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# Cycle–Pancyclism in Multipartite Tournaments I

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**Abstract**

Let $T$ be a Hamiltonian multipartite tournament with $n$ vertices and $\gamma$ a Hamiltonian cycle of $T$. We prove that for every $k$, $4 \leq k \leq \frac{n+4}{2}$, there exists a cycle $\mathcal{C}$ of length $l(\mathcal{C}) \in \{k-3, k-2, k-1, k\}$, whose intersection with the arcs of $\gamma$ is at least $l(\mathcal{C}) - 3$. In some cases the result is best possible.

**Keywords:** Multipartite Tournament, Cycle, Pancyclism, Cycle-Pancyclism.

**2000 Mathematics Subject Classification:** 05C20

## 1. Introduction

The subject of pancyclism in tournaments has been studied by several authors (e.g. [1],[4]). Two types of pancyclism have been considered. A tournament $T$ is *vertex-pancyclic* if given any vertex $v$ there are cycles of every length containing $v$. Similarly, a tournament $T$ is *arc-pancyclic* if given any arc $e$ there are cycles of every length containing $e$. It is well known that a Hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In a previous paper [8] we introduced the concept of *cycle-pancyclism* to study questions such as the following. Given a cycle $\mathcal{C}$, what is the maximum number of arcs which a cycle of length $k$ contained in the tournament induced by the vertices of $\mathcal{C}$, $T[V(\mathcal{C})]$, has in common with $\mathcal{C}$? Clearly, to study this kind of question it is sufficient to consider a Hamiltonian tournament where $\mathcal{C}$ is a Hamiltonian cycle of $T$.

In this paper we consider a Hamiltonian multi-partite tournament $T$ with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set $A$. And we will assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a Hamiltonian cycle of $T$. Let $\mathcal{C}_k$ denote a directed cycle of length $k$. For a cycle $\mathcal{C}_k$ we denote $\mathcal{I}_\gamma(\mathcal{C}_k) = |A(\gamma) \cap A(\mathcal{C}_k)|$, or simply $\mathcal{I}(\mathcal{C}_k)$ when $\gamma$ is understood. Let $f(n,k,T) = \max\{\mathcal{I}_\gamma(\mathcal{C}_k) | C_k \subset T\}$ and $f(n,k) = \min\{f(n,k,T) | T$ is a Hamiltonian tournament with $n$ vertices$\}$. In [8, 9, 10] we proved that $f(n,k)$ is either $k-3$ or $k-4$, depending on weather $n \geq 2k-4$ or $n < 2k-4$, respectively.

It is well known that a Hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (i.e. it has cycles of every even length) but not necessarily arc-bipancyclic (see e.g. [4, 14, 20]). Cycle-pancyclism has also been studied in bipartite tournaments. In [6] it was proved that for even $k$ in the range $4 \leq k \leq \frac{n+4}{2}$, there exist a cycle $\mathcal{C}$ of length $l(\mathcal{C}) \in \{k-2, k\}$ intersecting $\gamma$ in at least $l(\mathcal{C}) - 3$ arcs. For the case of $k > \frac{n+4}{2}$, it was proved in [7] that the intersection is at least $l(\mathcal{C}) - 4$ arcs.

The study of pancyclism in multipartite tournaments was initiated by Bondy in 1976 [2]. Multipartite tournaments satisfy a restricted type of vertex-pancyclism: It is known that every partite set of a strongly connected $p$-partite tournament has at least one vertex that lies on cycles of each length $m$ for $m \in \{3, 4, \ldots, p\}$ [13]. Several extensive surveys on cycles in multipartite tournaments are [5, 16, 17, 19]. Most of the results about cycles in $p$-partite tournaments concern cycles of length at most $p$ or for the case of regular or almost regular multipartite tournaments.

In this paper we prove that, for every $k$, $4 \leq k \leq \frac{n+4}{2}$, for every Hamiltonian multipartite tournament, there exists a cycle $\mathcal{C}$ of length $l(\mathcal{C}) \in \{k-3, k-2, k-1, k\}$, whose intersection with the arcs of $\gamma$ is at least $l(\mathcal{C}) - 3$. As explained at the end of the paper, in some cases this result is tight. These results were originally announced in [12].

**Some related results.**

In [16] the following two open problems are mentioned, from [15].

- Conjecture 2.31. Let $D$ be a regular $p$-partite tournament with $p \geq 5$. Then $D$ contains a strongly connected sub-tournament of order $p$.
- Problem 2.32. Determine further sufficient condition for (strongly connected) $p$-partite tournaments to contain a strong sub-tournament of order $c$ for some $4 \leq c \leq p$.

In [18] Conjecture 2.31 is solved, actually, it is proved that every almost regular $p$-partite tournament with $p \geq 5$ contains a strongly connected subtournament of order $c$ for every $c \in \{3, 4, \ldots, p\}$. As a consequence of our result, we obtain a related result, in case that $D$ is Hamiltonian and not necessarily regular. Another consequence of this result is similar to Problem 2.32:

- Let $T$ be a Hamiltonian $p$-partite tournament of order $n$, and $4 \leq k \leq \frac{n+4}{2}$. Then $T$ contains a Hamiltonian multipartite sub-tournament of order $v$ with $v \in \{k-3, k-2, k-1, k\}$.

The rest of this paper is organized as follows. In Section 2 some notation and basic results needed in the rest of the paper are introduced. The proof of the main result, i.e., that there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$, for $n \geq 2k - 4$ appears in Section 3, 4, 5 and 6. Sections 3 through 5 contain special cases (for particular values of $n$ and $k$). The general case is left to Section 6.

## 2. Preliminaries

In all of this paper all notation is taken modulo $n$. Recall that in this paper we consider a Hamiltonian multi-partite tournament $T$ with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set $A$, and $\gamma = (0, 1, \ldots, n-1, 0)$ is an arbitrary Hamiltonian cycle of $T$. Vertex $u$ is *adjacent* to vertex $v$ iff either $(u, v) \in A(T)$ or $(v, u) \in A(T)$. We denote by $\mathcal{C}_l$ a cycle of length $l$. For a cycle $\mathcal{C}$, let $\mathcal{I}(\mathcal{C}) = |A(\mathcal{C}) \cap A(\gamma)|$.

### 2.1. Chords and Quasichords

A *c-ordered pair* is an ordered pair $(u, v)$ with $l\langle u, \gamma, v\rangle = c$ and a *-c-ordered pair* is an ordered pair $(u, v)$ with $l\langle v, \gamma, u\rangle = c$, where $\langle u, \gamma, v\rangle$ denotes the $uv$–directed path contained in $\gamma$, and $l\langle u, \gamma, v\rangle$ is its length. A *chord* of a cycle $C$ is an arc not in $C$ with both terminal vertices in $C$. The *length* of a chord $f = (u, v)$ of $C$, denoted $l(f)$, is equal to the length of $\langle u, C, v\rangle$. We say that $f$ is a $c$-chord if $l(f) = c$ and $f = (u, v)$ is a $-c$-chord if $l\langle v, C, u\rangle = c$. Observe that if $f$ is a $c$-chord then it is also a $-(n-c)$-chord. For the following definition, notice that if $u$ is not adjacent to $v$ then $u$ is adjacent to $v-1$ and $u$ is adjacent to $v+1$, because $T$ is a multi-partite tournament.

**Definition 2.1.** (Quasichord) *An ordered pair $(u, v)$ is a* quasichord *of $\gamma$ if the following properties are satisfied:*

1. *if $u$ is adjacent to $v$ in $T$ then $(u, v) \in A(T)$*
2. *if $u$ is not adjacent to $v$ in $T$ then $\{(u, v-1), (u, v+1)\} \subseteq A(T)$.*

*If $l(u, \gamma, v) = c$ we say that $f$ is a $c$-quasichord, and that it is a $-(n-c)$-quasichord.*

Observe that every chord of $\gamma$ is also a quasichord of $\gamma$, and that it may be that for a pair of vertices $u, v$, both $(u, v)$ and $(v, u)$ are quasichords of $\gamma$.

**Remark 1.** *If $(u, v)$ is not a quasichord then at least one of the following three ordered pairs is in $A$: $(v, u)$, $(v-1, u)$, $(v+1, u)$.*

**Definition 2.2.** (Quasicycle) *An $s$-*quasicycle *$\mathcal{Q}_m$ is a succession $u_0, u_1, \ldots, u_{m-1}, u_0$ of distinct vertices such that $m \geq 3$ and there exists $\{i_1, \ldots, i_s\} \subseteq \{0, 1, \ldots, m-1\}$ that satisfies*

1. *for each $i \in \{\{0, 1, \ldots, m-1\} - \{i_1, \ldots, i_s\}\}$, the arc $(u_i, u_{i+1})$ is in $A(\gamma)$.*
2. *for each $j \in \{i_1, \ldots, i_s\}$, the pair $(u_j, u_{j+1})$ is a quasichord of $\gamma$.*

*We say that the* length *of the quasicycle $Q_m$ is $l(\mathcal{Q}) = m$.*

**Definition 2.3.** *Let $B$ be a set of chords of $\gamma$. We say that $B$* induces *a cycle of $T$ with respect to $\gamma$, if there exists a cycle $\mathcal{C}$, $A(\mathcal{C}) \subseteq A(\gamma) \cup B$ such that $B \subseteq A(\mathcal{C})$. And we say that $\mathcal{C}$ is the cycle of $T$* induced *by $B$ with respect to $\gamma$, and denote it $\gamma[B]$.*

Notice that if such a cycle exists then it is unique.

**Definition 2.4.** *If $B \subseteq A(T)$ induces a cycle with respect to $\gamma$ then we will consider the following order for $B$; see Figure 1. Let $(u_1, v_1)$ be any arc of $B$. Once $(u_i, v_i)$ is defined, we denote by $(u_{i+1}, v_{i+1})$ the arc of $B$ such that $u_{i+1}$ is the first vertex of $\mathcal{C}$ (in the order implied by $\mathcal{C}$) after $u_i$ that is the initial vertex of some arc of $B$. Thus we have the set $\{(u_i, v_i) : i \in \{1, 2, \ldots, n\}\}$. We denote by $T_i = (v_i, \gamma, u_{i+1}) \subseteq \gamma$. Also, we denote by $T_i' = (z_i, \gamma, v_i)$ where $z_i$ is the first vertex of the cycle induced by $B$ after $v_i$ traversing $\gamma$ in the opposite direction. Finally, $l_i = l(T_i)$ and $l_i' = l(T_i')$.*

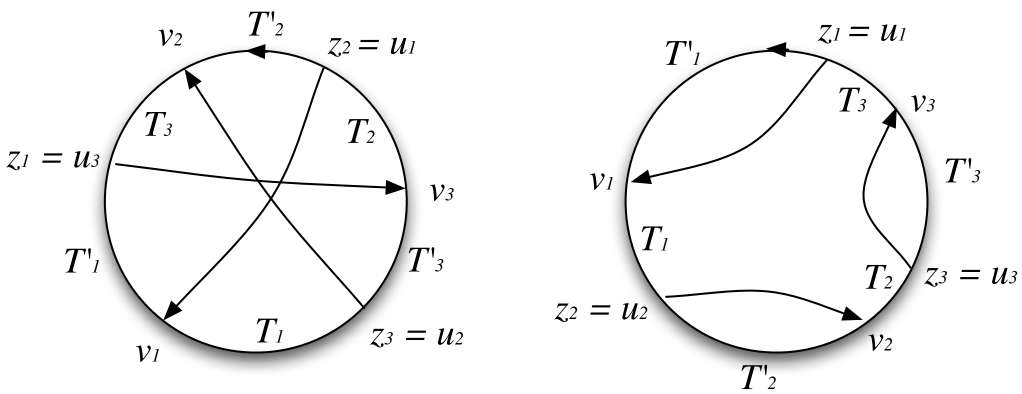


Figure 1: Examples with $|B| = 3$

**Lemma 2.1.** *If there exists a 1-quasicycle $\mathcal{Q}_{l+1}$ in $T$, then at least one of the following two conditions holds:*

1. *there exists a cycle $\mathcal{C}_{l+1}$ with $\mathcal{I}(\mathcal{C}_{l+1}) = l$;*
2. *there exists a cycle $\mathcal{C}_{l+2}$ with $\mathcal{I}(\mathcal{C}_{l+2}) = l+1$, and there exists a cycle $\mathcal{C}_l$ with $\mathcal{I}(\mathcal{C}_l) = l-1$.*

*Proof.* Let $\mathcal{Q}_{l+1} = (0, 1, \ldots l, 0)$ be a 1-quasicycle with $(l, 0)$ a quasichord of $\gamma$. When $l$ is adjacent to 0 we have that $(l, 0) \in A(T)$, and then it suffices to take $\mathcal{C}_{l+1} = \gamma[(l, 0)]$. When $l$ is not adjacent to 0 we have that $\{(l, n-1), (l, 1)\} \subseteq A(T)$. We take $\mathcal{C}_{l+2} = \gamma[(l, n-1)]$ and $\mathcal{C}_l = \gamma[(l, 1)]$. □

**Lemma 2.2.** *If there exists a 2-quasicycle $\mathcal{Q}_{l+2}$ in $T$, then there exists a cycle $\mathcal{C}_{h(l)}$ with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 2$, $h(l) \in \{l+2, l+3\}$, $l = l_1 + l_2$, where $l_1 = l(T_1)$, and $l_2 = l(T_2)$.*

*Proof.* Let $\mathcal{Q}_{l+2} = (0, x, x+1, \ldots, x+l_1 = y, z, z+1, \ldots, z+l_2 = n = 0)$ be a 2-quasicycle with $(0, x)$ and $(y, z)$ quasichords.

**Case 1.** $l_1 \geq 1$ and $l_2 \geq 1$.

If $\{(0, x), (y, z)\} \subseteq A(T)$ then let $\mathcal{C}_{l+2} = \gamma[\{(0, x), (y, z)\}]$.

If $(0, x) \in A(T)$ and $(y, z) \notin A(T)$ then let $\mathcal{C}_{l+3} = \gamma[\{(0, x), (y, z-1)\}]$.

If $(0, x) \notin A(T)$ and $(y, z) \in A(t)$ then let $\mathcal{C}_{l+3} = \gamma[\{(0, x-1), (y, z)\}]$.

If $(0, x) \notin A(T)$ and $(y, z) \notin A(t)$ then let $\mathcal{C}_{l+2} = \gamma[\{(0, x+1), (y, z-1)\}]$.

**Case 2.** $l_1 \geq 1$ and $l_2 = 0$. In this case $z = 0$.

If $\{(0, x), (y, 0)\} \subseteq A(T)$ then let $\mathcal{C}_{l+2} = \gamma[\{(0, x), (y, 0)\}]$.

If $(0, x) \in A(T)$ and $(y, 0) \notin A(T)$ then let $\mathcal{C}_{l+3} = \gamma[\{(0, x), (y, n-1)\}]$.

If $(0, x) \notin A(T)$ and $(y, 0) \in A(t)$ then let $\mathcal{C}_{l+3} = \gamma[\{(0, x-1), (y, 0)\}]$.

If $(0, x) \notin A(T)$ and $(y, 0) \notin A(t)$ then let $\mathcal{C}_{l+2} = \gamma[\{(0, x+1), (y, z-1)\}]$.

**Case 3.** $l_1 = 0$ and $l_2 \geq 1$. This is analogous to Case 2.

**Case 4.** $l_1 = 0$ and $l_2 = 0$. This case is impossible due to the definition of quasicycle.

□

**Lemma 2.3.** *Assume there exists a 3-quasicycle $\mathcal{Q}_{l+3}$ in $T$ with $l+3 \geq 4$, $l = l_1 + l_2 + l_3$, such that for each $i \in \{1, 2, 3\}$ if $l(T_i) = 0$ then $l(T_i') \geq 2$. Then there exists a cycle $\mathcal{C}_{h(l)}$ with $h(l) \in \{l+3, l+3+\Delta, l+3+2\Delta\}$, for a fixed $\Delta$, $\Delta \in \{-1, +1\}$, and with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 3$.*

*Proof.* Let $f_i = (u_i, v_i)$ with $i \in \{1, 2, 3\}$ the three quasichords of $\mathcal{Q}_{l+3}$.

**Case 1.** $f_i \in A(T)$ for all $i \in \{1, 2, 3\}$.

Let $\mathcal{C}_{h(l)} = \gamma[\{f_1, f_2, f_3\}]$. Clearly $h(l) = l+3$ and $\mathcal{I}(\mathcal{C}_{h(l)}) = l$.

**Case 2.** exactly one $f_i \notin A(T)$, say $f_1$ is not in $A(T)$ and $\{f_2, f_3\} \subseteq A(T)$.

If $l(T_1) \geq 1$ then $\mathcal{C}_{h(l)} = \gamma[\{(u_1, v_1+1), f_2, f_3\}]$, clearly $h(l) = l+2$, $\Delta = -1$ and $\mathcal{I}(\mathcal{C}_{h(l)}) = h(l) - 3$.

If $l(T_1) = 0$ it follows from the lemma hypothesis that $l(T_1') \geq 2$ and $\mathcal{C}_{h(l)} = \gamma[\{(u_1, v_1-1), f_2, f_3\}]$, clearly $h(l) = l+4$, $\Delta = +1$ and $\mathcal{I}(\mathcal{C}_{h(l)}) = h(l) - 3$.

**Case 3.** there exist two quasichords in $\{f_1, f_2, f_3\}$, say $f_1, f_2$, such that $\{f_1, f_2\} \cap A(T) = \emptyset$, and $f_3 \in A(T)$.

If $l(T_1) \geq 0$ and $l(T_2) > 0$ then let $\mathcal{C}_{h(l)} = \gamma[\{(u_1, v_1+1), (u_2, v_2+1), f_3\}]$. Clearly $h(l) = l+1 = l+3+2\Delta$, $\Delta = -1$ and $\mathcal{I}(\mathcal{C}_{h(l)}) = l-2$.

If $l(T_1) > 0$ and $l(T_2) = 0$ then let $\mathcal{C}_{h(l)} = \gamma[\{(u_1, v_1 + 1), (u_2, v_2 - 1), f_3\}]$. Clearly $h(l) = l + 3$ with $\mathcal{I}(\mathcal{C}_{h(l)}) = l$.

The case of $l(T_1) = 0$ and $l(T_2) > 0$ is analogous to the previous case.

If $l(T_1) = 0$ and $l(T_2) = 0$ then let $\mathcal{C}_{h(l)} = \gamma[\{(u_1, v_1 - 1), (u_2, v_2 - 1), f_3\}]$. Clearly $h(l) = l + 5 = l + 3 + 2\Delta$, $\Delta = +1$, and $\mathcal{I}(\mathcal{C}_{h(l)}) = l + 2$.

**Case 4.** $f_1, f_2, f_3 \notin A(T)$.

Since $4 \le l(\mathcal{Q}_{l+3}) = l + 3 < n - 3$ there exist $i, j \in \{1, 2, 3\}$ such that $l(T_i) > 0$ and $l(T'_j) > 1$.

If $i = j$ then there exists $k \in \{1, 2, 3\} - \{i\}$ such that $l(T_k) > 0$. Let

$$f'_s = \begin{cases} (u_s, v_s + 1) & \text{if } l(T_s) > 0 \\ (u_s, v_s - 1) & \text{if } l(T_s) = 0 \end{cases}$$

where $s \in \{1, 2, 3\} - \{i, k\}$. Let $\mathcal{C}_{h(l)} = \gamma[\{(u_k, v_k + 1), (u_i, v_i - 1), f'_s\}]$. Clearly $h(l) \in \{l + 2, l + 4\}$; if $l(T_s) > 0$ then $\Delta = -1$ ; if $l(T_s) = 0$ then $\Delta = +1$. Also $\mathcal{I}(\mathcal{C}_{h(l)}) \ge h(l) - 3$.

If $i \ne j$ let $k \in \{1, 2, 3\} - \{i, j\}$, and

$$f'_k = \begin{cases} (u_k, v_k + 1) & \text{if } l(T_k) > 0 \\ (u_k, v_k - 1) & \text{if } l(T_k) = 0 \end{cases}$$

Let $\mathcal{C}_{h(l)} = \gamma[\{(u_i, v_i + 1), (u_j, v_j - 1), f'_k\}]$. Clearly $h(l) \in \{l + 2, l + 4\}$; if $l(T_k) > 0$ then $\Delta = -1$; if $l(T_k) = 0$ then $\Delta = +1$. Also $\mathcal{I}(\mathcal{C}_{h(l)}) \ge h(l) - 3$.

□

Notice that in the previous lemma $h(l) = l + 5$ only in the case that two of the quasi-chords, say $f_1, f_2$ are not in $A(T)$, and $l(T_1) = l(T_2) = 0$.

## 2.2. Basic Properties

For any $a$, $2 \le a \le n-2$, denote by $t_a$ the largest integer such that $a + t_a(k-3) < n-1$, where $n$ is the number of vertices of $T$. The important case of $t_{k-2}$ is denoted by $t$ in the rest of the paper. Let $r$ be defined as follows: $r = n - [k - 2 + t(k-3)]$.

Notice the following facts.

- If $a \le b$, then $t_a \ge t_b$.
- $t \ge 0$.
- $2 \le r \le k - 2$.

- $n = k - 2 + t(k-3) + r$.

**Lemma 2.4.** *If the a-ordered pair with initial vertex* 0 *(recall that* 0 *is an arbitrary vertex of* $T$*) is a quasichord of* $\gamma$*, then at least one of the two following properties holds.*

(i) *Exists a cycle* $\mathcal{C}_{h(k)}$ *of length* $h(k) \in \{k-2, k-1, k\}$ *with* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.

(ii) *For every* $0 \leq i \leq t_a$, $(0, a + i(k-3))$ *is a quasichord.*

*Proof.* Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \ldots, t_a\} \mid (0, a + i(k-3)) \text{ is not a quasichord}\}.$$

**Case 1.** If 0 and $a + j(k-3)$ are adjacent, then $(a + j(k-3), 0) \in A$.

$$\mathcal{C}'_{k-1} = \langle a + (j-1)(k-3)), \gamma, a + j(k-3)\rangle \cup \langle a + j(k-3), 0, a + (j-1)(k-3)\rangle$$

is a 2-quasicycle of length $k-1$. By Lemma 2.2 there exists a cycle of length $h(k)$ with $h(k) \in \{k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.

**Case 2.** If 0 and $a + j(k-3)$ are not adjacent, then 0 and $a + j(k-3) + 1$ are adjacent, and also 0 and $a + j(k-3) - 1$ are adjacent. And since $(0, a + j(k-3))$ is not a quasichord, there are two possibilities.

**Case 2.1.** If $(a + j(k-3) - 1, 0) \in A$ then

$$\mathcal{C}'_{k-2} = \langle a + j(k-3) - 1, 0, a + (j-1)(k-3)\rangle \cup \langle a + (j-1)(k-3), \gamma, a + j(k-3)\rangle$$

is a 2-quasicycle of length $k-2$. By Lemma 2.2 there exists a cycle of length $h(k)$ with $h(k) \in \{k-2, k-1\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 2$.

**Case 2.2.** If $(a+j(k-3)+1, 0) \in A$ then let $\mathcal{C}' = \gamma[\{(a+j(k-3)+1, 0), (0, a+(j-1)(k-3))\}]$ when $(0, a+(j-1)(k-3)) \in A$, and $\mathcal{C}' = \gamma[\{(a+j(k-3)+1, 0), (0, a+(j-1)(k-3)-1)\}]$ when $(0, a + (j-1)(k-3)) \notin A$. Clearly $\ell(\mathcal{C}') \in \{k-1, k\}$, and $\mathcal{I}(\mathcal{C}') = \ell(\mathcal{C}') - 2$. □

The following is a consequence of Lemma 2.4.

**Corollary 2.5.** *At least one of the two following properties holds.*

(i) *Exists a cycle* $\mathcal{C}_{h(k)}$ *of length* $h(k)$ *such that* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$, *and* $h(k) \in \{k-2, k-1, k\}$.

(ii) *For every* $0 \leq i \leq t$, *every* $(0, (k-2) + i(k-3))$*-ordered pair is a quasichord.*

*Proof.* We consider two possible cases. Case 1, that $(0, k-2)$ is a quasichord. In this case, the result follows from Lemma 2.4. Case 2, $(0, k-2)$ is not a quasichord. If 0 and $k-2$ are adjacent then $(k-2, 0) \in A$ and $\gamma[(k-2, 0)]$ is a cycle $\mathcal{C}_{k-1}$ with $\mathcal{I}(\mathcal{C}_{k-1}) = k-2$. If 0 and $k-2$ are not adjacent then $(k-1, 0) \in A$ or $(k-3, 0) \in A$. When $(k-1, 0) \in A$ we have that $\mathcal{C}' = \gamma[(k-1, 0)]$ has length $k$ and $\mathcal{I}(\mathcal{C}') = k-1$. When $(k-3, 0) \in A$ we have that $\mathcal{C}' = \gamma[(k-3, 0)]$ has length $k-2$ and $\mathcal{I}(\mathcal{C}') = k-3$. □

## 3. The Cases $k = 4, 5$

**Theorem 3.1.** *For $n \geq 3$ there exists a cycle $\mathcal{C}_3$, $\mathcal{I}(\mathcal{C}_3) \geq 1$, or exists a cycle $\mathcal{C}_4$, $\mathcal{I}(\mathcal{C}_4) \geq 2$.*

*Proof.* Let $i = \min\{j \in V | (j, 0) \in A\}$. Observe that $i$ is well defined since $(n-1, 0) \in A$. If $i-1$ and 0 are adjacent, then by the choice of $i$ we have $(0, i-1) \in A$ and $\mathcal{C}_3 = (0, i-1, i, 0)$ is a cycle with $\mathcal{I}(\mathcal{C}_3) \geq 1$. If $i-1$ and 0 are not adjacent, then 0 and $i-2$ are adjacent, and by the choice of $i$, $(0, i-2) \in A$, and $\mathcal{C}_4 = (0, i-2, i-1, i, 0)$ is a cycle with $\mathcal{I}(\mathcal{C}_4) \geq 2$. □

**Theorem 3.2.** *For $n \geq 6$ there exists a cycle $\mathcal{C}_4$, $\mathcal{I}(\mathcal{C}_4) \geq 1$, or exists there exists a cycle $\mathcal{C}_5$, $\mathcal{I}(\mathcal{C}_5) \geq 2$, or there exists a cycle $\mathcal{C}_6$, $\mathcal{I}(\mathcal{C}_6) \geq 5$.*

*Proof.* We consider the following cases.

$n \equiv 2 \pmod 3$ Notice that $r_5 = n - 4$. Considering Lemma 2.4 we have that at least one of the following statements holds: $(0, n-4) \in A$ or $\{(0, n-5), (0, n-3)\} \subseteq A$. If $(0, n-4) \in A$ then $\mathcal{C}_5 = (0, n-4, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_5) = 4$; otherwise $(0, n-3) \in A$ and then $\mathcal{C}_4 = (0, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_4) = 3$.

$n \equiv 1 \pmod 3$ Notice that $r_5 = n - 3$ and by Lemma 2.4 we have that $(0, n-3) \in A$ or $\{(0, n-2), (0, n-4)\} \subseteq A$. If $(0, n-3) \in A$ then $\mathcal{C}_4 = (0, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_4) = 3$; otherwise $(0, n-4) \in A$ and then $\mathcal{C}_5 = (0, n-4, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_5) = 4$.

$n \equiv 0 \pmod 3$ Notice that $r_5 = n - 2$ and by Lemma 2.4 we have that $(0, n-5) \in A$ or $\{(0, n-4), (0, n-6)\} \subseteq A$. If $(0, n-5) \in A$ then $\mathcal{C}_6 = (0, n-5, n-4, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_6) = 5$; otherwise $(0, n-4) \in A$ (we are assuming $n \geq 6$) and then $\mathcal{C}_5 = (0, n-4, n-3, n-2, n-1, 0)$ satisfies $\mathcal{I}(\mathcal{C}_5) = 4$.

□

## 4. The case of $n = 2k - 4$

**Theorem 4.1.** *If $n = 2k - 4$ then exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 1$.*

*Proof.* Let $x$ and $y$ be two vertices of $T$ such that $l\langle x, \gamma, y\rangle = l\langle y, \gamma, x\rangle = k - 2$. Without loss of generality we can assume that $x = 0$, $y = k - 2$.

**Case 1.** 0 and $k-2$ are adjacent. If $(0, k-2) \in A$ then $\mathcal{C}_{k-1} = (0, k-2) \cup \langle k-2, \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-1}) = k-2$. If $(k-2, 0) \in A$ then $\mathcal{C}_{k-1} = \langle 0, \gamma, k-2\rangle \cup (k-2, 0)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-1}) = k - 2$.

**Case 2.** 0 and $k-2$ are not adjacent. It is clear that 0 and $k-3$ are adjacent. If $(0, k-3) \in A$ then $\mathcal{C}_k = (0, k-3) \cup \langle k-3, \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$. If $(k-3, 0) \in A$ then $\mathcal{C}_{k-2} = \langle 0, \gamma, k-3\rangle \cup (k-3, 0)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$. □

## 5. The cases $r = k-2$, $r = k-3$ and $r = k-4$

**Theorem 5.1.** *If $r = k-2$, $r = k-3$, or $r = k-4$ then exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

*Proof.* By Corollary 2.5, taking $i = t$, at least one of the two following assertions is valid:

1. exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$, or
2. the $(0, (k-2) + t(k-3))$-ordered pair is a quasichord.

If assertion 1 holds the theorem follows. Assume then that it does not hold.

For the case $r = k-2$ (resp. $r = k-3$, $r = k-4$): when 0 and $k - 2 + t(k-3)$ are adjacent we have that $(0, k-2+t(k-3)) \in A$ and $(0, k-2+t(k-3)) \cup \langle k-2+t(k-3), \gamma, 0\rangle$ is a cycle $\mathcal{C}$ of length $k-1$ (resp. $k-2$, $k-3$) with $\mathcal{I}(\mathcal{C}) \geq \ell(\mathcal{C}) - 3$.

When 0 and $k-2+t(k-3)$ are not adjacent we have that $(0, k-2+t(k-3)-1) \in A$ and $(0, k-2+t(k-3)-1) \cup \langle k-2+t(k-3)-1, \gamma, 0\rangle$ is a cycle $\mathcal{C}$ of length $k$ (resp. $k-1$, $k-2$) with $\mathcal{I}(\mathcal{C}) \geq \ell(\mathcal{C}) - 3$. □

**Corollary 5.2.** *If $t = 0$ then exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

*Proof.* If $t = 0$ then $n = k - 2 + r$, where $k-2 \leq r \leq k-2$ since $n \geq 2k-4$. It follows that $r = k-2$ and the result follows directly from Theorem 5.1. □

## 6. The General Case

In this section we assume that $r \leq k-5$, $k \geq 6$, $t \geq 1$, since the other cases have been considered in the previous sections.

The next lemma follows directly from Lemma 2.4.

**Lemma 6.1.** *If the $(0, k-2+\alpha)$-ordered pair is a quasichord, for each $\alpha$, $-(k-4) \leq \alpha \leq r$, then at least one of the two following properties holds.*

(i) *There exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

(ii) *For every $0 \leq i \leq t-1$, the $(0, k-2+\alpha+i(k-3))$-ordered pair is a quasichord.*

**Lemma 6.2.** *At least one of the two following properties holds.*

(i) *There exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

(ii) *All the following ordered pairs are quasichords.*

   (a) *Every $(k-2)$-ordered pair.*

(b) *Every* $(-r)$*-ordered pair.*

(c) *Every* $-(r+1)$*-ordered pair.*

(d) *Every* $(k-3)$*-ordered pair.*

*Proof.* The proof of (a) follows directly from Corollary 2.5 taking $i = 0$.

The proof of (b) follows from Corollary 2.5 taking $i = t$ and recalling that $n - r = k - 2 + t(k - 3)$.

To prove (c) suppose that (c) does not hold. Then exist $y, x$, $x = y + r + 1$, such that $(y, x)$ is not a quasichord. It follows from Remark 1 that at least one of the following chords is in $A$: $(x, y), (x - 1, y), (x + 1, y)$.

**Case (c.1).** $g_1 = (x, y) \in A$. By (a) we can take $a = k-2$ in Lemma 2.4; taking $i = t-1$ in the same lemma, we can consider the $(k-2+(t-1)(k-3))$-quasichord, namely $f_1$, that starts in $y + k - 4$. Notice that $f_1$ ends in $x$. By Lemma 2.2, considering the 2-quasicycle generated by $\{g_1, f_1\}$, we have that there exists a cycle $\mathcal{C}_{h(k)}$ with $h(k) \in \{k - 2, k - 1\}$ and $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 2$. (Notice that $\ell = k - 4$ in Lemma 2.2).

**Case (c.2).** $g_2 = (x + 1, y) \in A$. By Lemma 2.2, considering the 2-quasicycle generated by $\{g_2, f_1\}$, it follows that there exists a cycle $\mathcal{C}_{h(k)}$ with $h(k) \in \{k - 1, k\}$ and $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 2$. (Notice that $\ell = k - 3$ in Lemma 2.2).

**Case (c.3).** $g_3 = (x - 1, y) \in A$. The $(k - 2 + (t - 1)(k - 3))$-quasichord that starts in $y+k-5$ ends in $x-1$. It follows from Lemma 2.2, considering the 2-quasicycle generated by $\{g_3, f_2\}$, that there exists a cycle $\mathcal{C}_{h(k)}$ with $h(k) \in \{k-3, k-2\}$ and $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-2$.

Finally to prove (d) assume (d) does not hold, and we prove (i) of the lemma. Suppose without loss of generality that $(0, k - 3)$ is not a quasichord of $\gamma$. Case (d.1) $x = 0$ is adjacent to $k-3$. In this case, $(k-3, 0) \in A$ and $\mathcal{C}_{k-2} = \gamma[(k-3, 0)]$ has $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$. Case (d.2) $x = 0$ is not adjacent to $k - 3$. It follows that $(k - 2, 0) \in A$ or $(k - 4, 0) \in A$. If $(k - 2, 0) \in A$ then $\mathcal{C}_{k-3} = \gamma[(k - 2, 0)]$ has $\mathcal{I}(\mathcal{C}_{k-3}) = k - 4$. If $(k - 4, 0) \in A$ then $\mathcal{C}_{k-3} = \gamma[(k - 4, 0)]$ has $\mathcal{I}(\mathcal{C}_{k-3}) = k - 4$. □

**Lemma 6.3.** *Let* $-1 \leq i \leq r$. *If all the* $-r$*-ordered pairs,* $-(r+1)$*-ordered pairs,* $(k-3+i)$*-ordered pairs and* $(k-2+i)$*-ordered pairs are quasichords then at least one of the following properties holds.*

(i) *There exists a cycle* $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ *such that* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-3$.

(ii) *All the* $-(2r-i+1)$*-ordered pairs,* $-(2r-i+2)$*-ordered pairs and* $-(2r-i+3)$*-ordered pairs are quasichords.*

*Proof.* Assume that the hypothesis of the lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the $[(k - 3) + i]$-ordered pairs are quasichords and all the $[(k - 2) + i]$-ordered pairs are quasichords, it follows from Lemma 6.1 (taking $\alpha = i-1$) that every $[k-3+i+(t-$

1)(k−3)]-ordered pair is a quasichord, and that (taking $\alpha = i$) every $[k-2+i+(t-1)(k-3)]$-ordered pairs is a quasichord. Thus the following ordered pairs are quasichords of $T$: $(r, 0)$, $(r+1, 0)$, $(0, k-2+(t-1)(k-3)+i)$, $(0, k-2+(t-1)(k-3)+i-1)$.

Let: $x_1 = r$, $x_2 = r+1$, $x_3 = k-2+(t-1)(k-3)+i-1$, $x_4 = x_3+1$, $x_5 = x_4+k-3$, $x_6 = x_5+1$, $x_7 = x_5-1$ and $x_8 = x_7-1$. Thus, $(x_1, 0)$ is a $(-r)$-quasichord and $(0, x_4)$ is a $[(k-2)+(t-1)(k-3)+i]$-quasichord. Observe that:

- It follows from $x_5 = k-2+t(k-3)+i$, and $n = k-2+t(k-3)+r$ that $l\langle x_5, \gamma, 0\rangle = n - x_5 = r - i$.
- $l\langle x_6, \gamma, 0\rangle = r - i - 1$.
- $l\langle x_6, \gamma, x_1\rangle = 2r - i - 1$.
- $l\langle x_7, \gamma, x_1\rangle = 2r - i + 1$.
- $l\langle x_7, \gamma, x_2\rangle = 2r - i + 2$.
- $l\langle x_8, \gamma, x_2\rangle = 2r - i + 3$.
- $l\langle x_4, \gamma, x_7\rangle = k - 4$.
- $l\langle x_3, \gamma, x_8\rangle = k - 4$.

**I.** We first prove that every $-(2r-i+1)$-ordered pairs is a quasichord. Suppose that not every $-(2r-i+1)$-ordered pairs is a quasichord. We can assume w.l.o.g. that $(x_7, x_1)$ is not a $-(2r-i+1)$-quasichord. Then by Remark 1 we have that $\{(x_7, x_1), (x_5, x_1), (x_8, x_1)\} \cap A \neq \emptyset$. We consider the three possible cases.

**Case 1.** $(x_7, x_1) \in A$. In this case $\mathcal{Q}_{k-1} = (x_7, x_1, 0, x_4) \cup \langle x_4, \gamma, x_7\rangle$ is a 3-quasicycle with at most two pairs of consecutive vertices that are not necessarily arcs of $T$, namely, $f_1 = (x_1, 0)$ and $f_2 = (0, x_4)$. Notice that $\ell(T_1) = 0$ and $\ell(T_2) \neq 0$ ($T_1, T_2$ defined in Definition 2.4). It follows from Lemma 2.3 that there exists a cycle $\mathcal{C}_{h(l)}$, $h(l) \in \{l+1, , l+2, l+3, l+4\}$, with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 3$, and $l = k-4$. Notice that the case $h(l) = l+5$ in Lemma 2.3 does not occur because for this it is necessary that $\ell(T_1) = 0$ and $\ell(T_2) = 0$ (see Case 3 in the proof of the lemma).

**Case 2.** $(x_5, x_1) \in A$. Recall that $(x_1, 0)$ is a $(-r)$-quasichord. Hence we consider two possible subcases.

**Case 2.1.** $x_1$ is adjacent to 0. Then $(x_1, 0) \in A$ and $\mathcal{Q}_k = (x_5, x_1, 0, x_4) \cup \langle x_4, \gamma, x_5\rangle$ is a 3-quasicycle with at most one pair of consecutive vertices that is not necessarily and arc of $T$, namely, $f_3 = (0, x_4)$. Since $\ell(T_3) > 0$ it follows from Lemma 2.3 that there exists a cycle $\mathcal{C}_{h(l)}$, $h(l) \in \{l+3, , l+2\}$, with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 3$, and $l = k-3$.

**Case 2.2.** $x_1$ is not adjacent to 0. It follows from the definition of quasichord that $(x_1, 1)$ and $(x_1, n-1)$ are in $A$. Also, $(1, x_4+1)$ is a $[(k-2)+i+(t-1)(k-3)]$-quasichord. And then, $\mathcal{Q}_{k-1} = (x_5, x_1, 1, x_4+1) \cup \langle x_4, \gamma, x_5\rangle$ is a 3-quasicycle with at most one pair of consecutive vertices that is not necessarily and arc of $T$, namely, $f_1 = (1, x_4 + 1)$. Notice that $\ell(T_1) > 0$ and $k \geq 6$. It follows from Lemma 2.3 that there exists a cycle $\mathcal{C}_{h(l)}$, $h(l) \in \{l+3, l+2\}$, with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 3$, and $l = k - 4$.

**Case 3.** $(x_8, x_1) \in A$. Consider $\mathcal{Q}_{k-2} = (x_8, x_1, 0, x_4) \cup \langle x_4, \gamma, x_8 \rangle$ is a 3-quasicycle with at most two pairs of consecutive vertices that are not necessarily arcs of $T$, namely, $f_1 = (x_1, 0)$, and $f_2 = (0, x_2)$. Notice that $\ell(T_1) = 0$ and $\ell(T_2) > 0$. It follows from Lemma 2.3 that there exists a cycle $\mathcal{C}_{h(l)}$, $h(l) \in \{l+2, l+3, l+4, l+5\}$, with $\mathcal{I}(\mathcal{C}_{h(l)}) \geq h(l) - 3$, and $l = k - 5$.

**II.** Now we prove that every $-(2r - i + 2)$-ordered pairs is a quasichord. If not every $-(2r - i + 2)$-ordered pairs is a quasichord, we may assume without loss of generality that $(x_2, x_7)$ is not a $-(2r - i + 2)$-quasichord. It follows from Remark 1 that at least one of the following holds: $(x_7, x_2) \in A$, $(x_8, x_2) \in A$, $(x_5, x_2) \in A$. We proceed as in the proof of **I.**, changing $x_1$ for $x_2$).

**III.** Now we prove that every $-(2r - i + 3)$-ordered pairs is a quasichord. If not every $-(2r - i + 3)$-ordered pairs is a quasichord, we may assume without loss of generality that $(x_2, x_8)$ is not a $-(2r - i + 3)$-quasichord. It follows from Remark 1 that at least one of the following holds: $(x_8, x_2) \in A$, $(x_7, x_2) \in A$, $(x_8 - 1, x_2) \in A$. We proceed as in the proof of **I.**, changing $x_4$ for $x_3$). □

**Lemma 6.4.** *At least one of the following properties holds.*

(i) *There exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

(ii) *For any $x$, there exist at most $k - 4$ consecutive vertices in $\gamma$, say $x_1, \ldots, x_m$, such that for all $1 \leq i \leq m$, $(x_i, x)$ is a quasichord of $\gamma$.*

*Proof.* Assume that (i) does not hold, and assume for contradiction that (ii) does not hold. Let $x = 0$, without loss of generality. Thus there exist $k - 3$ consecutive vertices in $\gamma$, say $x_1, \ldots, x_{k-3}$, such that $(x_i, 0)$ is a quasichord of $\gamma$. It follows from Corollary 2.5 that for every $i$, $0 \leq i \leq t$, the pair $(0, k - 2 + i(k-3))$ is a quasichord of $\gamma$. Therefore, there exist $j, j'$ such that $x_j = k - 2 + j'(k-3)$ $(0 \leq j' \leq t)$ and $(0, y = k - 2 + j'(k-3) - (k-3)) = f_2$ is a quasichord of $\gamma$. We then have the 2-quasicycle $\mathcal{Q}_{k-1} = (0, y) \cup \langle y, \gamma, x_j \rangle \cup (x_j, 0)$, with quasichords $f_2$ and $f_1 = (x_j, 0)$. It follows from Lemma 2.2, taking $l = k - 3$, that there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$. □

**Lemma 6.5.** *If $x_0, x_1, x_2, x_3$ are vertices of $T$ such that $0 \leq x_0 < x_1 < x_3 < x_2 < n$, $(x_1, x_2)$ is not a quasichord of $\gamma$, $(x_1, x_0)$ and $(x_0, x_3)$ are quasichords of $\gamma$, and $l\langle x_3, \gamma, x_2 \rangle = k - 5$, then there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

*Proof.* Since $(x_1, x_2)$ is not a quasichord of $\gamma$, it follows from Remark 1 that at least one of the following holds: $(x_2, x_1) \in A$, $(x_2 - 1, x_1) \in A$, $(x_2 + 1, x_1) \in A$. We will consider the three possible cases.

**Case 1.** $(x_2, x_1) \in A$. In this case $\mathcal{Q}_{k-2} = (x_2, x_1, x_0, x_3) \cup \langle x_3, \gamma, x_2 \rangle$ is a 3-quasicycle, with one quasichord that is a chord of $A$ (namely, $(x_2, x_1)$). It follows from Lemma 2.3 taking $f_1 = (x_2, x_1)$, $f_2 = (x_1, x_0)$, $f_3 = (x_0, x_3)$, $l(T_1) = l(T_2) = 0$, $l = k - 5$, that there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ (notice that the first subcase of Case 3 in the proof of the lemma does not occur because $l(T_1) = l(T_2) = 0$, which is the only subcase which would give $h(k) = k - 4$).

**Case 2.** $(x_2 + 1, x_1) \in A$. In this case $\mathcal{Q}_{k-1} = (x_2 + 1, x_1, x_0, x_3) \cup \langle x_3, \gamma, x_2 + 1 \rangle$ is a 3-quasicycle, with quasichords $f_1 = (x_2 + 1, x_1)$, $f_2 = (x_1, x_0)$, and $f_3 = (x_0, x_3)$. Notice that $l(T_1) = l(T_2) = 0$ and $l(T_3) > 0$ (since $k > 5$). It follows from Lemma 2.3 taking the three quasichords $f_1, f_2, f_3$ and $l = k - 4$, that there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$ (notice that we don't obtain length $l + 5$ since $f_1 \in A$ and $l(T_3) > 0$).

**Case 3.** $(x_2 - 1, x_1) \in A$. In this case $\mathcal{Q}_{k-3} = (x_2 - 1, x_1, x_0, x_3) \cup \langle x_3, \gamma, x_2 - 1 \rangle$ is a 3-quasicycle, with quasichords $f_1 = (x_2 - 1, x_1)$, $f_2 = (x_1, x_0)$, and $f_3 = (x_0, x_3)$. It follows from Lemma 2.3 taking the three quasichords $f_1, f_2, f_3$ and $l = k - 6$, that there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1\}$ except when: (a) We are in the second case of the proof of the lemma; when exactly one $f_i \notin A$ and $l(T_i) \geq 1$. But the only possible such quasichord is $f_3$, and in this case we consider $\mathcal{C}_{k-2}$, the cycle of $\gamma$ induced by the chords $\{(x_2 - 1, x_1), (x_1, x_0), (x_0, x_3 - 1)\}$, and clearly $\mathcal{I}(\mathcal{C}_{k-2}) = k - 5$. (b) We are in the first subcase of the 3rd case of the proof of the lemma; i.e., that there exist two quasichords $g_1, g_2$ such that $g_i \notin A$ and $l(T_i) > 0$, $i = 1, 2$. Clearly this case is impossible. ☐

**Lemma 6.6.** *If every $k$-ordered pair and every $(-r)$-ordered pairs are quasichords then at least one of the two following properties holds.*

- (i) *There exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*
- (ii) *For every $\alpha$, $0 < \alpha(r+1) < k$, every $-(\alpha+1)(r+1)$-ordered pair is a quasichord.*

*Proof.* Assume that (ii) does not hold; we show that (i) holds. Let $\alpha$ be the least integer for which there does not exist an $-(\alpha+1)(r+1)$-quasichord. By Lemma 6.2, $\alpha > 0$. It follows that there exist vertices $x_1, x_2$ such that $l\langle x_2, \gamma, x_1 \rangle = (\alpha+1)(r+1)$ and $(x_1, x_2)$ is not a quasichord of $\gamma$. Let $x_0 \in V$ such that $l\langle x_2, \gamma, x_0 \rangle = r + 1$. It follows from the choice of $\alpha$ that $(x_1, x_0)$ is a quasichord (because it is an $-\alpha(r+1)$-quasichord). Let $x_3 \in V$ such that $l\langle x_0, \gamma, x_3 \rangle = k - 1 + (t-1)(k-3)$. Observe that $x_3 \in \langle x_1, \gamma, x_0 \rangle$ because $\alpha(r+1) < k$ and $t \geq 1$.

First, notice that every $(k-1)$-ordered pair is a quasichord. When there exist a $(k-1)$-ordered pair which is not a quasichord, we have the following. Assume without loss of generality, $(0, k-1)$ is not a quasichord. Then there are two cases. When 0 and $k-1$ are adjacent, $(k-1, 0) \in A$ and $\gamma[(k-1, 0)]$ is a cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$. When 0 and $k-1$ are not adjacent, we have $\{(k-1, 1), (k-1, n-1)\} \subseteq A$. And then $\gamma[(k-1, 1)]$ or $\gamma[(k-1, n-1)]$ satisfies (i).

Lemma 6.1 and the fact that every $(k-1)$-ordered pair is a quasichord imply that either (i) holds or that every $k-1+(t-1)(k-3)$-ordered pair is a quasichord. Thus we can assume that every $k-1+(t-1)(k-3)$-ordered pair is a quasichord. Hence we have that $(x_0, x_3)$ is a quasichord of $\gamma$. The lemma follows from Lemma 6.5. □

**Lemma 6.7.** *At least one of the following properties holds.*

(i) *There exists a cycle* $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ *such that* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-3$.

(ii) *For* $-1 \leq i \leq r$, *every* $-(2r+2-i)$*-ordered pair is a quasichord and every* $(k-2+i)$*-ordered pair is a quasichord.*

*Proof.* Suppose that (i) does not hold. We shall prove that property (ii) holds by induction on $i$. We start with $i=-1$ and $i=0$, namely, we prove that the following ordered pairs are quasichords:

(a) every $(k-3)$-ordered pair,

(b) every $(k-2)$-ordered pair,

(c) every $-(2r+3)$-ordered pair,

(d) every $-(2r+2)$-ordered pair.

In fact we also prove that the following ordered pairs are quasichords of $\gamma$:

(e) every $-(2r+4)$-ordered pair.

The proof of (a) follows from Lemma 6.2(d), while the proof of (b) follows from Lemma 6.2(a).

Let 0 be any vertex of $T$. By Lemma 6.2 (b) and (c) we have that $(r, 0)$ and $(r+1, 0)$ are quasichords. It follows from Corollary 2.5 that the following is a quasichord $(0, k-2+(t-1)(k-3)$. And by Lemma 6.2(d), and Lemma 2.4 that $(0, t(k-3))$ is a quasichord. Notice that these two quasichords have consecutive end-points in $\gamma$.

Since 0 is an arbitrary vertex of $T$, we can prove that (c), (d) and (e) hold:

**Part (c).** every $-(2r+3)$-ordered pair is a quasichord. Assume $(r+1, n-r-2)$ is not a quasichord. Let $x_2 = n-(r+2)$, $x_1 = r+1$, and $x_0 = k-2+(t-1)(k-3)$. We have already noticed that $(0, x_0)$ and $(x_1, 0)$ are quasichords. Observe that $l\langle x_0, \gamma, x_2\rangle = k-5$. Then the claim follows from Lemma 6.5.

**Part (d).** every $-(2r+2)$-ordered pair is a quasichord. Assume $(r, n-(r+2))$ is not a quasichord. Let $x_2 = n-(r+2)$, $x_1 = r$, and $x_0 = k-2+(t-1)(k-3)$. Then the claim follows from Lemma 6.5.

**Part (e).** every $-(2r+4)$-ordered pair is a quasichord. Assume $(r+1, n-(r+3))$ is not a quasichord. Let $x_2 = n-(r+3)$, $x_1 = r+1$, and $x_0 = t(k-3)$. Then the claim follows from Lemma 6.5.

Assume that the lemma holds for each $i'$, $i' \leq i$ and we prove it for $i+1$; namely, we prove:

($\alpha$) Every $(k-1+i)$-ordered pair is a quasichord,

($\beta$) Every $-(2r+1-i)$-ordered pair is a quasichord.

**Proof of ($\alpha$)**

It follows from the inductive hypothesis that for each $j$, $0 \leq j \leq i$, every $(k-2)+j$-ordered pair and every $(k-3)+j$-ordered pair is a quasichord. Hence, by Lemmas 6.2 and 6.3, every $-(2r-j+1)$-ordered pair, $-(2r-j+2)$-ordered pair and every $-(2r-j+3)$-ordered pair is a quasichord. That is, for each $j$, $0 \leq j \leq i+2$, every $-(2r-j+3)$-ordered pair is a quasichord. These are $(i+3)$-quasichords with initial vertices consecutive in $\gamma$ (taking a fixed vertex $x$ and the $-(2r-j+3)$-quasichords that end in $x$, where $0 \leq j \leq i+2$).

Assume for contradiction that $(0, x_3)$ with $x_3 = k-1+i$ is not a $(k-1+i)$-quasichord. Let $x_0 = n-(2r-i-1)$. Hence letting $x_2 = 2$, we have that $(x_2, x_0)$ is a $-(2r-(i-1))$-quasichord (taking $j = i+2$ in the previous assertion).

Let us show that $x_0 \in \langle x_3+1, \gamma, n-1\rangle$:

$$l\langle x_0, \gamma, 0\rangle = 2r - i - 1,$$

$$\begin{aligned} l\langle x_3, \gamma, x_0\rangle &= n-(k+i-1+2r-i-1) \\ &= k-2+t(k-3)+r-(k+i-1+2r-i-1) \\ &\geq t(k-3)+r-2r \geq k-3-r. \end{aligned}$$

Since we are assuming $r \leq k-4$ then $l\langle x_3, \gamma, x_0\rangle \geq 1$. Hence $l\langle x_0, \gamma, 0\rangle \geq 1$, because $r \geq 1$.

Now, there exists an $x$ such that $x \in \langle x_2, \gamma, x_3-1\rangle$ and such that $(x, x_0)$ is not a quasichord (this is a direct consequence of Lemma 6.4 and the fact that the number of vertices in $\langle x_2, \gamma, x_3-1\rangle$ is at least $k-3$). Let $x_4$ be the smallest (the nearest to 0 in $\gamma$) such vertex.

Let $x_1 = 0$. We will prove that $x_4-i-3 \in \langle x_1, \gamma, x_4-3\rangle$. Since for each $j$, $0 \leq j \leq i+2$, every $-(2r-j+3)$-quasichord is in $T$, it follows that $\{(2, x_0), (3, x_0), \ldots, (i+4, x_0)\}$ are quasichords. Hence, the election of $x_4$ implies $x_4 \geq i+5$ and then $x_4-i-3 \geq 2 = x_2$.

Finally, since $l\langle x_4, \gamma, x_3\rangle + l\langle x_1, \gamma, x_4-i-3\rangle = k-4$ then we consider the following $\mathcal{Q}_{k-1} = (x_4-i-3, x_0, x_4) \cup \langle x_4, \gamma, x_3\rangle \cup (x_3, x_1) \cup \langle x_1, \gamma, x_4-i-3\rangle$. We prove the following

**Claim.** there exists a vertex $z \in \langle x_2, \gamma, x_3-1\rangle$ such that $(x_0, z) \in A$.

Assume for contradiction that there is no such $z$. By the definition of $x_4$ we have that for each $y \in \langle 2, \gamma, x_4-1\rangle$ it holds that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. And by the assumption it holds that for each $y \in \langle x_4, \gamma, x_3-1\rangle$ we have that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. (Notice that $l\langle x_2, \gamma, x_3-1\rangle \geq k-3$, so we have $k-2$ consecutive vertices $y \in \langle x_0+1, \gamma, x_3\rangle$, such that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$.) Let $z$ be the first vertex in $\langle x_2, \gamma^{-1}, x_0\rangle$ (where $\gamma^{-1}$ denotes the inverse traversal of $\gamma$) such that $(x_0, z) \in A$ (it exists since $(x_0, x_0 + 1) \in A$). Now consider $P = (x_0, z) \cup \langle z, \gamma, z + k - 3\rangle$; by the previous observation if $z+k-3$ is adjacent to $x_0$ then $(z+k-3, x_0) \in A$ and then $\mathcal{C}_{k-1} = P \cup (z+k-3, x_0)$ is a cycle of $T$ with $\mathcal{I}(\mathcal{C}_{k-1}) \geq k-4$. If $z+k-3$ is not adjacent to $x_0$ then $z+k-4$ is adjacent to $x_0$ and by the previous observation we have that $(z+k-4, x_0) \in A$ ($k-3 \geq 3$ since $k \geq 6$) and then $\mathcal{C}_{k-2} = (x_0, z) \cup \langle z, \gamma, z + k - 4\rangle \cup (z + k - 4, x_0)$ is a cycle of $T$ with $\mathcal{I}(\mathcal{C}_{k-2}) \geq k-5$. Contradicting our initial assumption. Thus the previous claim holds.

Let $x_5$ be the first vertex in $\langle 2, \gamma, x_3 - 1\rangle$ such that $(x_0, x_5) \in A$ (such a vertex exists by the previous Claim). Notice that for each $y \in \langle 2, \gamma, x_5 - 1\rangle$ if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. Now consider $x_5 - i - 3$; since $l\langle x_1, \gamma, x_4 - i - 3\rangle + l\langle x_4, \gamma, x_3\rangle = k - 4$ we have that $l\langle x_1, \gamma, x_5 - i - 3\rangle + l\langle x_5, \gamma, x_3\rangle = k - 4$. Consider now the two possible cases:

**Case 1.** $x_5 - i - 3$ is adjacent to $x_0$. By the definition of $x_5$ we have that $(x_5 - i - 3, x_0) \in A$. Let $f_1 = (x_3 - 1, x_1)$, $f_2 = (x_3, x_1)$, and $f_3 = (x_3 + 1, x_1)$. Since $(x_1, x_3)$ is not a quasichord, we have that at least one $f_i \in A$, $i = 1, 2, 3$.

**Case 1.1.** $f_1 \in A$. Let $\mathcal{C}_{k-2}$ the cycle of $T$ induced by $\{f_1, g_1, h\}$ where $g_1 = (x_5 - i - 3, x_0)$ and $h = (x_0, x_5)$. Clearly $\mathcal{I}(\mathcal{C}_{k-2}) = k - 5$. A contradiction.

**Case 1.2.** $f_2 \in A$. Let $\mathcal{C}_{k-1}$ the cycle of $T$ induced by $\{f_2, g_1, h\}$. Clearly $\mathcal{I}(\mathcal{C}_{k-1}) = k - 4$. A contradiction.

**Case 1.3.** $f_3 \in A$. Let $\mathcal{C}_k$ the cycle of $T$ induced by $\{f_3, g_1, h\}$. Clearly $\mathcal{I}(\mathcal{C}_k) = k - 3$. A contradiction.

**Case 2.** $x_5 - i - 3$ is not adjacent to $x_0$. Clearly we have that $x_5 - i - 4$ is adjacent to $x_0$. By the definition of $x_5$ we have that $(x_5 - i - 4, x_0) \in A$. Let $g_2 = (x_5 - i - 4, x_0)$. Since $(x_1, x_3)$ is not a quasichord we have the following three possibilities.

**Case 2.1.** $f_1 = (x_3 - 1, 0 = x_1) \in A$. Let $\mathcal{C}_{k-3}$ the cycle of $T$ induced by $\{f_1, g_2, h\}$. Clearly $\mathcal{I}(\mathcal{C}_{k-3}) = k - 6$. A contradiction.

**Case 2.2.** $f_2 = (x_3, x_1) \in A$. Let $\mathcal{C}_{k-2}$ the cycle of $T$ induced by $\{f_2, g_2, h\}$. Clearly $\mathcal{I}(\mathcal{C}_{k-2}) = k - 5$. A contradiction.

**Case 2.3.** $f_3 = (x_3 + 1, x_1) \in A$. Let $\mathcal{C}_{k-1}$ the cycle of $T$ induced by $\{f_3, g_2, h\}$. Clearly $\mathcal{I}(\mathcal{C}_{k-1}) = k - 4$. A contradiction.

**Proof of ($\beta$)**

Part ($\beta$ ) follows from Lemma 6.3 (taking $i + 1$ instead of $i$) and the following facts.

- Every $(k - 2 + i')$-ordered pair is a quasichord $(-1 \leq i' \leq i)$. It follows from the induction hypothesis.
- Every $(k - 1 + i')$-ordered pair is a quasichord $(-1 \leq i' \leq i)$. It follows from part ($\alpha$).

- Every $(-r)$-ordered pair and every $-(r+1)$-ordered pair is a quasichord. It follows from Lemma 6.2.

□

**Theorem 6.8.** *If $n \geq 2k-4$ then there exists a cycle $\mathcal{C}_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-3$.*

*Proof.* The case of $n = 2k-4$ is considered in Section 4. Assume that $n > 2k-4$ and assume for contradiction that there is no such cycle.

It follows from Lemma 6.7 that for each $i$, $-1 \leq i \leq r$, every $-(2r+2-i)$-ordered pair and every $(k-2+i)$-ordered pair is a quasichord . In particular all the following pairs are quasichords of $T$:

$$\{(0, k-3), (0, k-2), (0, k-1), (0, k), \ldots, (0, k+r-2)\}. \tag{1}$$

It follows from Lemma 6.2 that every $(-r)$-ordered pair is a quasichord, and by Lemma 6.7 that every $k$-ordered pair is a quasichord (taking $i = 2$). Since $r < k-4$, there exists integers $\alpha > 0$ such that $0 < \alpha(r+1) < k-3$. Let $\alpha_0 = \max\{\alpha \in \mathcal{N} \mid \alpha(r+1) < k-3\}$. Let $y = (\alpha_0+1)(r+1)$. It is easy to prove that $y \in \{k-3, k-2, \ldots, k+r-2\}$. By Lemma 6.6, $(y, 0)$ is a quasichord. On the other hand,(1) implies that $(0, y)$ is also a quasichord of $T$. It follows that 0 and $y$ are not adjacent andconsequently $(y, 1)$ and $(y, n-1)$ are arcs of $T$.Suppose first that $y > k-3$. As $k-3 \leq y-1 \leq k+r-3$, by (1), $(1, y)$ is a quasichord andas 1 and $y$ are adjacent, it follows $(1, y) \in A$, a contradiction with $(y, 1) \in A$.Suppose now that $y = k-3$. Since $k-3 < k-2 < k+r-2$, by (1) the $(k-2)$-orderedpair $(n-1, y)$ is a quasichord and as $n-1$ and $y$ are adjacent, it follows $(n-1, y) \in A$, a contradiction with $(y, n-1) \in A$. □

**Remark 2.** *The bound of this theorem is tight, since it coincides with the upper bounds proved in [8] for tournaments (multipartite with one vertex per part) and in [6] for bipartite tournaments, where it is proved that in some cases there are no cycles with larger intersection with $\gamma$.*

*Also, in general it is not possible to prove the result of this theorem for a set of options smaller than four (those of the set $\{k-3, k-2, k-1, k\}$). To see this, consider the cyclically 4-partite tournaments defined as follows. A tournament $T$ has vertices $V(T) = V_0 \cup V_1 \cup V_2 \cup V_3$ and $(x, y)$ is an arc of $T$ iff $x \in V_i$, $y \in V_{i+1}$ (modulo 4), for $i \in \{0, 1, 2, 3\}$. If $k \equiv i \pmod 4$ then all the cycles of $T$ have length congruent with $k-i \pmod 4$.*

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