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Independent domination by monochromatic paths in arc coloured bipartite tournaments

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Abstract

A digraph D is called an m-coloured digraph if its arcs are coloured with m colors. Let D be an m-coloured digraph. A subdigraph H of D is called *monochromatic* if all of its arcs are coloured alike and H will be called *quasi-monochromatic* if with at most one exception all of its arcs are coloured alike. A set $N \subseteq V(D)$ is said to be *dominating by monochromatic paths* whenever for every $w \in V(D) - N$ there exists a wN-monochromatic directed path; and N is said to be *independent by monochromatic paths* whenever for every pair of vertices $x, y \in N$ there is no monochromatic directed path between them. A *kernel by monochromatic paths* is a dominating independent by monochromatic paths set of vertices of D.

A well known result by Sands, Sauer and Woodrow proved in 1982 asserts that every 2-coloured digraph possesses a kernel by monochromatic paths, later in 1988 Shen Minggang proved that every m-coloured tournament satisfying that every subtournament of order 3 is quasi-monochromatic has a kernel by monochromatic paths, also he proved that the result is best possible for $m \geq 5$.

In this paper we proved an extension of the Shen Minggang's Theorem for bipartite tournaments. We prove that if D is an m-coloured bipartite tournament such that every cycle of length 4 and every transitive subtournament of order 4 is quasi-monochromatic then D has a kernel by monochromatic paths.

Keywords: kernel, kernel by monochromatic paths, bipartite tournament.

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1. Introduction

For general concepts we refer the reader to [1] and [2]. The topic of domination in graphs has been widely studied by several authors, a very complete study of this topic is presented in [13] and [14]. A special class of domination is the domination in digraphs, and it is defined as follows: Let D be a digraph, a set of vertices $S \subseteq V(D)$ is *dominating* whenever for every $w \in V(D) - S$ there exists a wS-arc in D. Dominating independent sets in digraphs (kernels in digraphs) have found many applications in different topics of Mathematics (see for example [4],[5],[7],[8] and [15]) and they have been studied by several authors, interesting surveys of kernels in digraphs can be found in [6] and [8]. Clearly the concepts of domination, independence and kernel by monochromatic paths in edge coloured digraphs are a generalization of those of domination, independence and kernel in digraphs. The study of the existence of kernels by monochromatic paths in edge-coloured digraphs starts with the Theorem of Sands, Sauer and Woodrow, proved in [21], which asserts that every 2-coloured digraph possesses a kernel by monochromatic paths; later we have the result of Shen Minggang [20] sentenced in the Abstract, in that work he also proved that his result is best possible for $m \geq 5$, in 2004 [11] the authors proved that the result is also best possible for $m \geq 4$; the question for $m = 3$ is still open: Does every 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic have a kernel by monochromatic paths? Sufficient conditions for the existence of kernels by monochromatic paths in edge coloured digraphs have been obtained mainly in nearly tournament digraphs (due to the difficulty of the problem) in several papers (see for example [9],[10],[11],[12], [20] and [21]). The concept of kernel by monochromatic paths is a generalization of the one of kernel, another interesting generalization is the concept of (k,l)-kernel introduced by Kwäsnik in [18]. Other results about (k,l)-kernels can be found in [16],[17], and [19]. In this paper we prove the following extension of the Shen Minggang's Theorem: If D is an arc coloured bipartite tournament such that every cycle of length 4 and every transitive subtournament of order 4 is quasi-monochromatic then D has a kernel by monochromatic paths.

2. Preliminaries

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. All the paths, cycles and walks considered in this paper will be directed paths, cycles or walks. If C is a walk we will denote its length by $l(C)$. Let S_1, S_2 be subsets of $V(D)$. The path $(u_0, u_1, \ldots, u_n)$ will be called S_1S_2-path whenever $u_0 \in S_1$ and $u_n \in S_2$. An arc (u_1, u_2) is *asymmetrical* (resp. *symmetrical*) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$), the asymmetrical part of D denoted by $\mathrm{Asym}(D)$ is the spanning subdigraph of D whose arcs are the asymmetrical arcs of D.

Let D be an m-coloured digraph, the closure of D denoted by $\mathcal{C}(D)$ is the digraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, and $(u,v) \in A(\mathcal{C}(D))$ if and only if there exists an uv-monochromatic path in D. Clearly N is a kernel by monochromatic paths of D if and only if N is a kernel of $\mathcal{C}(D)$.

A digraph D is said to be a *kernel-perfect digraph* whenever every one of its induced subdigraphs possesses a kernel.

The following result will be very useful to prove the main result of this paper:

Theorem 2.1. [3] *Let D be a digraph. If every cycle has a symmetrical arc then D is a kernel-perfect digraph.*

3. The Main Result

We start this section with some notation in order to make more efficient the proof of the main result.

Let D be a bipartite tournament. We will denote by C_4 the cycle of length 4 and by T_4 the transitive subtournament of order 4, that means $V(T_4) = \{u, v, w, x\}$, $A(T_4) = \{(u, v), (v, w), (w, x), (u, x)\}$.

Let D be a bipartite tournament such that every C_4 and every T_4 is quasi-monochromatic, $\{u, v, w, x\} \subseteq V(D)$. We will denote by $(\{u, v, w, x\}, c(f) = c(g) = i, c(h) = j, c([u, x]) = i)$ to mean that the set of vertices $\{u, v, w, x\}$ induces a C_4 or a T_4 (that is $\{(u, v), (v, w), (w, x)\} \subseteq A(D)$), $\{f, g, h\} \subseteq \{(u, v), (v, w), (w, x)\}$; f and g are arcs coloured i and h is an arc coloured j with $j \neq i$; thus the arc between u and x is coloured i. This argument is used repeatedly along the proof, so we introduce this notation to make more efficient the proof of the main result. So if $f = (u, v) \in A(D)$ $c(f) = i$ means that f is coloured i. Finally we will denote by $\vec{T}_8$ the 2-coloured bipartite tournament defined as follows: $V(\vec{T}_8) = \{s, t, u, v, w, x, y, z\}$, $A(\vec{T}_8) = \{(s, t), (s, x), (t, u), (t, y), (u, v), (u, z), (v, w), (v, s), (w, x), (w, t), (x, y), (x, u), (y, z), (y, v), (z, s), (z, w)\}$, the arcs $(s, t), (s, x), (t, u), (u, v), (u, z), (v, w), (x, u)$ and (z, w) are coloured 1 and each other arc in $\vec{T}_8$ is coloured 2, see Figure 1.

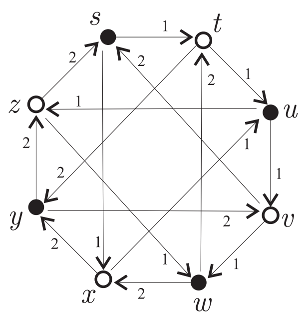

Figure 1: $\vec{T}_8$

Finally we will write m.p. instead of monochromatic path.

In this section we will prove that if D is an m-coloured bipartite tournament such that every C_4 and every T_4 is quasi-monochromatic and D contains no induced subdigraph isomorphic to $\vec{T}_8$ then each cycle of $\mathcal{C}(D)$, the closure of D, possesses a symmetrical arc and hence $\mathcal{C}(D)$ is kernel perfect. When D contains a subdigraph isomorphic to $\vec{T}_8$ we will prove that D is coloured with two colours and then D has a kernel by monochromatic paths.

The following easy observation will be used repeatedly along the proof without more explanations.

Observation 3.1. *Let D be a bipartite tournament.*

1. *If $C = (u_0, u_1, \ldots, u_n)$ is a walk in D then for each $i \in \{0, 1, \ldots, n\}$ and each $j \in \{0, 1, \ldots, n\}$, $|\{(u_i, u_j), (u_j, u_i)\} \cap A(D)| = 1$ if and only if $j - i \equiv 1 (\operatorname{mod} 2)$.*
2. *If every C_4 and every T_4 is quasi-monochromatic then every path of length 3 uses at most two colors.*

Lemma 3.2. *Let D be an m-coloured bipartite tournament such that every C_4 and every T_4 is quasi-monochromatic; and $u, v \in V(D)$ such that there exists a uv-m.p. and there is no vu-m.p. Then at least one of the following properties holds: (i) $(u, v) \in A(D)$, (ii) there exists a uv-path of length 2, (iii) there exists a uv-m.p. of length 4.*

Proof. Let be D, u, v as in the hypothesis and $T = (u = u_0, u_1, \ldots, u_n = v)$ a uv-m.p., say coloured 1. When n is odd it follows from Observation 3.1 that property (i) holds. When $n = 2$ or $n = 4$ clearly property (ii) (resp. (iii)) holds.

Now assume that n is even and $n \geq 6$.

When $(u_0, u_{n-1}) \in A(D)$ or $(u_1, u_n) \in A(D)$ we have that property (ii) holds. So we will assume that $(u_{n-1}, u_0) \in A(D)$ and $(u_n, u_1) \in A(D)$.

Now we will analyze the several possible cases and subcases.

Case (a). $(u_0, u_{n-3}) \in A(D)$.

In the case $C = (u = u_0, u_{n-3}, u_{n-2}, u_{n-1}, u_0)$ is a C_4, thus it is quasi-monochromatic. Since (u_{n-3}, u_{n-2}) and (u_{n-2}, u_{n-1}) are coloured 1 we have that (u_0, u_{n-3}) is coloured 1 or (u_{n-1}, u_0) is coloured 1. If (u_0, u_{n-3}) is coloured 1 then property (iii) holds. Thus we will assume that (u_{n-1}, u_0) is coloured 1.

Subcase (a.1). $(u_3, u_n) \in A(D)$.

In this subcase we have $C' = (u_1, u_2, u_3, u_n, u_1)$ is a C_4 and thus it is quasi-monochromatic. Since (u_1, u_2) and (u_2, u_3) are coloured 1 we have that (u_3, u_n) is coloured 1 or (u_n, u_1) is coloured 1. If (u_n, u_1) is coloured 1 then $(u_n, u_1) \cup (u_1, T, u_{n-1}) \cup (u_{n-1}, u_0)$ is a vu-m.p. (coloured 1), a contradiction. Thus (u_3, u_n) is coloured 1 and then $(u_0, u_1, u_2, u_3, u_n)$ is a uv-m.p. of length 4 and (iii) holds.

Subcase (a.2). $(u_n, u_3) \in A(D)$.

We have that $D[\{u_1, u_2, u_3, u_n\}]$ is isomorphic to T_4 and hence it is quasi-monochromatic. Since (u_1, u_2) and (u_2, u_3) are coloured 1 we have (u_n, u_3) is coloured 1 or (u_n, u_1) is coloured 1. In any case we get a vu-m.p., a contradiction.

Case (b). $(u_{n-3}, u_0) \in A(D)$.

Claim 1. (u_{n-1}, u_0) is coloured 1 or (u_{n-3}, u_0) is coloured 1.

Since $D[\{u_{n-3}, u_{n-2}, u_{n-1}, u_0\}] \cong T_4$ and, (u_{n-3}, u_{n-2}) and (u_{n-2}, u_{n-1}) are coloured 1, Claim 1 holds.

Subcase (b.1). $(u_3, u_n) \in A(D)$.

We have the cycle $(u_1, u_2, u_3, u_n, u_1)$ which is quasi-monochromatic (from the hypothesis). So (u_3, u_n) is coloured 1 or (u_n, u_1) is coloured 1. When (u_n, u_1) is coloured 1 we have from Claim 1 that $(u_n, u_1) \cup (u_1, T, u_{n-1}) \cup (u_{n-1}, u_0)$ is a vu-path coloured 1 or $(u_n, u_1) \cup (u_1, T, u_{n-3}) \cup (u_{n-3}, u_0)$ is a vu-path coloured 1, a contradiction. Thus (u_3, u_n) is coloured 1 and then $(u_0, u_1, u_2, u_3, u_n)$ is a uv-path coloured 1 and property (iii) holds.

Subcase (b.2). $(u_n, u_3) \in A(D)$.

Clearly $D[\{u_1, u_2, u_3, u_n\}] \cong T_4$ and thus it is quasi-monochromatic. It follows that (u_n, u_3) is coloured 1 or (u_n, u_1) is coloured 1. Now, from Claim 1 we have (u_{n-1}, u_0) is coloured 1 or (u_{n-3}, u_0) is coloured 1. In any case we get a vu-path coloured 1, a contradiction. □

Definition 3.1. *Let D be an m-coloured digraph. The pair $\widehat{\gamma} = (\gamma, \mathcal{P})$ will be called a γ-cycle whenever the following properties hold.*

1. *γ is a succesion of vertices of D $(u_0, u_1, \ldots, u_n = u_0)$ with $n \geq 1$.*
2. *For each $i \in \{1, 2, \ldots, n\}$ there is no $u_{i+1}u_i$-m.p.*
3. *$\mathcal{P}$ is a collection of paths $\mathcal{P} = \{P_i \,|\, i \in \{0, \ldots, n-1\}\}$ such that P_i is a u_iu_{i+1}-path with $l(P_i) \in \{1, 2, 4\}$.*
4. *If $l(P_i) = 2$, then $(u_i, u_{i+1}) \notin A(D)$, and*
5. *If $l(P_i) = 4$, then P_i is monochromatic and there is no u_iu_{i+1}-path of length lesser than 4.*

We will say that $\widehat{\gamma}$ has length n which will be denoted by $l(\widehat{\gamma}) = n$.

Theorem 3.3. *Let D be an m-coloured bipartite tournament such that, every C_4 is quasi-monochromatic, every T_4 is quasi-monochromatic and D has no induced subdigraph isomorphic to $\vec{T}_8$. Then D has no γ-cycle.*

Proof. We proceed by contradiction. Assume, for a contradiction that D has a γ-cycle and let $\widehat{\gamma} = (\gamma, \mathcal{P})$ such a cycle of minimum length. Let $\gamma = (u = u_0, u_1, \ldots, u_{n-1}, u_n = u_0)$ with $n \geq 1$; we have the following assertions.

1. For each $i \in \{0, 1, \ldots, n-1\}$ and each $j \geq 2$, if there is no $u_{i+j}u_i$-m.p. then there is no u_iu_{i+j}-m.p.

This follows directly from Lemma 3.2, the definition of γ-cycle and the choice of $\widehat{\gamma}$ as a γ-cycle of minimum length.

2. If $l(P_i) = 4$ for some $i \in \{0, 1, \ldots, n-1\}$ then there is no $u_{i+2}u_i$-m.p.

Assume without loss of generality that $P_i = (u_i = z_0, z_1, z_2, z_3, z_4 = u_{i+1})$ and it is coloured 1. Now, from the Observation 3.1 and the Definition 3.1 we have $(z_3, u_i = z_0) \in A(D)$ and $(z_4 = u_{i+1}, z_1) \in A(D)$.

Now we proceed by contradiction to prove 2. Suppose, for a contradiction that there exists a $u_{i+2}u_i$-m.p. and let $T = (u_{i+2} = x_0, x_1, \ldots, x_{k-1}, x_k = u_i)$ be such a path.

Notice that T and P_i are not coloured alike; otherwise $T \cup P_i$ would be a $u_{i+2}u_{i+1}$-m.p., a contradiction with the Definition 3.1. So we will assume that T is coloured 2.

Subcase 2.a. For some $j \in \{0, \ldots, k\}$ with $j \equiv k-1 (\mathrm{mod}\, 2)$ we have $(u_{i+1}, x_j) \in A(D)$. Let $j_0 = \max\{j \in \{0, \ldots, k\} \mid (u_{i+1}, x_j) \in A(D)\}$.

Claim 2. (u_{i+1}, x_{j_0}) is coloured 1. Recall that $(z_3, z_4 = u_{i+1}, x_{j_0}, x_{j_0+1})$ is a path of length 3, (z_3, z_4) is coloured 1 and (x_{j_0}, x_{j_0+1}) is coloured 2, thus (u_{i+1}, x_{j_0}) is coloured 1 or it is is coloured 2. When (u_{i+1}, x_{j_0}) is coloured 2 we have $(u_{i+1}, x_{j_0}) \cup (x_{j_0}, T, x_k = u_i)$ contains a $u_{i+1}u_i$-m.p., contradicting the Definition 3.1.

When $j_0 \leq k-2$, we have from the definition of j_0 that $(x_{j_0+2}, u_{i+1}) \in A(D)$ and then $(\{u_{i+1}, x_{j_0}, x_{j_0+1}, x_{j_0+2}\}, c(x_{j_0}, x_{j_0+1}) = c(x_{j_0+1}, x_{j_0+2}) = 2, c(u_{i+1}, x_{j_0}) = 1, c(x_{j_0+2}, u_{i+1}) = 2)$, which implies that $(u_{i+2}, T, x_{j_0+2}) \cup (x_{j_0+2}, u_{i+1})$ is coloured 2 and hence there exists a $u_{i+2}u_{i+1}$-m.p., a contradiction.

When $j_0 = k-1$, we have $(\{z_3, z_4, x_{j_0}, x_{j_0+1} = u_i\}, c(z_3, z_4) = c(u_{i+1}, x_{j_0}) = 1, c(x_{j_0}, x_{j_0+1}) = 2, c(z_3, u_i) = 1)$, and in the other hand we have $(\{z_4 = u_{i+1}, x_{j_0}, x_{j_0+1} = u_i = z_0, z_1\}, c(z_0, z_1) = c(u_{i+1}, x_{j_0}) = 1, c(x_{j_0}, x_{j_0+1}) = 2, c(z_4, z_1) = 1)$. Thus $(u_{i+1} = z_4, z_1, z_2, z_3, z_0 = u_i)$ is a $u_{i+1}u_i$-m.p. coloured 1, a contradiction. We conclude that the subcase 2.a is not possible.

Subcase 2.b. For each $j \in \{0, 1, \ldots, k\}$ with $j \equiv k-1 (\mathrm{mod}\, 2)$ we have $(x_j, u_{i+1}) \in A(D)$.

First notice that for each $j \in \{0, 1, \ldots, k\}$ the arc (x_j, u_{i+1}) is not coloured 2; otherwise for some $j \in \{0, 1, \ldots, k\}$ we have (x_j, u_{i+1}) is coloured 2 and hence $(u_{i+2} = x_0, T, x_j) \cup (x_j, u_{i+1})$ contains a $u_{i+2}u_{i+1}$-m.p., a contradiction. When $l(T) \geq 3$ we have that $D[\{x_{k-3}, x_{k-2}, x_{k-1}, u_{i+1}\}] \cong T_4$ and it has two arcs coloured 2 and two arcs not coloured 2 contradicting the hypothesis. Thus $l(T) \leq 2$. Moreover $l(T) = 2$ as $(u_{i+2}, u_{i+1}) \notin A(D)$.

We continue the proof of this case with several claims.

Claim 3. The arc between z_1 and u_{i+2} is coloured 2. We have $(\{u_{i+2} = x_0, x_1, x_2 = u_i = z_0, z_1\}, c(x_0, x_1) = c(x_1, x_2) = 2, c(z_0, z_1) = 1, c([z_1, u_{i+2}]) = 2)$.

Claim 4. The arc (z_4, z_1) is coloured 2. Clearly $(\{u_{i+2} = x_0, x_1, u_{i+1} = z_4, z_1\}, c(x_0, x_1) = c([z_1, u_{i+2}]) = 2, c(x_1, u_{i+1}) \neq 2, c(z_4, z_1) = 2)$.

Claim 5. $(u_{i+2}, z_1) \in A(D)$. Otherwise $(z_1, u_{i+2}) \in A(D)$ and $(u_{i+1} = z_4, z_1, u_{i+2}) \cup T$

contains a $u_{i+1}u_i$-m.p., a contradiction.

Claim 6. The arc between z_3 and u_{i+2} is coloured 1. We have $(\{u_{i+2}, z_1, z_2, z_3\}, c(z_1, z_2) = c(z_2, z_3) = 1, c(u_{i+2}, z_1) = 2, c([z_3, u_{i+2}]) = 1)$.

Claim 7. $(z_3, u_{i+2}) \in A(D)$. Otherwise $(u_{i+2}, z_3) \in A(D)$ and $(u_{i+2}, z_3, z_4 = u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction.

Claim 8. (z_3, u_i) is coloured 2. This follows from: $(\{z_3, u_{i+2} = x_0, x_1, x_2 = u_i\}, c(u_{i+2}, x_1) = c(x_1, x_2) = 2, c(z_3, u_{i+2}) = 1, c(z_3, u_i) = 2)$.

Claim 9. The arc between x_1 and z_2 is coloured 1. It is a consequence of $(\{x_1, x_2 = u_i = z_0, z_1, z_2\}, c(z_0, z_1) = c(z_1, z_2) = 1, c(x_1, z_0) = 2, c([x_1, z_2]) = 1)$.

Now, when $(x_1, z_2) \in A(D)$ we have that $D[\{x_1, z_2, z_3, z_0\}] \cong T_4$ and it is not quasi-monochromatic (as $c(x_1, z_2) = c(z_2, z_3) = 1$ and $c(z_3, u_i) = c(x_1, u_i) = 2$), a contradiction. When $(z_2, x_1) \in A(D)$ we have that $D[\{u_{i+2}, z_1, z_2, x_1\}] \cong T_4$ with $c(z_1, z_2) = c(z_2, x_1) = 1$ and $c(u_{i+2}, z_1) = c(u_{i+2}, x_1) = 2$. So it is not quasi-monochromatic, a contradiction. In any case we get a contradiction. Thus assertion 2 is proved.

3. If $l(P_i) = 4$ for some $i \in \{0, 1, \ldots, n-1\}$ then there is no u_iu_{i+2}-m.p. This follows directly from 1 and 2.

4. For each $i \in \{0, 1, \ldots, n-1\}$ we have $l(P_i) \neq 4$ i.e. $l(P_i) \in \{1, 2\}$. We proceed by contradiction. Assume, for a contradiction that there exists $i \in \{0, 1, \ldots, n-1\}$ such that $l(P_i) = 4$. From the Definition 3.1 P_i is monochromatic. We may assume $P_i = (u_i = z_0, z_1, z_2, z_3, z_4 = u_{i+1})$ and it is coloured 1. Also from the Definition 3.1 we have:

$$(z_3, u_i) \in A(D) \text{ and } (u_{i+1}, z_1) \in A(D) \qquad \text{... (4.1)}$$

Now we analyze the three possible cases: $l(P_{i+1}) = 1$, $l(P_{i+1}) = 2$ and $l(P_{i+1}) = 4$.

Case 4.a. $l(P_{i+1}) = 1$. Thus $(u_{i+1}, u_{i+2}) \in A(D)$ and from the assertion 3 (u_{i+1}, u_{i+2}) is not coloured 1, suppose it is coloured 2. Now $(\{z_2, z_3, z_4, u_{i+2}\}, c(z_2, z_3) = c(z_3, z_4) = 1, c(z_4, u_{i+2}) = 2, c([z_2, u_{i+2}]) = 1)$. When $(u_{i+2}, z_2) \in A(D)$ we have that $(u_{i+2}, z_2, z_3, z_4 = u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction. When $(z_2, u_{i+2}) \in A(D)$ we have that $(u_i = z_0, z_1, z_2, u_{i+2})$ is a u_iu_{i+2}-m.p., contradicting assertion 3.

Case 4.b. $l(P_{i+1}) = 2$. Let $P_{i+1} = (u_{i+1} = z_4, z_5, z_6 = u_{i+2})$. From the assertion 3 we have that P_{i+1} is not monochromatic coloured 1. Now we consider the two possible subcases.

Case 4.b.1. P_{i+1} is monochromatic. Assume without loss of generality that it is coloured 2.

Claim 10. The arc between z_3 and z_6 is coloured 2. Notice that $(\{z_3, z_4, z_5, z_6\}, c(z_4, z_5) = c(z_5, z_6) = 2, c(z_3, z_4) = 1, c([z_3, z_6]) = 2)$.

Claim 11. $(u_i = z_0, z_5) \in A(D)$. Assume, for a contradiction that $(z_0, z_5) \notin A(D)$, then $(z_5, z_0) \in A(D)$. Hence $D[\{z_3, z_4, z_5, z_0\}] \cong T_4$ and it is quasi-monochromatic. Since $c(z_3, z_4) = 1$ and $c(z_4, z_5) = 2$ we have that (z_3, z_0) and (z_5, z_0) are both coloured 1 or are both coloured 2. When (z_5, z_0) is coloured 2 we obtain $(u_{i+1} = z_4, z_5, z_0 = u_i)$ a $u_{i+1}u_i$-m.p., a contradiction. Hence $c(z_5, z_0) = c(z_3, z_0) = 1$. On the other hand we have $(\{z_4 = u_{i+1}, z_5, z_0, z_1\}, c(z_5, z_0) = c(z_0, z_1) = 1, c(z_4, z_5) = 2, c(z_4, z_1) = 1)$. So we obtain $(u_{i+1} = z_4, z_1, z_2, z_3, z_0 = u_i)$ a $u_{i+1}u_i$-m.p., a contradiction.

Now we consider the two possibilities: $(z_6, z_3) \in A(D)$ or $(z_3, z_6) \in A(D)$. When $(z_6, z_3) \in A(D)$ we have $(z_0 = u_i, z_5, z_6 = u_{i+2}, z_3, z_0)$ is a C_4 and thus it is quasi-monochromatic. Now $c(z_5, z_6) = c(z_6, z_3) = 2$ we have that $c(z_0, z_5) = 2$ or $c(z_3, z_0) = 2$. If $c(z_0, z_5) = 2$ then $(u_i = z_0, z_5, z_6 = u_{i+2})$ is a u_iu_{i+2}-m.p. contradicting assertion 3. If $c(z_3, z_0) = 2$ then $(u_{i+2} = z_6, z_3, z_0 = u_i)$ is a $u_{i+2}u_i$-m.p., contradicting assertion 2.

When $(z_3, z_6) \in A(D)$ we have $(\{z_1, z_2, z_3, z_6\}, c(z_1, z_2) = c(z_2, z_3) = 1, c(z_3, z_6) = 2, c[z_1, z_6] = 1)$. If $(z_1, z_6) \in A(D)$ then $(u_i = z_0, z_1, z_6 = u_{i+2})$ is a u_iu_{i+2}-m.p. coloured 1, contradicting assertion 3. If $(z_6, z_1) \in A(D)$ then $(u_{i+2} = z_6, z_1, z_2, z_3, z_4 = u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., contradicting the Definition 3.1.

Case 4.b.2. P_{i+1} is not monochromatic. Since $\{z_3, z_4, z_5, z_6\}$ induces a C_4 or a T_4 which from the hypothesis is quasi-monochromatic, $c(z_3, z_4) = 1$ and $P_{i+1} = (z_4, z_5, z_6)$ is not monochromatic, it follows that $c([z_3, z_6]) = 1$. If $(z_6, z_3) \in A(D)$ then $(u_{i+2} = z_6, z_3, z_4 = u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction. If $(z_3, z_6) \in A(D)$ then $(u_i = z_0, z_1, z_2, z_3, z_6 = u_{i+2})$ is a u_iu_{i+2}-m.p., contradicting assertion 3.

We conclude that Case 4.b is not possible.

Case 4.c. $l(P_{i+1}) = 4$. Let $P_{i+1} = (u_{i+1} = z_4, z_5, z_6, z_7, z_8 = u_{i+2})$. From the assertion 3 we have that P_{i+1} is not coloured 1. Assumme without loss of generality that P_{i+1} is not coloured 2.

Claim 12. $(z_0, z_5) \in A(D)$. Proceeding by contradiction suppose that $(z_5, z_0) \in A(D)$. Recall that from (4.1) we have that $(z_3, u_i = z_0) \in A(D)$ and $(z_4 = u_{i+1}, z_1) \in A(D)$. Thus $D[\{z_3, z_4, z_5, z_0\}] \cong D[\{z_4, z_5, z_0, z_1\}] \cong T_4$ and they are quasi-monochromatic. Since $c(z_3, z_4) = c(z_0, z_1) = 1$ and $c(z_4, z_5) = 2$ we have that $c(z_5, z_0) = c(z_3, z_0) = c(z_4, z_1)$ and they are coloured 1 or 2. If they are coloured 2 then $(u_{i+1} = z_4, z_5, z_0 = u_i)$ is a $u_{i+1}u_i$-m.p., a contradiction. If they are coloured 1 then $(u_{i+1} = z_4, z_1, z_2, z_3, z_0 = u_i)$ is a $u_{i+1}u_i$-m.p., again a contradiction.

Claim 13. (z_0, z_5) is not coloured 2. This follows directly from assertion 3.

Claim 14. The arc between z_0 and z_7 is coloured 2. We have $(\{z_0, z_5, z_6, z_7\}, c(z_5, z_6) = c(z_6, z_7) = 2, c(z_0, z_5) \neq 2, c([z_0, z_7]) = 2)$.

When $(z_0, z_7) \in A(D)$ we have $(u_i = z_0, z_7, z_8 = u_{i+2})$ a u_iu_{i+2}-m.p., contradicting assertion 3. When $(z_7, z_0) \in A(D)$ we obtain $(u_{i+1} = z_4, z_5, z_6, z_7, z_0 = u_i)$ a $u_{i+1}u_i$-m.p., again a contradiction. Thus Case 4c is not possible.

We conclude that $l(P_i) \in \{1, 2\}$ for each $i \in \{0, 1, \ldots, n-1\}$.

5. Let $i \in \{0, 1, \ldots, n-1\}$; if $l(P_i) = 1$ then there is no $u_{i+2}u_i$-m.p. of length at least 2. Assume, for a contradiction that there exists $i \in \{0, 1, \ldots, n-1\}$ such that $l(P_i) = 1$ and there is a $u_{i+2}u_i$-m.p. of length at least 2. Also we assume that

P_i is coloured 1 and $T = (u_{i+2} = x_0, x_1, \ldots, x_k = u_i)$ is a $u_{i+2}u_i$-m.p. From the Definition 3.1 we have that T is not coloured 1, say it is coloured 2. Now $(\{x_{k-2}, x_{k-1}, x_k = u_i, u_{i+1}\}, c(x_{k-2}, x_{k-1}) = c(x_{k-1}, x_k) = 2, c(u_i, u_{i+1}) = 1, c([x_{k-2}, u_{i+1}]) = 2)$. If $(u_{i+1}, x_{k-2}) \in A(D)$ then $(u_{i+1}, x_{k-2}, x_{k-1}, x_k = u_i)$ is a $u_{i+1}u_i$-m.p., a contradiction. If $(x_{k-2}, u_{i+1}) \in A(D)$ then $(u_{i+2}, T, x_{k-2}) \cup (x_{k-2}, u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., again a contradiction.

6. Let $i \in \{0, 1, \ldots, n-1\}$; if $l(P_{i+1}) = 1$ then there is no $u_{i+2}u_i$-m.p. of length at least 2. Proceeding by contradiction suppose that there exists $i \in \{0, 1, \ldots, n-1\}$ such that $l(P_{i+1}) = 1$, P_{i+1} is coloured 1 and $T = (u_{i+2} = x_0, x_1, \ldots, x_k = u_i)$ is a $u_{i+2}u_i$-m.p. coloured 2 (notice that T is not coloured 1 as there is no $u_{i+1}u_i$-m.p.). Now we have $(\{u_{i+1}, u_{i+2} = x_0, x_1, x_2\}, c(u_{i+2} = x_0, x_1) = c(x_1, x_2) = 2, c(u_{i+1}, u_{i+2}) = 1, c([u_{i+1}, x_2]) = 2)$. If $(u_{i+1}, x_2) \in A(D)$ then $(u_{i+1}, x_2) \cup (x_2, T, u_i)$ is a $u_{i+1}u_i$-m.p., a contradiction. If $(x_2, u_{i+1}) \in A(D)$ then $(u_{i+2} = x_0, x_1, x_2, u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction.

7. For each $i \in \{0, 1, \ldots, n-1\}$ there exists $j(i) \in \{i, i+1, i+2\}$ such that $l(P_{j(i)}) \neq 1$, (i.e. the paths P_i, P_{i+1}, P_{i+2} cannot have length 1 all of them at the same time). Proceeding by contradiction suppose that there exists $i \in \{0, 1, \ldots, n-1\}$ such that $l(P_i) = l(P_{i+1}) = l(P_{i+2}) = 1$.

Case 7.a. $c(u_i, u_{i+1}) \neq c(u_{i+1}, u_{i+2})$. Say $c(u_i, u_{i+1}) = 1$ and $c(u_{i+1}, u_{i+2}) = 2$. Notice that $\{u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$ induces a C_4 or a T_4 and since $c(u_i, u_{i+1}) \neq c(u_{i+1}, u_{i+2})$ then (u_{i+2}, u_{i+3}) and the arc between u_i and u_{i+3} are both coloured 1 or they are both coloured 2.

Claim 15. $(u_i, u_{i+3}) \in A(D)$. Otherwise $(u_{i+3}, u_i) \in A(D)$ and we have the following: When (u_{i+2}, u_{i+3}) and (u_{i+3}, u_i) are coloured 1 we get $(u_{i+2}, u_{i+3}, u_i, u_{i+1})$ a $u_{i+2}u_{i+1}$-m.p., a contradiction. When (u_{i+2}, u_{i+3}) and (u_{i+3}, u_i) are coloured 2 we obtain $(u_{i+1}, u_{i+2}, u_{i+3}, u_i)$ a $u_{i+1}u_i$-m.p., a contradiction.

Claim 16. For each $i \in \{0, 1, \ldots, n-1\}$, there exists a $u_{i+3}u_i$-m.p. Suppose, for a contradiction that there is some $i \in \{0, 1, \ldots, n-1\}$ such that there is no $u_{i+3}u_i$-m.p. Since (u_i, u_{i+3}) is a m.p., this assumption contradicts assertion 1.

Let $T = (u_{i+3} = x_0, x_1, \ldots, x_k = u_i)$ be a $u_{i+3}u_i$-m.p. We have that $D[\{x_{k-1}, x_k = u_i, u_{i+1}, u_{i+2}\}]$ is quasi-monochromatic (as it induces a T_4 or a C_4). Since $c(u_i, u_{i+1}) = 1$ and $c(u_{i+1}, u_{i+2}) = 2$ we have that T is coloured 1 or 2. Recall that (u_{i+2}, u_{i+3}) is coloured 1 or 2.

When (u_{i+2}, u_{i+3}) is coloured 1, we have that T is not coloured 1; otherwise $(u_{i+2}, u_{i+3}) \cup T \cup (u_i, u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction. Thus T is coloured 2. Now $(\{u_{i+2}, u_{i+3} = x_0, x_1, x_2\}, c(u_{i+3} = x_0, x_1) = c(x_1, x_2) = 2, c(u_{i+2}, u_{i+3}) = 1, c([u_{i+2}, x_2]) = 2)$. If $(u_{i+2}, x_2) \in A(D)$ then $(u_{i+1}, u_{i+2}, x_2) \cup (x_2, T, x_k)$ is a $u_{i+1}u_i$-m.p., a contradiction. If $(x_2, u_{i+2}) \in A(D)$ then $(u_{i+3} = x_0, x_1, x_2, u_{i+2})$ is a $u_{i+3}u_{i+2}$-m.p., a contradiction.

When (u_i, u_{i+3}) is coloured 2; we have that T is not coloured 2, otherwise $(u_{i+1}, u_{i+2}, u_{i+3}) \cup T$ is a $u_{i+1}u_i$-m.p., a contradiction. Thus T is coloured 1. Now

$(\{u_{i+2}, u_{i+3} = x_0, x_1, x_2\}, c(x_0, x_1) = c(x_1, x_2) = 1, c(u_{i+2}, u_{i+3}) = 2, c([u_{i+2}, x_2]) = 1)$. If $(u_{i+2}, x_2) \in A(D)$ then $(u_{i+2}, x_2) \cup (x_2, T, x_k = u_i) \cup (u_i, u_{i+1})$ is a u_{i+2}, u_{i+1}-m.p., a contradiction. If $(x_2, u_{i+2}) \in A(D)$ then $(u_{i+3} = x_0, x_1, x_2, u_{i+2})$ is a u_{i+3}, u_{i+2}-m.p., a contradiction.

Case 7.b. (u_i, u_{i+1}) and (u_{i+1}, u_{i+2}) are coloured alike. In this case (u_i, u_{i+1}, u_{i+2}) is a $u_i u_{i+2}$-m.p., contradicting assertion 1 (notice that from assertion 5, there is no $u_{i+2}u_i$-m.p. of length at least 2, on the other hand there is no $u_{i+2}u_i$-m.p. of length 1 as D is bipartite).

8. For each $i \in \{0, 1, \ldots, n-1\}$, $l(P_i) = 2$. Proceeding by contradiction suppose that there exists $i \in \{0, 1, \ldots, n-1\}$ such that $l(P_i) = 1$ (recall assertion 4) and suppose that $(u_i, u_{i+1}) = P_i$ is coloured 1.

Case 8.a. $l(P_{i+1}) = 1$. From 7 we have $l(P_{i+2}) \neq 1$ and from 4 we obtain $l(P_{i+2}) = 2$.

Claim 17. There is no $u_{i+2}u_i$-m.p. of length at least 2. Follows directly from 5.

Claim 18. (u_{i+1}, u_{i+2}) is not coloured 1. Otherwise (u_i, u_{i+1}, u_{i+2}) is a m.p., contradicting assertion 1 (recall Claim 17).

Let $c(u_{i+1}, u_{i+2}) = 2$, $P_{i+2} = (u_{i+2}, x, u_{i+3})$, $c(u_{i+2}, x) = a$ and $c(x, u_{i+3}) = b$.

Claim 19. There is no $u_{i+3}u_{i+1}$-m.p. of length at least 2. Follows directly from 5.

Claim 20. $(u_{i+3}, u_{i+1}) \in A(D)$. Otherwise it follows from Claim 19 that there is no $u_{i+3}u_{i+1}$-m.p. but (u_{i+1}, u_{i+3}) is a m.p., contradicting assertion 1.

Now we analyze the several possibilities for colors a and b.

When $a = b = 2$.

In this case we have $(\{u_i, u_{i+1}, u_{i+2}, x\}, c(u_{i+1}, u_{i+2}) = c(u_{i+2}, x) = 2, c(u_i, u_{i+1}) = 1, c([u_i, x]) = 2)$. If $(x, u_i) \in A(D)$ then $(u_{i+1}, u_{i+2}, x, u_i)$ is a $u_{i+1}u_i$-m.p., a contradiction. Thus $(u_i, x) \in A(D)$ and then $(\{u_i, x, u_{i+3}, u_{i+1}\}, c(u_i, x) = c(x, u_{i+3}) = 2, c(u_i, u_{i+1}) = 1, c(u_{i+3}, u_{i+1}) = 2)$. Hence $(u_{i+3}, u_{i+1}, u_{i+2})$ is a $u_{i+3}u_{i+2}$-m.p., a contradiction.

When $a = b$ and $a \neq 2$.

In this case we have $(\{u_{i+1}, u_{i+2}, x, u_{i+3}\}, c(u_{i+2}, x) = c(x, u_{i+3}) = a$, $c(u_{i+1}, u_{i+2}) = 2$, $c(u_{i+3}, u_{i+1}) = a)$; and hence $(u_{i+2}, x, u_{i+3}, u_{i+1})$ is a $u_{i+2}u_{i+1}$-m.p., a contradiction.

When $a \neq b$.

Since $(u_i, u_{i+1}, u_{i+2}, u_{i+3})$ is a path of length 3, $c(u_i, u_{i+1}) = 1$ and $c(u_{i+1}, u_{i+2}) = 2$ we have $a \in \{1, 2\}$. If $a = 1$ then $b = 2$ (arguing as above). Now $(\{u_{i+1}, u_{i+2}, x, u_{i+3}\}, c(u_{i+1}, u_{i+2}) = c(x, u_{i+3}) = 2, c(u_{i+2}, x) = 1, c(u_{i+3}, u_{i+1}) = 2)$ thus $(u_{i+3}, u_{i+1}, u_{i+2})$ is a $u_{i+3}u_{i+2}$-m.p., a contradiction. If $a = 2$ then we have $(\{u_{i+1}, u_{i+2}, x, u_{i+3}\}, c(u_{i+1}, u_{i+2}) = c(u_{i+2}, x) = 2, c(x, u_{i+3}) = b, c(u_{i+3}, u_{i+1}) = 2)$; hence $(u_{i+3}, u_{i+1}, u_{i+2})$ is a m.p., a contradiction.

Case 8.b. $l(P_{i+1}) = 2$. Let $P_{i+1} = (u_{i+1}, x, u_{i+2})$.

Claim 21. $(u_{i+2}, u_i) \in A(D)$. Suppose, for a contradiction that $(u_{i+2}, u_i) \notin A(D)$ then from assertion 5 we have that there is no $u_{i+2}u_i$-m.p. From assertion 1 there is no u_iu_{i+2}-m.p., a contradiction (as $(u_i, u_{i+2}) \in A(D)$).

Claim 22. $c(u_{i+1}, x) = c(x, u_{i+2}) = 1$. Let $a = c(u_{i+1}, x)$ and $b = c(x, u_{i+2})$. If $a \neq b$ then (u_{i+2}, u_i, u_{i+1}) is a m.p., a contradiction (notice that $C_4 = (u_i, u_{i+1}, x, u_{i+2}, u_i)$ is quasi-monochromatic). Thus $a = b$. If $a \neq 1$ then $c(u_{i+2}, u_i) = a$ (recall $c(u_i, u_{i+1}) = 1$), and it follows $(u_{i+1}, x, u_{i+2}, u_i)$ is a m.p., a contradiction.

Claim 23. $c(u_{i+2}, u_i) \neq 1$. Otherwise $(u_{i+1}, x, u_{i+2}, u_i)$ is a m.p., a contradiction.

Claim 24. $l(P_{i-1}) = 2$. From Case 8.a we have that $l(P_{i-1}) \neq 1$, thus from 4 $l(P_{i-1}) = 2$. Let $P_{i-1} = (u_{i-1}, y, u_i)$.

Claim 25. $(u_{i+1}, u_{i-1}) \in A(D)$. Otherwise it follows from assertion 6 that there is no $u_{i+1}u_{i-1}$-m.p. Moreover from assertion 1 there is no $u_{i-1}u_{i+1}$-m.p. contradicting that $(u_{i-1}, u_{i+1}) \in A(D)$.

Claim 26. $c(u_{i-1}, y) = c(y, u_i) = 1$. Consider the cycle of length 4 $C' = (u_{i-1}, y, u_i, u_{i+1}, u_{i-1})$ and let $a = c(u_{i-1}, y)$, $b = c(y, u_i)$. If $a \neq b$ then $c(u_i, u_{i+1}) = c(u_{i+1}, u_{i-1})$ (as C' is quasi-monochromatic). Thus (u_i, u_{i+1}, u_{i-1}) is a m.p., a contradiction. Hence $a = b$. When $a \neq 1$ we have $c(u_{i+1}, u_{i-1}) = a$ (as C' is quasi-monochromatic), and then $(u_{i+1}, u_{i-1}, y, u_i)$ is a m.p., a contradiction. Hence $a = 1$.

Claim 27. $c(u_{i+1}, u_{i-1}) = 2$. We have $\{u_{i+2}, u_i, u_{i+1}, u_{i-1}\}$ induces a T_4 or a C_4 (and thus it is quasi-monochromatic). Since $c(u_i, u_{i+1}) = 1$ and $c(u_{i+2}, u_i) = 2$ it follows that $c(u_{i+1}, u_{i-1})$ is 1 or 2. If $c(u_{i+1}, u_{i-1}) = 1$ then (u_i, u_{i+1}, u_{i-1}) is a m.p., a contradiction. Hence $c(u_{i+1}, u_{i-1}) = 2$.

Claim 28. The arc between u_{i-1} and u_{i+2} is coloured 2. Clearly we have $(\{u_{i+2}, u_i, u_{i+1}, u_{i-1}\}, c(u_{i+2}, u_i) = c(u_{i+1}, u_{i-1}) = 2, c(u_i, u_{i+1}) = 1, c([u_{i-1}, u_{i+2}]) = 2)$.

We conclude the proof of the Case 8.b as follows: If $(u_{i-1}, u_{i+2}) \in A(D)$ then $(u_{i+1}, u_{i-1}, u_{i+2}, u_i)$ is a m.p., a contradiction. If $(u_{i+2}, u_{i-1}) \in A(D)$ then $D[\{u_{i+2}, u_{i-1}, y, u_i\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction. Thus this case is not possible and assertion 8 holds.

9. Let $i \in \{0, 1, \ldots, n-1\}$, if P_i is 2-coloured then there is no $u_{i+2}u_i$-m.p. Proceeding by contradiction suppose that there exists $i \in \{0, 1, \ldots, n-1\}$ such that P_i is 2-coloured and there exists a $u_{i+2}u_i$-m.p. Let $T = (u_{i+2} = x_0, x_1, \ldots, x_k = u_i)$ be such a path, a the colour of T, $P_i = (u_i, z_i, u_{i+1})$, $c(u_i, z_i) = 1$ and $c(z_i, u_{i+1}) = 2$. Now, $D[\{x_{k-1}, x_k = u_i, z_i, u_{i+1}\}]$ is quasi-monochromatic. Since $c(u_i, z_i) = 1$ and $c(z_i, u_{i+1}) = 2$ then $c([u_{i+1}, x_{k-1}]) = c(x_{k-1}, x_k) = a$. If $(u_{i+1}, x_{k-1}) \in A(D)$ then $(u_{i+1}, x_{k-1}, x_k = u_i)$ is a m.p., a contradiction. If $(x_{k-1}, u_{i+1}) \in A(D)$ then $(u_{i+2} = x_0, T, x_{k-1}) \cup (x_{k-1}, u_{i+1})$ is a m.p., a contradiction.

10. For each $i \in \{0, 1, \ldots, n-1\}$, if P_i is 2-coloured then there is no u_iu_{i+2}-m.p. Let $i \in \{0, 1, \ldots, n-1\}$ be such that P_i is 2-coloured. It follows from assertion 9 that there is no $u_{i+2}u_i$-m.p. Thus from assertion 1 there is no u_iu_{i+2}-m.p.

11. There exists $i \in \{0, 1, \ldots, n-1\}$ such that P_i is monochromatic. Proceeding by

contradiction suppose that for each $i \in \{0, 1, \ldots, n-1\}$ P_i is 2-coloured. Let $P_i = (u_i, z_i, u_{i+1})$ with $c(u_i, z_i) = a_i$, $c(z_i, u_{i+1}) = b_i$ and $a_i \neq b_i$.

Claim 29. For each $i \in \{2, \ldots, n-1\}$ the following two assertions hold: (i) The arc between z_0 and u_i is coloured b_0. (ii) $(z_0, u_i) \in A(D)$.

To proof Claim 29 we proceed by induction on i.

First let $i = 2$. Since $D[\{z_0, u_1, z_1, u_2\}]$ is quasi-monochromatic, $c(u_1, z_1) \neq c(z_1, u_2)$ we have $c(z_0, u_1) = c([z_0, u_2]) = b_0$. Now to prove (ii) assume, for a contradiction that $(u_2, z_0) \in A(D)$; then (u_2, z_0, u_1) is a m.p., a contradiction.

Now suppose that for each i, with $2 \leq i \leq n-1$ assertions (i) and (ii) hold.

Consider $i+1$ we will prove assertions (i) and (ii) for $i+1$. Since $D[\{z_0, u_i, z_i, u_{i+1}\}]$ is quasi-monochromatic and $c(u_i, z_i) \neq c(z_i, u_{i+1})$ we have $c(z_0, u_i) = c([z_0, u_{i+1}]) = b_0$. Now assume, for a contradiction that $(u_{i+1}, z_0) \in A(D)$; then (u_{i+1}, z_0, u_i) is a $u_{i+1}u_i$-m.p., a contradiction. Thus Claim 29 is proved.

Clearly it follows from Claim 29 and the contradiction hypothesis that $(u_0, z_0, u_{n-1}, z_{n-1}, u_n = u_0)$ is a non quasi-monochromatic cycle, a contradiction.

12. For each $i \in \{0, 1, \ldots, n-1\}$, P_i is monochromatic. Proceeding by contradiction suppose that there exists some $i \in \{0, 1, \ldots, n-1\}$ such that $P_i = (u_i, z_i, u_{i+1})$ is not monochromatic. Let $c(u_i, z_i) = 1$, $c(z_i, u_{i+1}) = 2$ and $c(u_{i+1}, z_{i+1}) = a$.

Claim 30. The arc between u_i and z_{i+1} is coloured a. We have $D[\{u_i, z_i, u_{i+1}, z_{i+1}\}]$ is quasi-monochromatic. Since $c(u_i, z_i) \neq c(z_i, u_{i+1})$ it follows that $c(u_{i+1}, z_{i+1}) = c([u_i, z_{i+1}])$. Thus $c([u_i, z_{i+1}]) = a$.

Claim 31. $(u_i, z_{i+1}) \in A(D)$. Otherwise $(z_{i+1}, u_i) \in A(D)$ and (u_{i+1}, z_{i+1}, u_i) is a m.p., a contradiction.

Claim 32. (z_{i+1}, u_{i+2}) is not coloured a. Assume, for a contradiction that $c(z_{i+1}, u_{i+2}) = a$; then (u_i, z_{i+1}, u_{i+2}) is a m.p., contradicting assertion 10.

13. For each $i \in \{0, 1, \ldots, n-1\}$ there is no $u_{i+2}u_i$-m.p. We proceed by contradiction, suppose that there exists $i \in \{0, 1, \ldots, n-1\}$ such that there is a $u_{i+2}u_i$-m.p. Let $T = (u_{i+2}, x_0, x_1, \ldots, x_k = u_i)$ a $u_{i+2}u_i$-m.p. Assume that P_i is coloured 1 and P_{i+1} is coloured a.

Claim 33. T is not coloured 1. Otherwise $T \cup P_i$ is a $u_{i+2}u_i$-m.p., a contradiction.

Let $c(T) = 2$.

Claim 34. $a \neq 2$. Otherwise $P_{i+1} \cup T$ is a $u_{i+1}u_i$-m.p., a contradiction.

Now we consider the four possible cases:

Case 13.a. $l(T) \geq 4$ and for some $j \in \{1, \ldots, k-3\}$, $(u_{i+1}, x_j) \in A(D)$.

Claim 35. $c(u_{i+1}, x_j) \neq 2$. Otherwise $(u_{i+1}, x_j) \cup (x_j, T, u_i)$ is a m.p., a contradiction.

Claim 36. The arc between x_{j+2} and u_{i+1} is coloured 2. Clearly we have $(\{u_{i+1}, x_j, x_{j+1}, x_{j+2}\}, c(x_j, x_{j+1}) = c(x_{j+1}, x_{j+2}) = 2, c(u_{i+1}, x_j) \neq 2, c([x_{j+2}, u_{i+1}]) = 2)$. When

$(x_{j+2}, u_{i+1}) \in A(D)$ we have $(u_{i+2}, T, x_{j+2}) \cup (x_{j+2}, u_{i+1})$ a m.p., a contradiction. When $(u_{i+1}, x_{j+2}) \in A(D)$ we get $(u_{i+1}, x_{j+2}) \cup (x_{j+2}, T, u_i)$ a m.p., a contradiction again. Thus Case 13.a is not possible.

Case 13.b. $l(T) \geq 6$ and for each $j \in \{1, \ldots, k-3 \mid j \equiv 1(\bmod 2)\}$, $(x_j, u_{i+1}) \in A(D)$.

Claim 37. $c(x_{k-5}, u_{i+1}) \neq 2$. Otherwise $(u_{i+2}, T, x_{k-5}) \cup (x_{k-5}, u_{i+1})$ is a m.p., a contradiction.

Claim 38. $c(x_{k-3}, u_{i+1}) = 2$. We have $(\{x_{k-5}, x_{k-4}, x_{k-3}, u_{i+1}\}, c(x_{k-5}, x_{k-4}) = c(x_{k-4}, x_{k-3}) = 2, c(x_{k-5}, u_{i+1}) \neq 2, c([x_{k-3}, u_{i+1}]) = 2)$. Thus we have $(u_{i+2}, T, x_{k-3}) \cup (x_{k-3}, u_{i+1})$ a m.p., a contradiction. Thus this case is impossible.

Case 13.c. $l(T) = 4$ and $(x_1, u_{i+1}) \in A(D)$.

Claim 39. (x_1, u_{i+1}) is coloured a. We have $(\{u_{i+1}, z_{i+1}, u_{i+2}, x_1\}, c(u_{i+1}, z_{i+1}) = c(z_{i+1}, u_{i+2}) = a, c(u_{i+2}, x_1) = 2, c(x_1, u_{i+1}) = a)$.

Claim 40. The arc between x_2 and z_{i+1} is coloured 2. We have $(\{z_{i+1}, u_{i+2}, x_1, x_2\}, c(u_{i+2}, x_1) = c(x_1, x_2) = 2, c(z_{i+1}, u_{i+2}) = a, c([x_2, z_{i+1}]) = 2)$.

Claim 41. The arc between x_3 and u_{i+1} is coloured 1. Clearly $(\{x_3, u_i, z_i, u_{i+1}\}, c(u_i, z_i) = c(z_i, u_{i+1}) = 1, c(x_3, u_i) = 2, c([x_3, u_{i+1}]) = 1)$.

Claim 42. The arc between z_i and x_2 is coloured 2. It follows from $(\{x_2, x_3, u_i, z_i\}, c(x_2, x_3) = c(x_3, u_i) = 2, c(u_i, z_i) = 1, c([z_i, x_2]) = 2)$.

Claim 43. $(x_2, z_{i+1}) \in A(D)$. Otherwise $(z_{i+1}, x_2) \in A(D)$ and then $D[\{x_1, u_{i+1}, z_{i+1}, x_2\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction (recall $a \neq 2$, $c(x_1, u_{i+1}) = c(u_{i+1}, z_{i+1}) = a$ and $c(x_1, x_2) = c(z_{i+1}, x_2) = a$).

Claim 44. $(u_{i+1}, x_3) \in A(D)$. Otherwise $(x_3, u_{i+1}) \in A(D)$, thus $D[\{x_1, x_2, x_3, u_{i+1}\}] \cong T_4$ with $c(x_1, x_2) = c(x_2, x_3) = 2$ and $c(x_1, u_{i+1}) \neq 2$ and $c(x_3, u_{i+1}) \neq 2$, a contradiction.

Claim 45. $(z_i, x_2) \in A(D)$. Otherwise $(x_2, z_i) \in A(D)$, and then $D[\{x_2, z_i, u_{i+1}, x_3\}]$ is not quasi-monochromatic, a contradiction (recall $c(x_2, z_i) = c(x_2, x_3) = 2$ and $c(z_i, u_{i+1}) = c(u_{i+1}, x_3) = 1$.

Claim 46. The arc between z_i and u_{i+2} is coloured 2. We have $(\{z_i, x_2, z_{i+1}, u_{i+2}\}, c(z_i, x_2) = c(x_2, z_{i+1}) = 2, c(z_{i+1}, u_{i+2}) = a, c([z_i, u_{i+2}]) = 2)$.

Claim 47. $a = 1$. Since $D[\{z_i, u_{i+1}, z_{i+1}, u_{i+2}\}]$ is quasi-monochromatic, $c(z_i, u_{i+1}) = 1$ and the arc between z_i and u_{i+2} is coloured 2, it follows that $a \in \{1, 2\}$; and we have $a \neq 2$.

Claim 48. $(u_{i+2}, z_i) \in A(D)$. Otherwise $(z_i, u_{i+2}) \in A(D)$ and $D[\{z_i, u_{i+2}, x_1, u_{i+1}\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction (recall $c(z_i, u_{i+2}) = c(u_{i+2}, x_1) = 2$ and $c(x_1, u_{i+1}) = c(z_i, u_{i+1}) = 1$).

Claim 49. The arc between u_i and z_{i+1} is coloured 2. Clearly $(\{u_i, z_i, x_2, z_{i+1}\}, c(z_i, x_2) = c(x_2, z_{i+1}) = 2, c(u_i, z_i) = 1, c([u_i, z_{i+1}]) = 2)$.

Claim 50. $(z_{i+1}, u_i) \in A(D)$. Otherwise $(u_i, z_{i+1}) \in A(D)$. Thus $D[\{u_i, z_{i+1}, u_{i+2}, z_i\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction (recall $c(u_i, z_{i+1}) = c(u_{i+2}, z_i) = 2$ and $c(z_{i+1}, u_{i+2}) = c(u_i, z_i) = 1$).

Claim 51. The arc between u_i and x_1 is coloured 1. We have $(\{x_1, u_{i+1}, x_3, u_i\}, c(x_1, u_{i+1}) = c(u_{i+1}, x_3) = 1, c(x_3, u_i) = 2, c([u_i, x_1]) = 1)$.

Claim 52. $(u_i, x_1) \in A(D)$. Otherwise $(x_1, u_i) \in A(D)$ and then $D[\{z_{i+1}, u_{i+2}, x_1, u_i\}] \cong T_4$ and it is not quasi-monochromatic (recall $c(z_{i+1}, u_{i+2}) = c(x_1, u_i) = 1$ and $c(u_{i+2}, x_1) = c(z_{i+1}, u_i) = 2$), a contradiction.

Claim 53. The arc between u_{i+2} and x_3 is coloured 1. We have $(\{u_{i+2}, z_i, u_{i+1}, x_3\}, c(z_i, u_{i+1}) = c(u_{i+1}, x_3) = 1, c(u_{i+2}, z_i) = 2, c([u_{i+2}, x_3]) = 1)$.

Claim 54. $(x_3, u_{i+2}) \in A(D)$. Otherwise $(u_{i+2}, x_3) \in A(D)$ and then $D[\{z_{i+1}, u_{i+2}, x_3, u_i\}] \cong T_4$ and it is not quasi-monochromatic ($c(z_{i+1}, u_{i+2}) = c(u_{i+2}, x_3) = 1$ and $c(x_3, u_i) = c(z_{i+1}, u_i) = 2$), a contradiction.

We conclude the Case 13.c by considering $H = D[\{u_i, z_i, u_{i+1}, z_{i+1}, u_{i+2}, x_1, x_2, x_3\}]$.

From the previous Claims we have $H \cong \vec{T}_8$, contradicting the hypothesis of Theorem 3.3. Thus Case 13.c is not possible. (See fig. 2.)

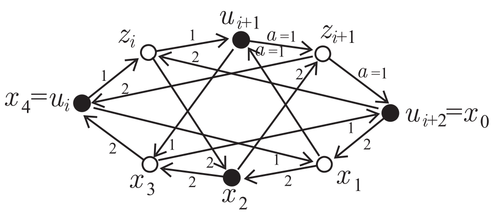

Figure 2:

Case 13.d. $l(T) = 2$.

Claim 55. The arc between u_{i+1} and x_1 is coloured 1. We have $(\{x_1, u_i, z_i, u_{i+1}\}, c(u_i, z_i) = c(z_i, u_{i+1}) = 1, c(x_1, u_i) = 2, c([u_{i+1}, x_1]) = 1)$.

Claim 56. $a = 1$. Clearly $D[\{u_{i+1}, z_{i+1}, u_{i+2}, x_1\}]$ is quasi-monochromatic. Since $c(u_{i+2}, x_1) = 2$ and $c([u_{i+1}, x_1)] = 1$ we have that $a \in \{1, 2\}$. Recalling that $a \neq 2$ we conclude $a = 1$.

Claim 57. $c([u_i, z_{i+1}]) = 2$ (The arc between u_i and z_{i+1} is coloured 2). It follows from: $(\{z_{i+1}, u_{i+2}, x_1, u_i\}, c(u_{i+2}, x_1) = c(x_1, u_i) = 2, c(z_{i+1}, u_{i+2}) = 1, c([u_i, z_{i+1}]) = 2)$.

Claim 58. $c([z_i, u_{i+2}]) = 2$. This follows from: $(\{u_{i+2}, x_1, u_i, z_i\}, c(u_{i+2}, x_1) = c(x_1, u_i) = 2, c(u_i, z_i) = 1, c([z_i, u_{i+2}]) = 2)$.

Claim 59. $(u_{i+1}, x_1) \in A(D)$. Proceeding by contradiction suppose $(x_1, u_{i+1}) \in A(D)$. First notice that $(u_i, z_{i+1}) \in A(D)$; otherwise $D[\{x_1, u_{i+1}, z_{i+1}, u_i\}] \cong T_4$ and it is not quasi-monochromatic (as $c(x_1, u_{i+1}) = c(u_{i+1}, z_{i+1}) = 1$ and $c(x_1, u_i) =$

$c(z_{i+1}, u_i) = 2)$, a contradiction. Also we have $z_i, u_{i+2} \in A(D)$; otherwise $D[\{u_i, z_{i+1}, u_{i+2}, z_i\}] \cong T_4$ and it is not quasi-monochromatic (because $c(u_i, z_{i+1}) = c(u_{i+2}, z_i) = 2$ and $c(z_{i+1}, u_{i+2}) = c(u_i, z_i) = 1)$, a contradiction. Hence $D[\{z_i, u_{i+2}, x_1, u_{i+1}\}] \cong T_4$ and it is not quasi-monochromatic (because $c(z_i, u_{i+2}) = c(u_{i+2}, x_1) = 2$ and $c(x_1, u_{i+1}) = c(z_i, u_{i+1}) = 1)$, a contradiction.

Claim 60. $(z_{i+1}, u_i) \in A(D)$. Otherwise $(u_i, z_{i+1}) \in A(D)$ and $D[\{u_{i+1}, x_1, u_i, z_{i+1}\}] \cong T_4$ which is not quasi-monochromatic (as $c(u_{i+1}, x_1) = c(u_{i+1}, z_{i+1}) = 1$ and $c(u_i, z_{i+1}) = c(x_1, u_i) = 2)$, a contradiction.

Claim 61. $(u_{i+2}, z_i) \in A(D)$. Otherwise $(z_i, u_{i+2}) \in A(D)$ and then $D[\{z_{i+1}, u_i, z_i, u_{i+2}\}] \cong T_4$ and it is not quasi-monochromatic (because $c(u_i, z_i) = c(z_{i+1}, u_{i+2}) = 1$ and $c(z_{i+1}, u_i) = c(z_i, u_{i+2}) = 2)$, a contradiction.

We conclude the Case 13.d by observing that $D[\{u_{i+2}, z_i, u_{i+1}, x_1\}] \cong T_4$ and it is not quasi-monochromatic (because $c(z_i, u_{i+1}) = c(u_{i+1}, x_1) = 1$ and $c(u_{i+2}, z_i) = c(u_{i+2}, x_1) = 2)$, a contradiction. Thus Case 13.d is not possible.

14. For each $i \in \{0, 1, \ldots, n-1\}$ there is no $u_i u_{i+2}$-m.p. Follows directly from 1 and 13.

15. For each $i \in \{0, 1, \ldots, n-1\}$, P_i and P_{i+1} have different colour. Follows from 14.

16. $n \geq 3$. If $n = 2$ then $P_0 \cup P_1$ is a C_4 and it is not quasi-monochromatic, a contradiction.

17. $n \neq 4$. Assume, for a contradiction that $n \leq 3$; from 16 $n = 3$. Let 1 be the colour of P_0 and 2 the colour of P_1. From 15 the colour of P_2 is different to 1 and 2; let 3 be the colour of P_2. Now $c([z_2, u_1]) = 1$ because $(\{z_2, u_0, z_0, u_1\}, c(u_0, z_0) = c(z_0, u_1) = 1, c(z_2, u_0) = 3, c([z_2, u_1]) = 1)$. Also $c([z_2, u_1]) = 2$ as $(\{u_1, z_1, u_2, z_2\}, c(u_1, z_1) = c(z_1, u_2) = 2, c(u_2, z_2) = 3, c([z_2, u_1]) = 2)$. Thus $c([z_2, u_1]) = 1$ and $c([z_2, u_1]) = 2$, a contradiction.

18. For each $i \in \{0, 1, \ldots, n-1\}$, $(z_{i+1}, u_i) \in A(D)$. Proceeding by contradiction suppose that $(u_i, z_{i+1}) \in A(D)$. Let $Q = (u_i, z_{i+1}, u_{i+2})$, $\gamma' = (u_0, u_1, \ldots, u_{i-1}, u_i, u_{i+2}, u_{i+3}, \ldots, u_n = u_0)$ (γ' is obtained from γ by deleting u_{i+1}) and $P' = (P_0, P_1, \ldots, P_{i-1}, Q, P_{i+2}, \ldots, P_{n-1})$ (P' is obtained from P by deleting P_i and P_{i+1} and adding Q). Clearly (γ', P') is a γ-cycle whose length is less than n, a contradiction.

19. For each $i \in \{0, 1, \ldots, n-1\}$, $(u_{i+2}, z_i) \in A(D)$. Proceeding by contradiction suppose that $(z_i, u_{i+2}) \in A(D)$. Let $Q = (u_i, z_i, u_{i+2})$, $\gamma' = (u_0, u_1, \ldots, u_i, u_{i+2}, \ldots, u_n = u_0)$ and $P' = (P_0, P_1, \ldots, P_{i-1}, Q, P_{i+2}, \ldots, P_{n-1})$. Clearly (γ', P') is a γ-cycle whose length is less than n, a contradiction.

We may assume (assertion 15) that P_0 is coloured 1, P_1 is coloured 2 and P_{n-1} is coloured a with $a \neq 1$.

20. $c(z_0, u_{n-1}) = a$. From assertion 18 on $i = n-1$ we have $(z_0, u_{n-1}) \in A(D)$. Thus $(\{u_0, z_0, u_{n-1}, z_{n-1}\}, c(u_{n-1}, z_{n-1}) = c(z_{n-1}, u_0) = a, c(u_0, z_0) = 1, c(z_0, u_{n-1}) = a)$.

21. $c(u_2, z_0) = 2$. From assertion 19 on $i = 0$ we have $(u_2, z_0) \in A(D)$. Hence $(\{u_2, z_0, u_1, z_1\}, c(u_1, z_1) = c(z_1, u_2) = 2, c(z_0, u_1) = 1, c(u_2, z_0) = 2)$.

22. $c(z_1, u_0) = 1$. From assertion 18 on $i = 0$ we have $(z_1, u_0) \in A(D)$. Then we have $(\{u_0, z_0, u_1, z_1\}, c(u_0, z_0) = c(z_0, u_1) = 1, c(u_1, z_1) = 2, c(z_1, u_0) = 1)$.

23. $a \neq 2$. Otherwise $(u_2, z_0, u_{n-1}, z_{n-1}, u_0)$ is a m.p. contradicting assertion 13.

24. $c([u_2, z_{n-1}]) = a$. We have $(\{u_2, z_0, u_{n-1}, z_{n-1}\}, c(z_0, u_{n-1}) = c(u_{n-1}, z_{n-1}) = a, c(u_0, z_0) = 2, c(u_2, z_{n-1}) = a)$.

Finally we conclude the proof of the Theorem 3.3 as follows:

When $(u_2, z_{n-1}) \in A(D)$ we have $D[\{z_1, u_2, z_{n-1}, u_0\}]$ is not quasi-monochromatic (because $c(z_1, u_0) = 1$, $c(z_1, u_2) = 2$ and $c(u_2, z_{n-1}) = a$ with $a \neq 1$ and $a \neq 2$), a contradiction.

When $(z_{n-1}, u_2) \in A(D)$ we obtain $D[\{z_{n-1}, u_2, z_0, u_1\}]$ is not quasi-monochromatic (as $c(z_{n-1}, u_2) = a$, $c(u_2, z_0) = 2$ and $c(z_0, u_1) = 1$), again a contradiction. Thus D has no γ-cycle. □

Theorem 3.4. *Let D be an m-coloured bipartite tournament such that every C_4 and every T_4 is quasi-monochromatic. If D has an induced subdigraph isomorphic to $\vec{T}_8$ then $m = 2$.*

Proof. Let H be a 2-coloured subdigraph of D isomorphic to $\vec{T}_8$. Suppose that H is as in Figure 1. Let $\{U, W\}$ the partition of $V(D)$ that defines D as a bipartite digraph. Suppose w.l.o.g. that $\{s, u, w, y\} \subseteq U$ and $\{t, v, x, z\} \subseteq W$.

Assume, for a contradiction that there is an arc (a, b) of D such that $c(a, b) = 3$. Observe that $(a, b) \notin A(H)$. We analyze the following cases.

Case 1. $a \in \{s, t, u, w, x, y\}$ or $b \in \{s, u, v, w, y, z\}$. In this case is easy to find a 3-coloured path of length 3 in D, a contradiction.

Case 2. $a = v$. We have $(\{t, u, v, b\}, c(t, u) = c(u, v) = 1, c(v, b) = 3, c([t, b]) = 1)$ and $(\{t, y, v, b\}, c(t, y) = c(y, v) = 2, c(v, b) = 3, c([t, b]) = 2)$, a contradiction.

Case 3. $a = z$. We have $(\{t, u, z, b\}, c(t, u) = c(u, z) = 1, c(z, b) = 3, c([t, b]) = 1)$ and $(\{t, y, z, b\}, c(t, y) = c(y, z) = 2, c(z, b) = 3, c([t, b]) = 2)$, a contradiction.

Case 4. $b = t$. We have $(\{a, t, u, v\}, c(t, u) = c(u, v) = 1, c(a, t) = 3, c([a, v]) = 1)$ and $(\{a, t, y, v\}, c(t, y) = c(y, v) = 2, c(a, t) = 3, c([a, v]) = 2)$, a contradiction.

Case 5. $b = x$. We have $(\{a, x, u, z\}, c(x, u) = c(u, z) = 1, c(a, x) = 3, c([a, z]) = 1)$ and $(\{a, x, y, z\}, c(x, y) = c(y, z) = 2, c(a, x) = 3, c([a, z]) = 2)$, a contradiction.

Now, we may assume that every arc that insides in $V(H)$ is coloured 1 or 2.

Case 6. $a \in U - \{s, u, w, y\}$. If $(t, a) \in A(D)$ then (s, t, a, b) or (w, t, a, b) is a 3-coloured path of length 3, a contradiction. So, $(a, t) \in A(D)$. Also, if $(v, a) \in A(D)$ then (u, v, a, b) or (y, v, a, b) is a 3-coloured path of length 3, a contradiction. Hence, $(a, v) \in A(D)$. Now, since $D[\{a, v, s, t\}] \cong T_4$, $c(v, s) = 2$, $c(s, t) = 1$ and every T_4 in D is quasi-monochromatic, then $c(a, v) = c(a, t)$. If $c(a, v) = 1$, then $D[\{a, t, y, v\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction. Finally, if $c(a, v) = 2$, then $D[\{a, t, u, v\}] \cong T_4$

and it is not quasi-monochromatic, a contradiction.

Case 7. $a \in W - \{t, v, x, z\}$. If $(b, t) \in A(D)$ then (a, b, t, u) or (a, b, t, y) is a 3-coloured path of length 3, a contradiction. Hence, $(t, b) \in A(D)$. Now, if $(b, v) \in A(D)$ then (a, b, v, w) or (a, b, v, s) is a 3-coloured path of length 3, a contradiction. Then, $(v, b) \in A(D)$. Since $D[\{v, w, t, b\}] \cong T_4$, $c(v, w) = 1$, $c(w, t) = 2$ and every T_4 in D is quasi-monochromatic, then $c(t, b) = c(v, b)$. If $c(t, b) = 1$, then $D[\{t, y, v, b\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction. If $c(t, b) = 2$, then $D[\{t, u, v, b\}] \cong T_4$ and it is not quasi-monochromatic, a contradiction. □

Theorem 3.5. *Let D be an m-coloured bipartite tournament such that every C_4 and every T_4 is quasi-monochromatic, then D has a kernel by monochromatic paths.*

Proof. Suppose that D has no induced subdigraph isomorphic to $\vec{T}_8$. We affirm that every cycle of $\mathcal{C}(D)$ possesses a symmetrical arc. Otherwise, if Γ is an asymmetrical cycle of $\mathcal{C}(D)$ clearly Γ is a γ-cycle of D, contradicting Theorem 3.3. From Theorem 2.1 we have that $\mathcal{C}(D)$ is kernel perfect and hence D has a kernel by monochromatic paths.

On the other hand, if D has an induced subdigraph isomorphic to $\vec{T}_8$ then Theorem 3.4 implies that D is 2-coloured and it follows from the Sands, Sauer and Woodrow's result, mentioned in the abstract, that D has a kernel by monochromatic paths. □

Remark 3.6. *The bipartite tournament D_1 in Figure 3 shows that the condition over C_4 can not be removed from the Theorem 3.5. D_1 is a 3-coloured bipartite tournament which contains no induced subdigraph isomorphic to T_4, (x, u, y, v, x) is a C_4 in D_1 that is not quasi-monochromatic and D_1 has no kernel by monochromatic paths.*

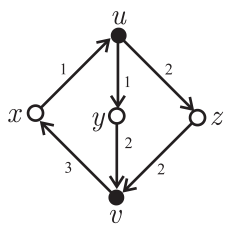

Figure 3: D_1

Remark 3.7. *The bipartite tournament D_2 in Figure 4 shows that we can not remove the condition about T_4 from the Theorem 3.5. D_2 is a 3-coloured bipartite tournament such that every C_4 is quasi-monochromatic, $\{x, v, y, w\}$ induces a T_4 which it is not quasi-monochromatic and D_2 has no kernel by monochromatic paths.*

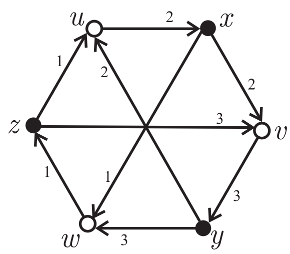

Figure 4: D_2

Remark 3.8. *For each $m \geq 3$ there is an m-coloured bipartite tournament which satisfies that every C_4 and every T_4 is quasimonochromatic. For example, let U and W two nonempty disjoint sets such that U has m elements; we put all the possible arcs from W to U and for each vertex $u \in U$ we use the same colour for all the arcs that inside in u, and for different vertices we use different colours. So we obtain an m-coloured bipartite tournament which has no C_4 and T_4 as an induced subdigraph.*

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