AKCE J. Graphs. Combin., 6, No. 2 (2009), pp. 307-313

Odd-Graceful Labelings of Trees of Diameter 5

Christian Barrientos
Department of Mathematics
Clayton State University
Morrow, GA 30260, USA
e-mail: *chr_barrientos@yahoo.com*

Communicated by: Joseph Gallian
Received 9 June 2008; accepted 22 April 2009

Abstract

A difference vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy the weight $|f(x) - f(y)|$. A difference vertex labeling f of a graph G of size n is odd-graceful if f is an injection from $V(G)$ to $\{0, 1, ..., 2n - 1\}$ such that the induced weights are $\{1, 3, ..., 2n - 1\}$. We show here that any forest whose components are caterpillars is odd-graceful. We also show that every tree of diameter up to five is odd-graceful.

Keywords: Odd-graceful labeling, α-labeling, trees of diameter 5.

2000 Mathematics Subject Classification: Primary 05C78

1. Introduction

Let G be a graph of order m and size n, a *difference vertex labeling* of G is an assignment f of labels to the vertices of G that induces for each edge xy a label or *weight* given by the absolute value of the difference of its vertex labels. Graceful labelings are a well-known type of difference vertex labeling; a function f is a *graceful labeling* of a graph G of size n if f is an injection from $V(G)$ to the set $\{0, 1, ..., n\}$ such that, when each edge xy of G has assigned the weight $|f(x) - f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1, 2, ..., n\}$. A graph that admits a graceful labeling is said to be *graceful*.

When a graceful labeling f of a graph G has the property that there exists an integer λ such that for each edge xy of G either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, f is named an α-labelings and G is said to be an α-graph. From the definition it is possible to deduce that an α-graph is necessarily bipartite and that the number λ (called the *boundary value* of f) is the smaller of the two vertex labels that yield the edge with weight 1. Some examples of α-graphs are the cycle C_n when $n \equiv 0 \pmod 4$, the complete bipartite graph $K_{m,n}$, and caterpillars (i.e., any tree with the property that the removal of its end vertices leaves a path).

A little less restrictive than α-labelings are the odd-graceful labelings introduced by Gnanajothi in 1991 [4]. A graph G of size n is *odd-graceful* if there is an injection $f: V(G) \to \{0, 1, 2, ..., 2n-1\}$ such that the set of induced weights is $\{1, 3, ..., 2n-1\}$. In this case, f is said to be an *odd-graceful labeling* of G. One of the applications of these labelings is that trees of size n, with a suitable odd-graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{n,n}$. In Figure 1 we show an odd-graceful tree of size 6 together with its embedding in the circular arrangement used to produce the cyclic decomposition of $K_{6,6}$. Once the labeled tree has been embedded, successive 60° (counterclockwise) rotations produce the desired cyclic decomposition of $K_{6,6}$.

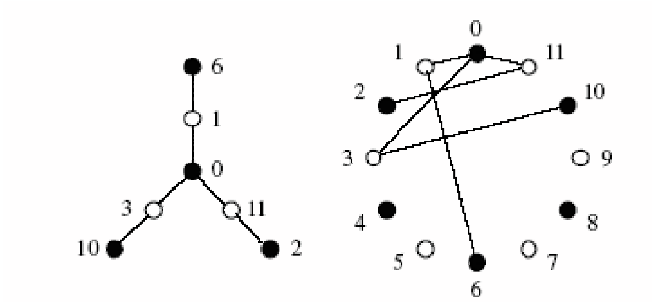

Figure 1: Cyclic decomposition of $K_{6,6}$

Gnanajothi [4] proved that the class of odd-graceful graphs lies between the class of α-graphs and the class of bipartite graphs; she proved that every α-graph is also odd-graceful. The reverse case does not work, for example the odd-graceful tree shown in Figure 1 is the smallest tree without an α-labeling. Since many families of α-graphs are known, the most attractive examples of odd-graceful graphs are those without an α-labeling or where an α-labeling is unknown; for instance, Gnanajothi [4] proved that the following are odd-graceful graphs: C_n when $n \equiv 2 \pmod 4$, the disjoint union of C_4, the prism $C_n \times K_2$ if and only if n is even, and trees of diameter 4 among others. Eldergil [2] proved that the one-point union of any number of copies of C_6 is odd-graceful. Seoud, Diab, and Elsakhawi [5] showed that a connected n-partite graph is odd-graceful if and only if $n = 2$ and that the join of any two connected graphs is not odd-graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian' survey [3].

Gnanajothi [4] conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. The author has extended this up to trees with order up to 12[1]. In this paper we prove that all trees of diameter 5 are odd-graceful and that any forest whose components are caterpillars is odd-graceful.

[1] Odd-graceful labelings of trees of order 11 and 12 can be found at http://cims.clayton.edu/cbarrien/research

2. Odd-Graceful Forests

In this section we study $forests$ that accept odd-graceful labelings. Recall that a forest with more than one component cannot be graceful because it has "too many vertices". Gnanajothi [4] proved that every α-graph is odd-graceful. In fact, let G be an α-graph of size n. Suppose that f is an α-labeling of G such that $\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}$, where $\{A, B\}$ is the bipartition of $V(G)$. An odd-graceful labeling of G is given by

$$g(x) = \begin{cases} 2f(x), & x \in A \\ 2f(x) - 1, & x \in B. \end{cases}$$

In Figure 2 we show an example of an α-labeling of a caterpillar of size 10, together with its corresponding odd-graceful labeling. We use these labelings in the proof of Theorem 1.

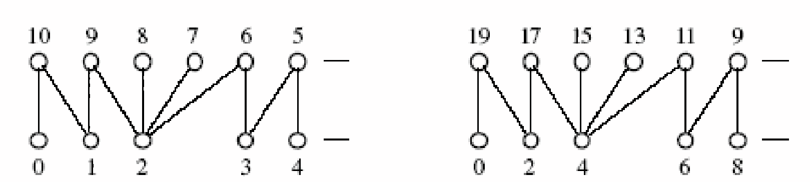

Figure 2: Odd-graceful labeling of a caterpillar

Theorem 1. *Any forest whose components are caterpillars is odd-graceful.*

Proof. Let F_i be a caterpillar of size $n_i \geq 1$, for $1 \leq i \leq k$. Let $u_i, v_i \in V(F_i)$ such that $d(u_i, v_i) = diam(F_i)$; so identifying v_i with u_{i+1}, for each $1 \leq i \leq k-1$, we have a caterpillar F of size $\sum_{i=1}^{k} n_i = n$. Now we proceed to find both, the α-labeling of F and its corresponding odd-graceful labeling, using the scheme shown in Figure 2. Once the odd-graceful labeling has been obtained, we disengage each caterpillar F_i from F, keeping their labels; in this form, the weights induced are $\{1, 3, ..., 2n-1\}$. To eliminate the overlapping of labels we subtract 1 from each vertex label of F_i when i is even, in this way the weights remain the same and the labels assigned on u_{i+1} and v_i differ by one unit. Therefore, the labeling of the forest $\bigcup_{i=1}^{k} F_i$ is odd-graceful. □

In Figure 3 we show an example of this construction using the odd-graceful labeling obtained in Figure 2.

The procedure used in this proof can be extended to the disjoint union of graphs with α-labelings. In fact, suppose that the concatenation of blocks $B_1, B_2, ..., B_k$ results in a graph G whose block-cutpoint graph is a path; in [1] we proved that G is an α-graph

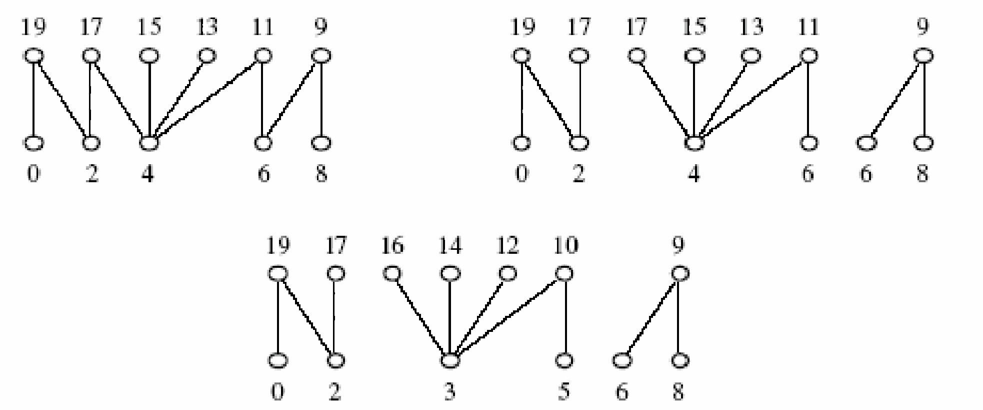

Figure 3: Odd-graceful labeling of a forest

provided that each B_i is an α-graph. Transforming this α-labeling into an odd-graceful labeling and disconnecting G into blocks, the disjoint union of these blocks is odd-graceful.

Theorem 2. *The disjoint union of blocks that accept α-labelings is odd-graceful.*

As consequence, any forest which components are α-trees is odd-graceful.

3. Odd-Graceful Trees of Diameter Five

Every tree of diameter at most 3 is a caterpillar, therefore it is odd-graceful. Gnanajothi [4] proved that every rooted tree of height 2 (that is, diameter at most 4) is odd-graceful. In the next theorem, we represent trees of diameter 5 as rooted trees of height 3 and prove that they are odd-graceful.

Let T be a tree of diameter 5; T can be represented as a rooted tree of height 3 by using any of its two central vertices as the root vertex. Note that only one of the vertices in level 1 has descendants in level 3; this vertex will be located in the right extreme of level 1. Now, within each level, the vertices are placed from left to right in such a way that their degrees are increasing. In the proof of the next theorem we use this type of representation of T, that is, assuming that v (one of the two central vertices) is the root.

Theorem 3. *All trees of diameter five are odd-graceful.*

Proof. Let T be a tree of diameter 5 and size n. Suppose that T has been drawn according to the previous description. Let $v_{i,j}$ denote the ith vertex of level j, for $j = 1, 2, 3$; this vertex is placed at the right of $v_{i+1,j}$. Consider the labeling f of the vertices within each level given by recurrence as follows: $f(v) = 0$, where v is the root of T, $f(v_{1,1}) = 2n - 2\deg(v) + 1$, $f(v_{1,2}) = 2$, $f(v_{1,3}) = 3$, now the labels are set for the initial vertices of each level and $f(v_{i,j}) = f(v_{i-1,j}) + d(v_{i,j}, v_{i-1,j})$ where $i \geq 2$, $1 \leq j \leq 3$, and $d(v_{i,j}, v_{i-1,j})$ represents the distance between the vertices $v_{i,j}$ and $v_{i-1,j}$.

We claim that f is an odd-graceful labeling of T. In fact, let us see that there is no overlapping of labels. On level 0 the label used is 0 and on level 2 all labels are even, being 2 the smallest label used here. On levels 1 and 3 the labels used are odd; on level 1 the labels used are $2n-1, 2n-3, ..., 2n-2\deg(v)+1$, while on level 3 the labels used are $3, 5, ...$ We want to prove that the largest label on level 3 is less than the smallest label on level 1.

Suppose that k is the number of vertices on level 3; thus the weights on level 3 edges are $1, 3, ..., 2k-1$; if $v_{t,2}$ is the last son of $v_{1,1}$ that has sons on level 3, then the weight $2k-1$ must be obtained on the edge $v_{t,2}v_{k,3}$. Since $f(v_{k,3}) = f(v_{t,2})+2k-1 \leq 2\deg(v_{1,1})+2k-3$, we claim that $f(v_{k,3}) < f(v_{1,1})$. In fact, since $\deg(v)+\deg(v_{1,1}) < n-k+2$, we may conclude that $2\deg(v_{1,1})+2k-3 < 2n-2\deg(v)-1$. Hence, the largest label on level 3 is less than the smallest label on level 1, which implies that there is no overlapping of labels.

As a consequence of the fact that labels used in consecutive levels have different parity, each weight obtained is an odd number not exceeding $2n-1$. Suppose that $v_{i+1,j}$ and $v_{i,j}$ have the same father x, by definition of f, the edges $xv_{i+1,j}$ and $xv_{i,j}$ have consecutive weights. If $v_{i+1,j}$ and $v_{i,j}$ have different father, x and y, respectively, then $|f(y) - f(v_{i,j})| = |(f(x)+2) - (f(v_{i+1,j})+4)| = |f(x) - f(v_{i+1,j}) - 2|$. Thus, on level 1 the weights are $2n-1, ..., 2n-2\deg(v)+1$, on level 2, the weights are $2n-2\deg(v)-1, ..., 2k+1$, and on level 3 the weights are $2k-1, ..., 1$.

Therefore, f is an odd-graceful labeling of T. $\square$

In Figure 4 we present a scheme of this labeling for a tree of size 23.

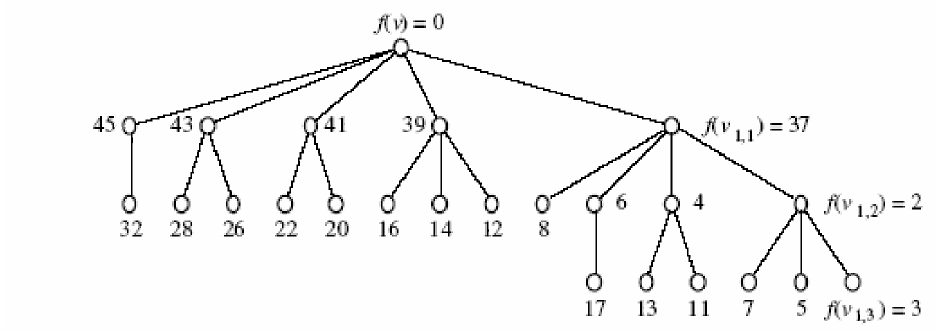

Figure 4: Odd-graceful tree of diameter 5

Similar arguments can be used to find odd-graceful labelings of trees of diameter 6; however we do not have a general labeling scheme for this case. So it is an open problem determining whether trees of diameter 6 are odd-graceful. In Figure 5, we give an example of an odd-graceful labeling for a tree of size 17 and diameter 6.

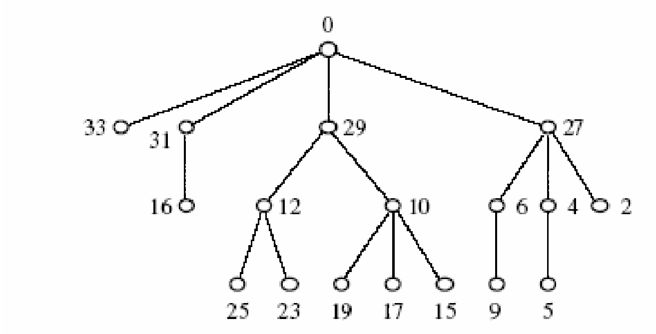

Figure 5: Odd-graceful tree of diameter 6

To conclude this section, we show in Figure 6 an odd-graceful labeling for a special type of tree of diameter 8, namely the star $S(n,4)$ with n spokes of length 4. The deletion of the vertices in the last row produces the star $S(n,3)$, a graceful labeling of this tree is obtained by subtracting $2n$ from the labels on the odd-numbered levels.

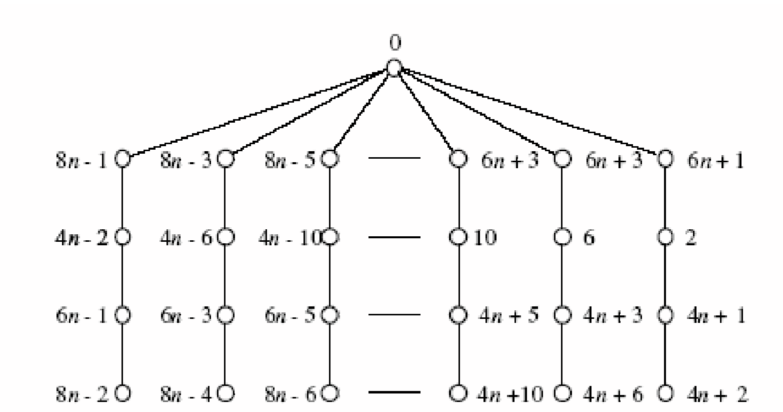

Figure 6: Odd-graceful labeling of the star $S(n,4)$

Acknowledgements

The author thanks Professor Mary Hudachek-Buswell for her help in the preparation of the manuscript. The author also thanks to referees for their fine job in reviewing the original version of this paper and their valuable observations.

References

[1] C. Barrientos, Graceful labelings of chain and corona graphs, *Bull. Inst. Combin. Appl.*, **34**(2002), 17-26

[2] P. Eldergill, *Decomposition of the Complete Graph with an Even Number of Vertices*, M. Sc. Thesis, McMaster University, 1997

[3] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combinatorics*, (2008), #DS6

[4] R.B. Gnanajothi, *Topics in Graph Theory*, Ph. D. Thesis, Madurai Kamaraj University, 1991

[5] M.A. Seoud, A.E.I. Abdel Maqsoud, and E.A. Elsahawi, On strongly-C harmoniuos, relatively prime, odd graceful and cordial graphs, *Proc. Math. Phys. Soc. Egypt*, **73**(1998), 33-55