightarrow V,$$

from S into V, such that (i) for every vertex $u \in S$, u is adjacent to $sh(u)$, and (ii) $sh(S) = S'$.

Definition 21. *A set S in a graph G is called a mobile dominating set if $N[S] = V$ and for every vertex $v \in V - S$, there exists a shift S' of S such that $v \in S'$ and S' is a dominating set.*

With mobile domination, we imagine placing a guard at each vertex of a dominating set S. If there is some trouble at a vertex $v \in V - S$, we need to move at least one guard from an adjacent vertex u to v, and possibly move several other guards at the same time, so that the resulting set of guards $sh(S)$ is a dominating set. The concept of mobile domination in graphs was introducted by Goddard, Hedetniemi and Hedetniemi in 2005 [27].

Theorem 22. *Every mobile dominating set S in a graph G is a (1,3:3) set.*

Proof. Let S be a mobile dominating set, and assume that there exists a vertex $v \in V - S$ with $sd(v, S) > 3$. Since S is a dominating set, assume that v is adjacent to a vertex $u \in S$ and whose next closest vertex $w \in S$ is at least distance four from v. Since we are assuming that $sd(v, S) > 3$, we can assume, in particular, that there are no vertices, other than u, within distance two of v. There are two cases to consider.

Case 1. A shortest path from v to w does not pass through vertex u. Let v, x, y be the first three vertices on a shortest path from v to w. Because this is a shortest path, we know that vertex v is not adjacent to vertex y. Since $sd(v, S) > 3$, it follows that x and

y are in $V - S$. And since S is a dominating set, x and y must be adjacent to at least one vertex in S. But this vertex cannot be other than vertex u, else $sd(v, S) \leq 3$. Thus, vertices v, x, and y are all adjacent to vertex u and to no other vertex in S.

But since S is a mobile dominating set, there must be a shift $sh(S)$ of the guards in S that includes the move of a guard in S to vertex $y \in V - S$ such that $sh(S)$ is still a dominating set. The only possibility for $sh(S)$ is to move the guard at vertex u to y. But in this case no vertex in $sh(S)$ can dominate vertex v, since v is not adjacent to y and there are no vertices, other than u, within distance two of v that can be moved to dominate v.

Case 2. A shortest path from v to w passes through vertex $u \in S$. Let the first three vertices on this path be v, u, and x. It follows that vertices v and x can only be adjacent to vertex $u \in S$, else $sd(v, S) \leq 3$. But since S is a mobile dominating set, there must be a shift $sh(S)$ of the guards in S that includes the move of a guard in S to vertex x such that $sh(S)$ is still a dominating set. The only possibility for $sh(S)$ is to move the guard at vertex u to x. But in this case no vertex in $sh(S)$ can dominate vertex v, since v is not adjacent to x and there are no vertices, other than u, within distance two of v that can be moved to dominate v. □

The fact that $id(S) = 3$ follows from Theorem 3, which asserts that for every dominating set S, $id(S) \leq 3$.

It only remains to show that the (1,3:3) bounds can be achieved. The set $S = \{2, 5\}$ in Figure 4 is a (1,3:3) mobile dominating set.

3.3. Secure domination

Definition 23. *A set S of vertices is a* secure dominating set *if $N[S] = V$ and for every vertex $v \in V - S$ there exists a vertex $u \in S$ adjacent to v such that the set obtained by deleting u from S and adding v is also a dominating set, that is, the set $S' = S - \{u\} \cup \{v\}$ is a dominating set.*

The context for secure dominating sets is that of assigning one guard to each vertex in a dominating set S. If there is some trouble at a vertex $v \in V - S$, we are required to move a guard to v from an adjacent vertex $u \in S$. We require, however, that after the guard at u has moved to v, the new placement of guards still forms a dominating set.

Notice the difference between mobile and secure dominating sets. Every secure dominating set is a mobile dominating set since the movement of only one guard at u to a neighboring vertex $v \in V - S$ suffices to produce another dominating set, and it is not necessary, as in mobile domination, to move more than one guard. But some mobile dominating sets are not secure. For example, in the cycle C_5 of length five, with vertices labeled in order $1, 2, 3, 4, 5$, the set $S = \{1, 3\}$ is a mobile dominating set, but is not a secure dominating set, since neither the guard at vertex 1 nor the guard at vertex 3 can move to vertex 2 and still create a dominating set.

The concept of a secure dominating set is due to Cockayne, Favaron and Mynhardt [11]; see also Klostermeyer and MacGillivray [41] and Mynhardt, Swart and Ungerer [47]. The proof of the following theorem is virtually the same as the proof for mobile dominating sets and is omitted in the interests of brevity.

Theorem 24. *Every secure dominating set S in a graph G is a (1,3:3) set.*

3.4. Maximal enclaveless sets

Definition 25. *A set S of vertices is called* enclaveless *if no vertex $u \in S$ satisfies the condition that $N[u] \subseteq S$, that is, every vertex $u \in S$ is adjacent to at least one vertex in $V - S$.*

Notice that if a set S is enclaveless, then the complement $V - S$ must be a dominating set.

Proposition 26. *Every maximal enclaveless set S in a graph G is a dominating set.*

Proof. Let S be a maximal enclaveless set in a graph G. Assume that S is not a dominating set. Then there exists a vertex $v \in V - S$ that is not adjacent to any vertex in S. Since we are assuming that every graph G is connected, we can assume that vertex v is adjacent to at least one other vertex $w \in V - S$. If we add vertex v to the set S, it will not become an enclave in S, since it would be adjacent to a vertex $w \in V - S$. But since vertex v is not adjacent to any vertex in S, adding it to S would not make any vertex in S an enclave. Therefore, S is not a maximal enclaveless set, contradicting our assumption. □

Theorem 27. *Every maximal enclaveless set is a (1,3:3) set.*

Proof. Let S be a maximal enclaveless set in a graph G, and assume that there exists a vertex $v \in V - S$ with $sd(v, S) \geq 4$. Since S is a dominating set, assume that v is adjacent to a vertex $u \in S$ and no other vertex in S is within distance 3 of v. There are two cases to consider.

Case 1. A shortest path from v to a second nearest vertex $w \in S$ does not pass through vertex u. Let v, x, y be the first three vertices on a shortest path from v to w. Since $sd(v, S) > 3$, it follows that x and y are in $V - S$. And since S is a dominating set, x and y must be adjacent to at least one vertex in S. But this vertex cannot be other than vertex u, else $sd(v, S) \leq 3$. Thus, vertices v, x, and y are all adjacent to vertex u and to no other vertex in S. But this means that if we add vertex x to S we will still not have an enclave in S. Vertex x will be adjacent to at least one vertex in $V - S$ (either v or y), and vertex $u \in S$ will still be adjacent to a vertex (either v or y) in $V - S$. Therefore $S \cup \{x\}$ is enclaveless, contradicting the maximality of S.

Case 2. A shortest path from v to a second vertex $w \in S$ passes through vertex $u \in S$. Let the first four vertices on this path be v, u, x and y, where $v, x, y \in V - S$. It follows that vertices v and x cannot be adjacent to a vertex in S other than u, else $sd(v, S) \leq 3$.

But this means that if we add vertex x to S we will still have an enclaveless set; vertex x will be adjacent to at least one vertex $y \in V - S$ and vertex u will still be adjacent to at least one vertex $v \in V - S$. Therefore $S \cup \{x\}$ is enclaveless, contradicting the maximality of S. □

The fact that $id(S) = 3$ follows from Theorem 3, which asserts that for every dominating set S, $id(S) \leq 3$.

It only remains to show that the (1,3:3) bounds can be achieved. The set $S = \{2, 4, 5, 8\}$ in Figure 5 is a (1,3:3) maximal enclaveless set.

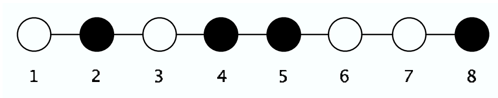

Figure 5: A (1,3:3) maximal enclaveless set

4. (1,3:2) Dominating Sets

In this section we present three types of dominating sets that are (1,3:2) sets.

4.1. Vertex covers

Definition 28. *A set $S \subset V$ is called a vertex cover if for every edge $uv \in E$ either $u \in S$ or $v \in S$ (or both).*

In a nontrivial connected graph G, every vertex u is incident to at least one edge $uv \in E$. Since every vertex cover S must contain either u or v, it follows that every vertex $u \in V$ is either in S or is adjacent to a vertex in S. Thus, every vertex cover is a dominating set, i.e. $dd(S) = 1$.

In addition, it can be seen that the complement $V - S$ of every vertex cover S is an independent set, and conversely, the complement of every independent set is a vertex cover.

Theorem 29. *Every vertex cover S in a graph G is a (1,3:2) set.*

Proof. Let S be a vertex cover in a graph G. Assume that there exists a vertex $v \in V - S$ with $sd(v, S) \geq 4$. Since S is a dominating set, assume that v is adjacent to a vertex $u \in S$. There are two cases to consider.

Case 1. A shortest path from v to a second vertex $w \in S$ does not pass through vertex u. Let v, x be the first two vertices on a shortest path from v to w. If $x \in S$ then $sd(v, S) = 1$, contradicting our assumption that $sd(v, S) \geq 4$. But if $x \notin S$, then S cannot be a vertex cover, since neither v nor x is in S.

Case 2. A shortest path from v to a second vertex $w \in S$ passes through vertex $u \in S$. Let the first four vertices on this path be v, u, x and y. If either x or y are vertices in S, then $sd(v, S) \leq 3$, contradicting our assumption that $sd(v, S) \geq 4$. On the other hand, if $x, y \notin S$, then S is not a vertex cover. □

It remains to show that $id(S) \leq 2$. But this follows from the observation that a shortest path between any two vertices in S cannot contain two consecutive vertices in $V - S$, else S is not a vertex cover.

The set $S = \{2, 4\}$ in Figure 6 is an example of a vertex cover that achieves the (1,3:2) bounds.

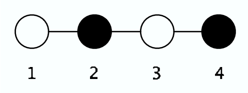

Figure 6: A (1,3:2) vertex cover

4.2. Weakly connected {independent} domination

We say that a set S is *weakly connected* if the graph $G_w[S] = (N[S], E \cap ((S \times S) \cup (S \times N(S)))$ is connected. The vertex set of $G_w[S]$ consists of all vertices in S together with all vertices in $V - S$ that are adjacent to at least one vertex in S, while the set of edges in $G_w[S]$ consists of all edges in $E(G)$ that are incident to at least one vertex in S.

It is easy to determine if a given set S of vertices is weakly connected. Simply remove all edges in G that connect two vertices in $V - S$. If the resulting graph is connected, then S is weakly connected.

Definition 30. *A set S of vertices is a* weakly connected {independent} dominating set *if $N[S] = V$ and S is a weakly connected {independent} set.*

Weakly connected domination was first introduced and studied by Dunbar, Grossman, Hattingh, Hedetniemi and McRae [18]. For other papers on weakly connected domination see [51, 17, 16].

Notice that the simple dominating set $S = \{2, 5\}$ in the P_6:2,5 example is not a weakly connected dominating set. Indeed, the (1,4:3) bounds cannot be achieved by any weakly connected {independent} dominating set.

Theorem 31. *Every weakly connected {independent} dominating set S in a graph G is a (1,3:2) set.*

Proof. Let S be any weakly connected {independent} dominating set in a connected graph G. The fact that $id(S) = 2$ follows from the observation that any path in $G_w[S]$ between two vertices in S cannot contain two consecutive vertices in $V - S$. Thus, any shortest path in $G_w[S]$ between two vertices in S has length at most two. From this observation it follows immediately that $sd(S) \leq 3$, since every vertex $v \in V - S$ must be dominated by at least one vertex $u \in S$. Therefore, vertex v is distance one to at least one vertex in S and distance at most three to a second vertex in S.

It remains to show that the (1,3:2) bounds can be achieved. The set $S = \{2, 4\}$ in Figure 6 is a (1,3:2) weakly connected {independent} dominating set. □

4.3. Global offensive alliances

We say that a set S is an *offensive alliance* if for every vertex $v \in N(S) \cap (V - S)$, $|N(v) \cap S| \geq |N[v] \cap (V - S)|$

Definition 32. *A set S of vertices is a* global offensive alliance *if $N[S] = V$ and the set S is an offensive alliance.*

In a global offensive alliance S, every vertex $v \in V - S$ has at least as many neighbors in S $(N(S) \cap V - S)$ as it has in $V - S$ including itself $(|N[v] \cap V - S|)$. This implies that every global alliance is a dominating set.

The concept of alliances in graphs, including offensive alliances, was first introduced and studied by Kristiansen, Hedetniemi and Hedetniemi in [42] and later in [37]. For other papers on alliances see [21, 50].

Theorem 33. *Every global offensive alliance S in a graph G is a (1,3:2) set.*

Proof. Let S be a global offensive alliance in a graph G. If every vertex in S has a neighbor in S, then $id(S) = 1$. So assume that there is a vertex $u \in S$ every neighbor of which is in $V - S$. Since we are assuming throughout this paper that $|S| \geq 2$, let $w \in S$ be a vertex in S closest to u and suppose that $d(u, w) \geq 3$. Let u, x, y be the first three vertices on a shortest path from u to w, where $x, y \in V - S$. Since x has a neighbor $y \in V - S$, and since S is a global offensive alliance, it follows that x must have at least one other neighbor, say z, in S. But this contradicts our assumption that vertex $w \in S$ is closest to u in S. Therefore, $d(u, S - \{u\}) \leq 2$ and $id(S) \leq 2$.

Since every global alliance is a dominating set, we know that every vertex $v \in V - S$ is at distance one from at least one vertex $u \in S$. But since $id(S) \leq 2$, every vertex in S is within distance two of another vertex in S, and therefore $sd(v, S) \leq 3$. Therefore, $sd(S) \leq 3$. □

The example P_4:2,4 given above in Figure 6 for vertex covers and weakly connected dominating sets, suffices to show that global offensive alliances can achieve the (1,3:2) bounds. The set $S = \{2, 4\}$ is a global offensive alliance.

5. (1,2:3) Dominating Sets

In this section we present the only example we know of a (1,2:3) set.

5.1. Global defensive alliances

Definition 34. *A set S of vertices is a* defensive alliance *if for every vertex $u \in S$, $|N[u] \cap S| \geq |N[u] \cap V - S|$.*

In a defensive alliance S, every vertex in S has at least as many neighbors in its closed neighborhood in S as it has in $V - S$. As with offensive alliances, defensive alliances were first introduced and studied by Kristiansen, Hedetniemi and Hedetniemi in [42] and later in [37]. For other papers on defensive alliances see [25, 29, 30].

Definition 35. *A set S of vertices is a* global defensive alliance *if $N[S] = V$ and the set S is a defensive alliance.*

Theorem 36. *Every global defensive alliance S in a graph G is a (1,2:3) set.*

Proof. Let S be a global defensive alliance in a graph G. Let $v \in V - S$. Because every global defensive alliance is a dominating set, assume that v is adjacent to a vertex $u \in S$. Let v, x be the first two vertices on a shortest path from v to a vertex in S other than u. If $x \in S$, then $sd(v, S) = 1$, so assume that $x \in V - S$. Since S is a dominating set, vertex x must be dominated by at least one vertex, say w in S. There are two cases. If $w \neq u$ then $sd(v, S) \leq 2$. But if $w = u$, then since u has at least two neighbors in $V - S$, it must have at least one neighbor, say $z \in S$. But then $d(v, z) = 2$ and $sd(v, S) \leq 2$. Thus, in all cases, $sd(v, S) \leq 2$.

The fact that $id(S) \leq 3$ follows from Theorem 3; for any dominating set S, $id(S) \leq 3$. □

The set $S = \{1, 4\}$ in Figure 7 is an example of a global defensive alliance that achieves the (1,2:3) bounds.

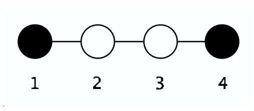

Figure 7: A (1,2:3) global defensive alliance

6. (1,2:2) Dominating Sets

In this section we present four types of (1,2:2) sets.

6.1. Maximal k-dependent sets

In Section 2.3 we defined k-dependent sets.

Proposition 37. *For any $k \geq 1$, every maximal k-dependent set S in a graph G is a dominating set.*

Proof. If S is a maximal k-dependent set and there exists a vertex $v \in V - S$ with $d(v, S) = 2$ then it would be an isolated vertex in the induced subgraph $G[S \cup \{v\}]$. Thus, the set $S' = S \cup \{v\}$ is also a k-dependent set, and therefore S is not a maximal k-dependent set. □

The situation for maximal k-dependent sets is a bit different than for k-dependent dominating sets.

Theorem 38. *Every maximal k-dependent set S in a graph G, for $k \geq 1$, is a (1,2:2) set.*

Proof. Since every maximal k-dependent set S, for $k \geq 1$, is a dominating set, we know that $dd(S) = 1$. Assume that $sd(S) > 2$, that is, there exists a vertex $v \in V - S$ with $d(v, S - \{u\}) \geq 3$. Since S is a maximal k-dependent set, we know that the set $S \cup \{v\}$ is not k-dependent. This means that either the degree of v in $G[S \cup \{v\}]$ is greater than k, in which case vertex v has at least two neighbors in S, or v has exactly one neighbor, say u in S, but adding v to S causes u to have more than k neighbors in $S \cup \{v\}$. In this case, vertex v has a second neighbor in S at most distance two from v.

It remains to show that $id(S) \leq 2$ for every maximal k-dependent set. Let $u \in S$ be any vertex in S, and assume that $d(u, S - \{u\}) > 2$. In this case u is an isolated vertex in $G[S]$. Since G is assumed to be a connected graph, we can assume that u has a neighbor $v \notin S$. If v has a neighbor in S other than u, then $d(u, S - \{u\}) \leq 2$. Therefore we can assume that the only neighbor of v in S is u. But in this case, $S \cup \{v\}$ is still a k-maximal set, contradicting the maximality of S. Therefore, $id(S) \leq 2$. □

It is easy to see that the set $S = \{2, 3, 5\}$ in Figure 8 is a maximal 1-dependent set that achieves the (1,2:2) bounds.

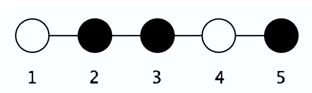

Figure 8: A (1,2:2) maximal 1-dependent set

6.2. P_3 domination

Definition 39. *A set S of vertices is a P_3-*dominating set *if $N[S] = V$ and every vertex $v \in V - S$ forms a path P_3 with two vertices in S.*

Since every vertex $v \in V - S$ forms a P_3 with two vertices in a P_3 dominating set S, it is immediate that $sd(S) \leq 2$ and that every vertex in S is distance at most two from another vertex in S.

Theorem 40. *Every P_3-dominating set S in a graph G is a (1,2:2) set.*

The set $S = \{2, 3, 5\}$ in Figure 8 is an example of a P_3 dominating set achieving the (1,2:2) bounds.

6.3. Maximal internally strong sets

In Section 2.7 we defined a set S to be internally strong if the subgraph induced by S consists of a disjoint union of complete subgraphs.

Proposition 41. *Every maximal internally strong set S in a graph G is a dominating set.*

Proof. If S is a maximal internally strong set and there exists a vertex $v \in V - S$ with $d(v, S) = 2$ then it would be an isolated vertex in the induced subgraph $G[S \cup \{v\}]$. Thus, the set $S = S \cup \{v\}$ is also an internally strong set, and therefore S is not maximal. □

In fact one can prove an even stronger result.

Theorem 42. *Every maximal internally strong set is a P_3-dominating set.*

Proof. We need to show that if S is a maximal internally strong set, then $sd(S) \leq 2$. Let $v \in V - S$ be a vertex whose secondary distance is at least three ($sd(v, S) \geq 3$). Since we know from Proposition 41 that S is a dominating set, we know that vertex v is adjacent to at least one vertex $u \in S$ and v is not adjacent to any other vertex in S, since $sd(v, S) \geq 3$. There are now two cases to consider: vertex u is adjacent to another vertex in S or it isn't. If u is adjacent to a vertex $w \in S$ then $sd(v, S) = 2$, contradicting our assumption that $sd(v, S) \geq 3$. But if u is not adjacent to another vertex in S, then v together with u will form a K_2 (a regular connected component) in the subgraph $G[S \cup \{v\}]$, contradicting our assumption that S is a maximal internally strong set. □

From Theorems 40 and 42 we can therefore conclude the following.

Corollary 43. *Every maximal internally strong set is a (1,2:2) set.*

Notice that although every maximal internally strong set is a dominating set, it is not necessarily the case that every internally strong dominating set is maximal internally strong.

It only remains to show that the (1,2:2) bounds can be achieved. The set $S = \{2, 3, 5\}$ in Figure 8 is a maximal internally strong (1,2:2) set.

6.4. Global powerful alliance

Definition 44. *A set S is a graph G is called a powerful alliance if it is both a defensive and an offensive alliance. A powerful alliance S is called global if $N[S] = V$.*

Powerful alliances were first studied by Brigham, Dutton, Haynes and Hedetniemi in [5, 4].

Although we have shown in previous sections that global offensive alliances are (1,3:2) sets and global defensive alliances are (1,2:3) sets, for global powerful alliances we get a different result.

Theorem 45. *Every global powerful alliance S in a graph G is a (1,2:2) set.*

Proof. Let S be a global powerful alliance in a graph G. Let $v \in V - S$. Since every global powerful alliance is a dominating set, we can assume that vertex v is adjacent to at least one vertex, say $u \in S$. Assume that $sd(v, S) > 2$ and let v, x, y be the first three vertices on a shortest path from v to a second vertex $w \in S$. There are two cases to consider.

Case 1. $x \notin S$.

In this case, since S is an offensive alliance, there must be at least two vertices in S adjacent to v, and thus $sd(v, S) = 1$, contradicting our assumption that $sd(v, S) > 2$.

Case 2. $x \in S$.

In this case, if $x \neq u$ then $sd(v, S) = 1$, again contradicting our assumption that $sd(v, S) > 2$. Therefore, we can assume that $x = u$. Consider then vertex y. If $y \in S$, then $sd(v, S) = 2$, contradicting our assumption that $sd(v, S) > 2$. Therefore, it must be the case that $y \notin S$. But in this case vertex u is adjacent to two vertices in $V - S$, and since S is a defensive alliance, vertex u must be adjacent to at least one other vertex in S. But this means that $sd(v, S) = 2$, again contradicting our assumption that $sd(v, S) > 2$.

It remains to show that $id(S) \leq 2$. Let $u \in S$ and assume that $d(u, S - \{u\}) > 2$. This means that every vertex adjacent to u must be in $V - S$. If the degree of u in G is at least two, then S cannot be a defensive alliance. Therefore we can assume that the degree of u in G is one. Let $v \in V - S$ be the only vertex adjacent to u. We can assume that the degree of v is at least two, else $G = K_2$ and $|S| = 1$ (we have always been assuming that $|S| \geq 2$). Therefore, let vertex v be adjacent to at least one other vertex w. If $w \in S$, then $d(u, S - \{u\}) = 2$, contradicting our assumption that $d(u, S - \{u\}) > 2$. Therefore, $w \in V - S$. But in this case since S is an offensive alliance, vertex v must be adjacent to at least one other vertex in S, which again means that $d(u, S - \{u\}) = 2$. □

The set $S = \{2, 3, 5\}$ in Figure 8 is a (1,2:2) global powerful alliance.

7. (1,2:1) Dominating Sets

In this section we present six types of (1,2:1) sets. In each case, the subgraph $G[S]$ induced by such a set S cannot contain an isolated vertex, and thus $id(S) = 1$. And this means, in turn, that if each type of set is a dominating set, it must be a (1,2:1) set.

7.1. Total domination

Definition 46. *A set S of vertices is a* total dominating set *if $N(S) = V$, that is, every vertex $v \in V$ is adjacent to at least one vertex $u \neq v \in S$. Equivalently, a set S is a total dominating set if every vertex $v \in V - S$ is adjacent to at least one vertex $u \in S$, and every vertex $u \in S$ is adjacent to at least one other vertex $w \neq u \in S$.*

Since every vertex u in a total dominating set S is adjacent to at least one other vertex in S, the following result is immediately obvious.

Theorem 47. *Every total dominating set S in a graph G is a (1,2:1) set.*

The path $P_4 : 2, 3$ provides a simple example of a (1,2:1) total dominating set.

7.2. Connected domination

Definition 48. *A set S of vertices is a* connected dominating set *if $N[S] = V$ and the induced subgraph $G[S]$ is connected.*

Since every vertex in a connected dominating set S, with $|V| > |S| \geq 2$, has at least one neighbor also in S, the following result is obvious.

Theorem 49. *Every connected dominating set S in a graph G is a (1,2:1) set.*

The path $P_4 : 2, 3$ provides a simple example of a (1,2:1) connected dominating set.

7.3. Paired domination and maximal matchings

A *matching* in a graph $G = (V, E)$ is any set $M \subset E$ of edges, no two of which have a vertex in common. If M is a matching and uv is an edge in M, then we say that vertices u and v are *saturated* by M. Let $V(M)$ denote the set of all vertices saturated by edges in a matching M. We say that a graph G has a *perfect matching* if it has a matching M such that $V(M) = V$, that is, every vertex $v \in V$ is saturated by exactly one edge in M.

A set $S \subset V$ of vertices is called *paired* if the induced subgraph $G[S]$ has a perfect matching.

Definition 50. *A set S of vertices is a* paired dominating set *if $N[S] = V$ and S is a paired set.*

Paired domination was first defined and studied by Haynes and Slater [33]. Since then quite a few papers have appeared on paired domination, including the following [34, 34, 10, 48, 49].

Theorem 51. *Every maximal paired set is a (paired) dominating set, but the set of vertices in a paired dominating set need not be a maximal paired set.*

Since every vertex in a paired dominating set has a neighbor in S, the following result is obvious.

Theorem 52. *Every paired dominating set S in a graph G is a (1,2:1) set.*

It is easy to see that if a set S is a maximal paired set, then the set of edges in the perfect matching in $G[S]$ is a maximal matching, but the converse may not be true, that is, the set of vertices saturated by the edges of any maximal matching need not form a maximal paired set. It is also easy to see that the set of vertices saturated by any maximal matching is a paired dominating set.

Corollary 53. *The set of vertices saturated by the edges of any maximal matching is a (1,2:1) set.*

The path $P_4 : 2, 3$ provides a simple example of a (1,2:1) paired dominating set and a (1,2:1) maximal matching.

7.4. Maximal total/isolate-free matching

Definition 54. *A matching M is called a* total matching, or isolate-free *if the induced subgraph $G[V(M)]$ does not contain an isolated K_2. We assume that M has at least two edges.*

Theorem 55. *The set $V(M)$ of vertices saturated by any maximal total matching M is a (1,2:1) set.*

Proof. Let M be a maximal total matching, let $S = V(M)$, and assume that $dd(S) > 1$. Then there exists a vertex $v \in V - S$ at distance two from S, $d(v, S) = 2$. Let v, w, u be the vertices on a path of length two to a vertex $u \in S$, where $w \in V - S$. It follows that adding the edge vw to M creates a larger total matching, contradicting the maximality of M. Therefore, $dd(S) = 1$.

Since $V(M)$ is a paired dominating set, it follows from Theorem 52 that M is a (1,2:1) set. □

7.5. Maximal uniquely restricted matchings

Definition 56. *A matching M is called* uniquely restricted *if the induced subgraph $G[V(M)]$ contains only one perfect matching.*

Uniquely restricted matchings were first introduced and studied by Golumbic, Hirst and Lewenstein [28]; see also Levit and Mandrescu [44, 45, 46].

Theorem 57. *The set $V(M)$ of vertices saturated by any maximal uniquely restricted matching M is a (1,2:1) set.*

Proof. Let $S = V(M)$ be the vertices saturated by the edges in a maximal uniquely restricted matching, and assume that $dd(S) > 1$. This implies that there exists a vertex $v \in V - S$ with $d(v, S) = 2$. Let v, x, u be the vertices on a shortest path from v to S. Consider the augmented matching $M' = M \cup \{vx\}$. It follows that in the induced subgraph $G' = G[V(M')]$, vertex v is an endvertex adjacent only to vertex x. Therefore, every perfect matching of G' must contain the edge vx. This implies that if G' has two distinct perfect matchings, then so does G. This would contradict the assumption that M is a uniquely restricted matching. Therefore, M' is also a uniquely restricted matching, contradicting the maximality of M. Therefore, $dd(S) = 1$ and $V(M)$ is a dominating set. The fact that $V(M)$ is a (1,2:1) set is therefore obvious.

The vertices labeled $1, 2, 3, 4$ in a cycle of length 4, together with the maximal uniquely restricted matching $M = \{12\}$ provides an example of a (1,2:1) set. □

7.6. Total vertex cover

Definition 58. *A vertex cover S in a graph G is called a total vertex cover if the induced subgraph $G[S]$ contains no isolated vertices.*

This is similar to a total dominating set S in which we also require that the induced subgraph $G[S]$ contain no isolated vertices. It is interesting to observe that the complement $V - S$ of a total vertex cover S is a restrained independent set. Conversely, the complement of a restrained independent set is always a total vertex cover. This is similar to the observation that the complement of every vertex cover is an independent set, and conversely.

Since every total vertex cover is a dominating set and since every vertex u in a total vertex cover S is adjacent to at least one other vertex in S, the following result is immediately obvious.

Theorem 59. *Every total vertex cover S in a graph G is a (1,2:1) set.*

Recall, by comparison, that we previously proved that every vertex cover is a (1,3:2) set.

8. (1,1:2) Dominating Sets

The remaining three types of dominating sets S all have the property that every vertex in $V - S$ is adjacent to at least two vertices in S, and hence every vertex in S is at most distance two from another vertex in S. Therefore, all of these are examples of (1,1:2) sets.

8.1. 2-domination

Definition 60. *A set S of vertices is a* 2-dominating set *if every vertex $v \in V - S$ is dominated by at least two vertices in S, that is, $|N(v) \cap S| \geq 2$.*

Theorem 61. *Every 2-dominating set S in a graph G is a (1,1:2) set.*

8.2. Maximal acyclic sets

Proposition 62. *Every maximal acyclic set S in a graph G is a dominating set.*

Proof. If S is a maximal acyclic set and there exists a vertex $v \in V - S$ with $d(v, S) = 2$, then it would be an isolated vertex in the induced subgraph $G[S \cup \{v\}]$. Thus, the set $S \cup \{v\}$ is also an acyclic set, contradicting the maximality of S. □

Notice that while every maximal acyclic set is a dominating set, it is not necessarily the case that every acyclic dominating set is a maximal acyclic set. In fact, for maximal acyclic sets we get the following result.

Theorem 63. *Every maximal acyclic set is a (1,1:2) set.*

Proof. Every vertex v not in a maximal acyclic set S has to be adjacent to at least two vertices in S, else adding v to S would create a larger acyclic set. Thus every maximal acyclic set S satisfies $dd(S) = 1$ and $sd(S) = 1$. If a vertex $u \in S$ is an isolated vertex in $G[S]$, then every neighbor of u is in $V - S$. Let v be any neighbor of u in $V - S$. Since v must be adjacent to at least two vertices in S, it follows that $d(u, S - \{u\}) = 2$. Therefore, $id(S) \leq 2$. □

8.3. Maximal bipartite sets

Proposition 64. *Every maximal bipartite set S in a graph G is a dominating set.*

Proof. If S is a maximal bipartite set and there exists a vertex $v \in V - S$ with $d(v, S) = 2$, then it would be an isolated vertex in the induced subgraph $G[S \cup \{v\}]$. Thus, the set $S \cup \{v\}$ is also a bipartite set, contradicting the maximality of S. □

Notice that although every maximal bipartite set is a dominating set, it is not necessarily the case that every bipartite dominating set is a maximal bipartite set.

Theorem 65. *Every maximal bipartite set in a graph G is a (1,1:2) set.*

9. Summary

The following table summarizes the results in this paper.

Type of Dominating Set	$(dd, sd : id)$
1. dominating	$(1, 4 : 3)$
2. independent dominating	$(1, 4 : 3)$
3. k-dependent dominating	$(1, 4 : 3)$
4. acyclic dominating	$(1, 4 : 3)$
5. bipartite dominating	$(1, 4 : 3)$
6. odd and externally odd dominating	$(1, 4 : 3)$
7. internally strong dominating	$(1, 4 : 3)$
8. open irredundant dominating	$(1, 4 : 3)$
9. restrained dominating	$(1, 4 : 3)$
10. 1-*moveable dominating*	$(1, 4 : 3)$
11. 2-*maximal independent*	$(1, 3 : 3)$
12. mobile dominating	$(1, 3 : 3)$
13. secure dominating	$(1, 3 : 3)$
14. maximal enclaveless	$(1, 3 : 3)$
15. vertex cover	$(1, 3 : 2)$
16. weakly connected {independent} dominating	$(1, 3 : 2)$
17. global offensive alliance	$(1, 3 : 2)$
18. global defensive alliance	$(1, 2 : 3)$
19. maximal $k \geq 1$*-dependent set*	$(1, 2 : 2)$
20. P_3 *dominating*	$(1, 2 : 2)$
21. maximal internally strong	$(1, 2 : 2)$
22. global powerful alliance	$(1, 2 : 2)$
23. total dominating	$(1, 2 : 1)$
24. connected dominating	$(1, 2 : 1)$
25. paired dominating and maximal matching	$(1, 2 : 1)$
26. maximal total/isolate-free matching	$(1, 2 : 1)$
27. maximal uniquely restricted matching	$(1, 2 : 1)$
28. total vertex cover	$(1, 2 : 1)$
29. 2-*dominating*	$(1, 1 : 2)$
30. maximal acyclic	$(1, 1 : 2)$
31. maximal bipartite	$(1, 1 : 2)$

In a sequel to this paper, we shall study the secondary and internal distances of a wide variety of maximal or minimal sets having the property that they are all distance-2 dominating sets [35]. These will include those in the following table.

Distance-2 Dominating Sets
1. maximal 2-packing
2. maximal open 2-packing
3. maximal 1.5-packing
4. maximal open 1.5-packing
5. perfect neighborhood set
6. perfect open neighborhood set
7. 1-maximal nearly perfect set
8. maximal irredundant set
9. maximal open irredundant set
10. maximal open-open irredundant
11. maximal closed-open irredundant set
12. minimal external redundant set
13. minimal pnc maximal set
14. maximal restrained {independent} set
15. maximal induced matching
16. maximal acyclic matching
17. maximal total matching
18. maximal disconnected matching
19. minimal R-annihilated set

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