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Holes in $L(2,1)$-Coloring on Certain Classes of Graphs

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Abstract

The channel assignment problem is the problem of efficiently assigning frequencies to radio transmitters located at various places such that communications do not interfere. Griggs and Yeh [5] introduced a variation of the channel assignment problem known as the L(2,1)-colorings of graphs. An $L(2,1)$-*coloring* of a graph $G = (V, E)$ is a *vertex coloring* $f : V(G) \rightarrow \{0, 1, 2, \ldots, k\}$, $k \geq 2$ such that $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. The *span* of G, $\lambda(G)$, is the smallest integer k for which G has an L(2,1)-coloring. An L(2,1)-coloring f is *irreducible* if there does not exist an L(2,1)-coloring g such that $g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$. A *span coloring* is an L(2,1)-coloring whose greatest color is $\lambda(G)$. Let f be an L(2,1)-coloring that uses colors from 0 to k. Then $h \in (0, k)$ is a *hole* if there is no vertex v in $V(G)$ such that $f(v) = h$. In this paper, we investigate maximum number of holes in span colorings of certain classes of graphs. We give exact values for the maximum number of holes in a span coloring of a path, cycle, star, complete bipartite graph and characterize complete graphs in terms of their maximum number of holes. Upper bounds for an arbitrary graph and other classes of graphs are also given.

Keywords: Channel assignment problems; $L(2,1)$-colorings; full-colorings; holes.

2000 Mathematics Subject Classification: 05C12

1. Introduction

The channel assignment problem is the problem of efficiently assigning frequencies to radio transmitters located at various places such that communications do not interfere.

In 1988, F. S. Roberts [9] proposed (in a private communication with Griggs) the problem of efficiently assigning radio channels to transmitters at several locations, using non-negative integers to represent channels, so that close locations receive different integers, and channels of very close locations are at least 2 apart. This evolved into the study of L(2,1)-coloring of a graph first studied by Griggs and Yeh [5].

Let G be a simple graph with a non-empty, finite vertex set. More rigorously, an *L(2,1)-coloring* of $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2, \ldots, k\}$, $k \geq 2$ such that $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$.

The *span* of G, $\lambda(G)$ or simply λ, is the smallest integer k for which G has an $L(2,1)$-*coloring*. That is,

$$\lambda(G) = \min\{\max f(u) \ : \ u \in V(G), \text{ f is an } L(2,1)\text{-coloring}\}.$$

A span coloring is an $L(2,1)$-coloring whose greatest color is $\lambda(G)$.

The maximum label used by an $L(2,1)$-coloring f on a graph G, will be called the span of f and will be denoted by $span(f)$.

An $L(2,1)$-coloring f is *irreducible* if there does not exist an $L(2,1)$-coloring g such that $g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$.

An $L(2,1)$-coloring f is a full-coloring if $f : V(G) \to \{0, 1, 2, \ldots, \lambda(G)\}$ is onto.

Let f be an $L(2,1)$-coloring of a graph G that uses colors from 0 to k. Then, the integer h is a hole in f, if $h \in (0, k)$ and there is no vertex v in G such that $f(v) = h$. The index of G, denoted by $\rho(G)$, is the minimum number of colors not used in a span coloring of G. An $L(2,1)$-coloring using colors from 0 to k is a no-hole coloring if it uses all the labels from 0 to k. Fishburn and Roberts [4] introduced the parameter $\mu(G)$, which is defined to be the minimum integer k for which G has a no-hole coloring, if it exists. An inh-coloring is defined as an irreducible no-hole coloring.

Define $f(V) = \{k | f(u) = k, u \in V(G)\}$. Let $V_j = \{v \in V(G) : f(v) = j\}$ and $H = \{j \in (0, \lambda(G)) | \, |V_j| = 0\}$ for all j. That is, $h \in H$ implies that h is a hole in the span coloring f.

With motivation from the channel assignment problem, Fishburn and Roberts [4], Sakai [10], Cozzens and Wang [2] studied the question of what graphs are no-hole colorable while Laskar and Villalpando [6, 7], Villalpando [11] and Laskar, Matthews, Novick and Villalpando [8] studied inh-colorable graphs. With the same motivation, the question of determining the maximum number of holes in a span coloring of a graph G could be seen as a question of determining the minimum number of different frequencies required for interference free communications in a given network. In this paper, we study maximum number of holes in the span colorings of certain classes of graphs. If an irreducible span coloring f of a graph G has the maximum number of holes over all irreducible span colorings of G, then f shall be called a *maximum-hole coloring*.

2. Background

We present a few examples using the path P_5 to illustrate some of the basic definitions in $L(2,1)$-coloring and present some previous results that are used in this paper.

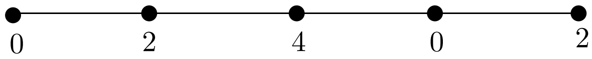

Figure 1: Irreducible span coloring with holes: holes = 1,3.

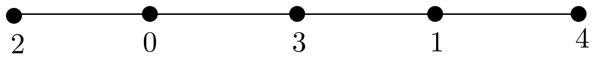

Figure 2: Irreducible full coloring (also an inh-coloring).

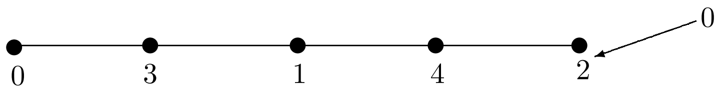

Figure 3: Reducible full coloring.

Proposition 1. [5] *Let P_n be a path on $n \geq 2$ vertices. Then,*

$$\lambda(P_n) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3, 4 \\ 4 & \text{if } n \geq 5. \end{cases}$$

Proposition 2. [5] *Let C_n be a cycle on $n \geq 3$ vertices. Then $\lambda(C_n) = 4$.*

Theorem 1. [5] *If T is a tree with maximum degree $\Delta \geq 1$, then $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$.*

Proposition 3. [4] *If G is not full colorable and $n \geq \lambda(G)+2$, then $\mu(G) \leq \lambda(G)+\rho(G)$.*

Corollary 1. [4] *G has a no-hole coloring if and only if $n \geq \lambda(G) + 1$.*

Theorem 2. [1] *If G is a graph with maximum degree Δ and $\lambda(G) = \Delta + 1$, then for any span coloring of G, a vertex of degree Δ , must be labeled 0 (or $\Delta + 1$) and its neighbors must be labeled $2 + i$ (or i), $i = 0, 1, \ldots, \Delta - 1$.*

3. Maximum Number of Holes in Paths, Cycles and Trees

Let $H_\lambda(G)$ denote the maximum number of holes in an irreducible span coloring of G and H the corresponding set of holes.

3.1. Paths

Lemma 1. [4], [11] *If G is a graph with span λ, then*

(i) $0, \lambda \notin H$ *and*

(ii) $h \in H$ *implies that h-1, h+1* $\notin H$.

Our first result is on paths P_n with n vertices.

For $n = 2$, $f(P_2) = \{0, 2\}$. Therefore, $\rho(P_2) = H_\lambda(P_2) = 1$.

For $n = 3$, $f(P_3) = \{0, 1, 3\}$ giving $\rho(P_3) = H_\lambda(P_3) = 1$.

Since P_4 is full colorable with span 3, $\rho(P_4) = H(P_4) = 0$.

We certainly cannot continue such listings. The following theorem considers all paths P_n with $n > 4$.

Theorem 3. *Let P_n be a path on n vertices, $n > 4$. Then*

(i) $\rho(P_n) = 0$ *and*

(ii) $H_\lambda(P_n) = 2$.

Proof. (i) All paths P_n, $n > 4$, are full colorable, thus $\rho(P_n) = 0$.

(ii) Order the vertices of P_n as $v_1, v_2, \dots, v_n$ where v_1 is the first vertex, v_n the last vertex and for every m, $m = 1, 2, \dots, n-1$, $v_m v_{m+1} \in E(P_n)$.

Consider the coloring f on $V(G)$ defined by:

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3, \\ 2 & \text{if } k \equiv 2 \pmod 3, \\ 4 & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Then, f defined as above is an irreducible span coloring on P_n with two holes, namely, 1 and 3. Thus $H_\lambda(P_n) \geq 2$.

To prove the reverse inequality, we recall from Lemma 1(i) that $0, 4 \in f(P_n)$ since $\lambda(P_n) = 4$ for $n > 4$. So, the only other labels left to be used in a span coloring of P_n are $1, 2$ and 3. By Lemma 1(ii), all these labels cannot be holes. That is, at least one of these labels must be in $f(P_n)$. Thus $H_\lambda(G) \leq 2$, and we conclude that $H_\lambda(P_n) = 2$. □

So, the minimum number of frequencies required for interference-free communications for transmitters placed in a linear corridor and distance being within m miles is 3. We now turn our focus on cycles.

3.2. Cycles

Lemma 2. *Let C_n be a cycle on n vertices and let f be an irreducible span coloring on C_n. Then there exists $v \in V(C_n)$ such that $f(v) = 1$ if and only if there exists $u \in V(C_n)$ such that $f(u) = 3$.*

Proof. We first note that every vertex in $V(C_n)$ is of degree two. Suppose $v \in V(C_n)$ such that $f(v) = 1$. Then, labels 0, 1, and 2 cannot be used for neighbors of v. So one of the neighbors of v must be labeled 3 since $\lambda(C_n) = 4$. This proves the forward direction.

Conversely, suppose there exists a vertex $u \in V(C_n)$ such that $f(u) = 3$. We again observe that the labels 2, 3 and 4 cannot be used for neighbors of u. Thus one of the neighbors of u must be labeled 1 proving the converse and hence the lemma. □

We now investigate the maximum number of holes in span colorings of cycles C_n on n vertices.

For $n = 3$, we have a K_3. So, $f(C_3) = \{0, 2, 4\}$ with $\rho(C_3) = H_\lambda(C_3) = 2$.

For $n = 4$, Figure 4 shows that $f(C_4) = \{0, 1, 3, 4\}$ giving $\rho(C_4) = H_\lambda(C_4) = 1$.

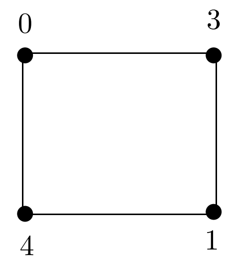

Figure 4: Maximum-hole coloring for C_4

Let $h_i(G)$ be the minimum number of holes over all irreducible $L(2,1)$-colorings of G.

The following theorem considers all cycles C_n with $n > 4$.

Theorem 4. *Let C_n be a cycle on n vertices, $n > 4$. Then*

(i) $\rho(C_n) = 0$ *except for* C_6,

(ii) $h_i(C_6) = 0$,

(iii) *if* $n \equiv 0 \pmod 3$, *then* $H_\lambda(C_n) = 2$,

(iv) *if* $n \not\equiv 0 \pmod 3$ *and* $n \equiv 0 \pmod 4$, *then* $H_\lambda(C_n) = 1$ *and*

(v) *if* $n \not\equiv 0 \pmod 3$ *and* $n \not\equiv 0 \pmod 4$ *then* $H_\lambda(C_n) = 0$.

Proof. (i) All cycles C_n, $n > 4$, are full colorable except for C_6. Hence, $\rho(C_n) = 0$ for all $n > 4$. In fact, $\rho(C_6) = 2$.

(ii) C_6 is inh-colorable as shown in Figure 5. Hence, $h_i(C_6) = 0$.

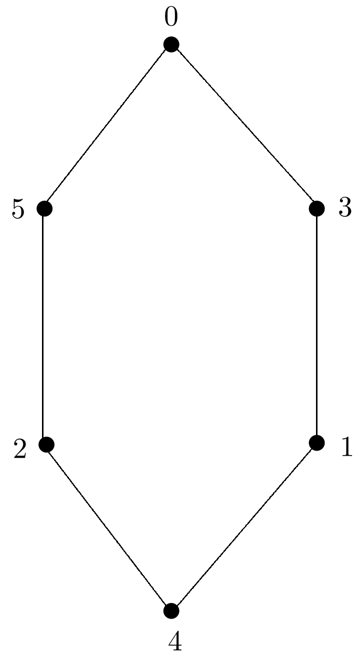

Figure 5: inh-coloring of C_6

(iii) Suppose $n \equiv 0 \pmod 3$. Order the vertices of C_n as $v_1, v_2, \dots, v_n$ where $v_n v_1 \in E(C_n)$ and for every m, $m = 1, 2, \dots, n-1$, $v_m v_{m+1} \in E(C_n)$. Define a coloring f on C_n as follows:

$$f(v_k) = \begin{cases} 2 & \text{if } k \equiv 1 \pmod 3, \\ 0 & \text{if } k \equiv 2 \pmod 3, \\ 4 & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Then, f is an irreducible span coloring on C_n with two holes, namely, 1 and 3. Thus $H_\lambda(C_n) \geq 2$.
Using the same arguments we used in proving the reverse inequality of Theorem 3, we see that $H_\lambda(C_n) \leq 2$, proving part (iii) of the theorem.

(iv) Suppose $n \not\equiv 0 \pmod 3$ but $n \equiv 0 \pmod 4$. We investigate two cases here.

Case 1. $n \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 4$.

From Lemma 1 and Lemma 2, we have only 3 possible sets of labels available for coloring any cycle on n vertices, $n > 4$. The sets are: $S_1 = \{0, 2, 4\}$ with two holes,
$S_2 = \{0, 1, 3, 4\}$ with one hole and
$S_3 = \{0, 1, 2, 3, 4\}$ with no holes.

We first show that S_1 cannot be used to color C_n in this case. Consider the ordering of the vertices of C_n described in the proof of part (iii) above. Attempting to color C_n by alternating the labels 0, 2 and 4, we realize that $f(v_n) = f(v_1)$ contradicting the definition of an $L(2,1)$-coloring. Thus, S_1 cannot be used to color C_n in this case. Therefore, $H_\lambda(C_n) < 2$ in this case. That is, $H_\lambda(C_n) \leq 1$.

We now investigate S_2. Again, considering the ordering of the vertices of C_n described in the proof of part (iii) above, we define a coloring f on C_n by

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 4, \\ 3 & \text{if } k \equiv 2 \pmod 4, \\ 1 & \text{if } k \equiv 3 \pmod 4, \\ 4 & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

Then, f defined as above is an irreducible span coloring of C_n with one hole. That is, $H_\lambda(C_n) \geq 1$ and we thus conclude that $H_\lambda(C_n) = 1$ for this case.

Case 2. $n \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 4$.

Following the ordering of the vertices of C_n above, attempting to color C_n by alternating 0, 2 and 4, we realize that $f(v_{n-1}) = f(v_1)$ and $f(v_n) = f(v_2)$ contradicting the definition of an $L(2,1)$-coloring. So, S_1 cannot be used to color C_n in this case. The same explanation in case 1 above proves that S_2 can be used to color C_n in this case. Thus, if $n \not\equiv 0 \pmod 3$ and $n \equiv 0 \pmod 4$, then $H_\lambda(C_n) = 1$.

(v) Suppose $n \not\equiv 0 \pmod 3$ and $n \not\equiv 0 \pmod 4$. In part (iv), we showed that if $n \not\equiv 0 \pmod 3$, then S_1 cannot be used to color C_n. All we have to show here is that, if $n \not\equiv 0 \pmod 4$, then S_2 cannot also be used to color C_n. Consider the ordering of the vertices of C_n described in part (iii) above. Color consecutive groups of 4 vertices by alternating the labels 0, 3, 1 and 4 starting with any of the labels. We notice that:

(a) for $n \equiv 1 \pmod 4$, $f(v_n) = f(v_1)$ contradicting the definition of an $L(2,1)$-coloring,

(b) for $n \equiv 2 \pmod 4$, $f(v_n) = f(v_2)$ also contradicting the definition of an $L(2,1)$-coloring and

(c) for $n \equiv 3 \pmod 4$, $|f(v_n) - f(v_1)| = 1$ also contradicting the definition of an $L(2,1)$-coloring.

Thus, S_2 cannot be used to label C_n in this case. Since C_n is full colorable for all $n > 4$ except for C_6, we conclude that only S_3 can be used to color C_n in this case, and hence, $H_\lambda(C_n) = 0$. □

Having found exact values for the maximum number of holes in span colorings of paths and cycles, we now investigate the maximum number of holes in span colorings of trees.

3.3. Trees

Lemma 3. *Let T be a star on n vertices, with $\Delta \geq 3$. Then $H_\lambda(T) = 1 = \rho(T)$.*

Proof. For a star T, $\lambda(T) = 1 + \Delta$. Let v be the central vertex with $f(v) = m$. Since v is a vertex of maximum degree, by Theorem 2, $m = 0$ or $m = \Delta + 1$. Thus, the neighbors of v would be labeled $2, 3, \ldots, 1 + \Delta$ or $0, 1, 2, \ldots, \Delta - 1$ respectively, giving $H_\lambda(T) = \rho(T) = 1$. □

Theorem 5. *Let T be a tree that is not a star with $\Delta \geq 2$. Then*

(i) $\lambda(T) = 1 + \Delta \Rightarrow H_\lambda(T) \leq 1$ *and*

(ii) $\lambda(T) = 2 + \Delta \Rightarrow H_\lambda(T) \leq 2$.

Proof. By Theorem 1, we have $\lambda(T) = 1 + \Delta$ or $\lambda(T) = 2 + \Delta$. We investigate both cases.

(i) Suppose $\lambda(T) = 1 + \Delta$. By Theorem 2 the maximum degree vertices must be 0 or $1 + \Delta$. Let v be a vertex of maximum degree in T. Consider the star $T' \subseteq T$ centered at v. By Lemma 3, $H_\lambda(T') = 1$. Since $\lambda(T') = \lambda(T)$ and $T' \subseteq T$, we see that every hole in T must be a hole in T'. That is, $H_\lambda(T) \leq H_\lambda(T') = 1$ and so $H_\lambda(T) \leq 1$.

(ii) Suppose $\lambda(T) = 2 + \Delta$. In this case, there are no restrictions for the labels of the maximum degree vertices. Let $T' \subseteq T$ be a star centered at some maximum degree vertex v. Set $f(v) = k$ where $k \neq 0$, $k \neq 1 + \Delta$ and $k \neq 2 + \Delta$. In labeling the neighbors of T', we must avoid only the labels $k - 1$, k and $k + 1$ creating two holes namely, $k - 1$ and $k + 1$ with the maximum label in T' being $2 + \Delta$. This coloring is clearly irreducible. In labeling the other vertices of T, the most we can do is to maintain these two holes since the maximum label in T' is $2 + \Delta = \lambda(T)$. Thus, $H_\lambda(T) \leq 2$. □

Remark 1. *From the proof of Theorem 5 above, we observe that the only possible scenario to have $H_\lambda(T) = 2$ is when $\lambda(T) = \Delta + 2$. But interestingly enough, we have not been able to find a tree T with span $\Delta + 2$ such that $H_\lambda(T) = 2$ other than the path P_n, $n > 4$. This remark motivates the following conjecture which is a characterization of trees in terms of the maximum number of holes.*

Conjecture 1. *Let T be a tree that is not a star. Then $H_\lambda(T) = 2$ if and only if $T = P_n$, $n > 4$.*

4. Maximum Number of Holes in Some Other Classes of Graphs

Theorem 6. *Let G be a simple graph on n vertices, $n > 1$. Then $H_\lambda(G) = n - 1$ if and only if $G = K_n$.*

Proof. Suppose $G = K_n$. K_n has a unique $L(2,1)$ optimum labeling f with $f(K_n) = \{0, 2, 4, \dots, 2n-2)\}$ and hence $\lambda(K_n) = 2n-2$ and the holes are the first $n-1$ odd numbers. This proves one part.

To prove the converse part, we apply induction on n. The result is obvious for $n = 2$. So assume that the result is true for graphs of order $n-1$. Let G be a graph of order n with $H_\lambda(G) = n-1$. Let v be any vertex of G. Now no graph on $n-1$ vertices can have more than $n-2$ holes in a λ-labeling (and in this case the labels must be $0, 2, \cdots, 2n-4$). Hence $G - v$ must be complete and this forces G also to be complete since otherwise $H_\lambda(G) = H_\lambda(G-v) = n-2$, a contradiction. This proves the converse and hence the theorem. □

Corollary 2. *Let G be a graph on n vertices. Then* $0 \leq H_\lambda(G) \leq n-1$.

Proof. Let $G = C_n$ where $n \not\equiv 0 \pmod 3$ and $n \not\equiv 0 \pmod 4$. Then by Theorem 4, $H_\lambda(G) = 0$, showing it is tight, since obviously $H_\lambda(G) \geq 0$ for any graph G. The second inequality follows directly from Theorem 6 since $G \subseteq K_n$. □

Proposition 4. *Let $K_{n,m}$ be a complete bipartite graph. Then* $H_\lambda(K_{n,m}) = 1$.

Proof. It is easy to see that $\lambda(K_{n,m}) = n+m$. There are $n+m+1$ labels in $[0, n+m]$ and $n+m$ vertices in $K_{n,m}$. Since no two vertices can have the same label, we observe that every span coloring of $K_{n,m}$ must have exactly $(n+m+1)-(n+m) = 1$ hole. Thus, $H_\lambda(K_{n,m}) = 1$. □

Corollary 3. *Let $K_{n_1,n_2,\dots,n_r}$ be a complete r-partite graph. Then* $H_\lambda(K_{n_1,n_2,\dots,n_r}) = r-1$.

Proof. This follows directly from Proposition 4 and by observing that $\lambda(K_{n_1,n_2,\dots,n_r}) = \sum_{i=1}^{r} n_i + (r-2)$. □

Corollary 4. *For every $m \geq 0$, there is a simple graph G with $H_\lambda(G) = m$.*

Proof. Let $m \geq 0$. Define G as the complete $(m+1)$-partite graph, $G = K_{n_1,n_2,\dots,n_{m+1}}$. Then by Corollary 3, $H_\lambda(G) = (m+1)-1 = m$. □

The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Let $\overline{G}$ be the complement of G. We know from Theorem 6 that $H_\lambda(G) \leq n-1$ and $H_\lambda(\overline{G}) \leq n-1$. So, $H_\lambda(G) + H_\lambda(\overline{G}) \leq 2(n-1)$. But $V(G) = V(\overline{G})$ and $E(G) + E(\overline{G}) = E(K_n)$. Could it be that $H_\lambda(G) + H_\lambda(\overline{G}) \leq n-1$ with equality only when $G = K_n$? This bound holds true for all the examples we have looked at so far but the proof is not yet settled. We state it here as a conjecture.

Conjecture 2. (Nordhaus-Gaddum type bound) *For any graph G with span $\lambda(G)$, $H_\lambda(G) + H_\lambda(\overline{G}) \leq n-1$, where $\overline{G}$ is the complement of G.*

Conjecture 3. (Generalized Nordhaus-Gaddum type bound) *If* G_1 *and* G_2 *are edge-disjoint graphs on the same vertex set, then* $H_\lambda(G_1) + H_\lambda(G_2) \leq H_\lambda(G_1 + G_2)$, *where the* $+$ *on the right stands for edge-disjoint union.*

5. Open Problems

In addition to the three conjectures already given, we propose two more problems.

Problem 1. [3] *Given a graph* G *with span* $\lambda(G)$, *develop an algorithm that computes the maximum number of holes in a span coloring of* G.

Problem 2. [3] *Is it possible to characterize certain classes of graphs in terms of* $H_\lambda(G)$*?*

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