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On Exponential Domination of $C_m \times C_n$

Mark Anderson*, Robert C. Brigham†, Julie R. Carrington*,
Richard P. Vitray* and Jay Yellen*

* Department of Mathematical Sciences, Rollins College, Winter Park, FL 32789

† Department of Mathematics, University of Central Florida, Orlando, FL 32816

e-mail: *JCarrington@rollins.edu*

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Abstract

An exponential dominating set of graph $G = (V, E)$ is a subset $D \subseteq V$ such that $\sum_{w \in D} (\frac{1}{2})^{d(v,w)-1} \geq 1$ for every $v \in V$, where $d(v, w)$ is the distance between vertices v and w. The exponential domination number, $\gamma_e(G)$, is the smallest cardinality of an exponential dominating set. Lower and upper bounds for $\gamma_e(C_m \times C_n)$ are determined and it is shown that $\lim_{m,n \to \infty} \frac{\gamma_e(C_m \times C_n)}{mn} \leq \frac{1}{13}$. Two connections are also established between exponential domination and distance-2 domination: (a) If D is an exponential dominating set of the infinite grid graph such that no two vertices in D are closer than distance 5, then D is a distance-2 dominating set; and (b) For sufficiently large m and n, every distance-2 dominating set of $C_m \times C_n$ is an exponential dominating set.

Keywords: Domination, exponential domination, grid graph.

2000 Mathematics Subject Classification: 05C69

1. Introduction

Given a graph $G = (V, E)$, a subset $D \subseteq V$, and a non-increasing function $\Gamma : \{0, 1, 2, ...\} \to [0, \infty)$, we say that D Γ-*dominates* G if, for every vertex $v \in V$, $\sum_{w \in D} \Gamma(d(v, w)) \geq 1$, where $d(v, w)$ is the distance between v and w. For the familiar distance-k domination, the function is $\Gamma(i) = 1$ for $n \leq k$ and $\Gamma(i) = 0$ otherwise. Dankelmann, et al. [1] introduce the concept of *exponential domination* for which the function Γ is defined by $\Gamma(i) = (\frac{1}{2})^{i-1}$, so that the domination contribution to a vertex falls exponentially as the dominator's distance increases. The *exponential domination number* of G, written $\gamma_e(G)$, is the cardinality of a smallest set D such that $\sum_{w \in D} (\frac{1}{2})^{d(v,w)-1} \geq 1$ for every $v \in V$. It has been suggested [1] that exponential domination is a model for the reliability of the spread of information or gossip. The assumption is that gossip heard directly from a source is totally reliable, while gossip passed from person to person loses half its credibility with each individual in the chain. Finding the exponential domination number in this application amounts to determining the minimum number of sources needed so that each person gets fully reliable information. The authors of [1] distinguish between *porous exponential domination*, where $d(u, v)$ is the length of the shortest $u - v$ path in G, and *nonporous exponential domination*, where $d(u, v)$ is the length of the shortest $u - v$ path in $V(G) - (D - \{u, v\})$. In this paper we consider only porous exponential domination and refer to it simply as exponential domination. Excellent references on the subject of domination are the two books by Haynes, Hedetniemi, and Slater [2, 3].

The graphs we consider are the finite grid graphs, $C_m \times C_n$, the Cartesian products of two cycles, where, without loss of generality, we assume $m \le n$. We identify the vertices with ordered pairs from the set $Z_m \times Z_n$ in the natural way. We also consider the *infinite grid graph*, $C_\infty \times C_\infty$, whose vertices are labeled with pairs from $Z \times Z$. In the figures representing graphs in this paper, we think of a vertex as being the center point of a square, two vertices being adjacent if the squares about them share an edge. A *sphere with radius* r about a vertex x is the set $S_r(x)$ of vertices whose distance from x is exactly r, a *ball with radius* r about x is the set $B_r(x)$ of vertices whose distance from x is at most r, and an *annulus with radii* r *and* R about x is the set $A_{r,R}(x)$ of vertices whose distance from x is at least r and at most R.

Observation 1. *(a) For $C_\infty \times C_\infty$, $|S_r(x)| = 4r$, for all $r \ge 1$; and (b) For $C_m \times C_n$, $|S_r(x)| = 4r$, for all $r \le \lfloor \frac{m-1}{2} \rfloor$; and $|S_r(x)| < 4r$, otherwise.*

2. A Lower Bound

To obtain a lower bound on $\gamma_e(C_m \times C_n)$, we consider the total exponential domination created by a single dominating vertex in $C_\infty \times C_\infty$, taken over the entire infinite grid graph. By Observation 1(a), a dominator in $C_\infty \times C_\infty$ contributes to the entire domination of the graph a total of exactly

$$\sum_{r=0}^{\infty} (4r) \left(\frac{1}{2}\right)^{r-1} = 18.$$

Using the above result permits determination of a rough lower bound for $\gamma_e(C_m \times C_n)$.

Theorem 2. *For all m and n, $\frac{\gamma_e(C_m \times C_n)}{mn} > \frac{1}{18}$.*

Proof. The computation above and Observation 1(b) imply that if D exponentially dominates $C_m \times C_n$, then

$$mn = |V| \le \sum_{v \in V} \left[\sum_{w \in D} (\tfrac{1}{2})^{d(v,w)-1}\right] = \sum_{w \in D} \left[\sum_{v \in V} (\tfrac{1}{2})^{d(v,w)-1}\right] < 18|D|. \qquad \square$$

We can increase this bound slightly by noticing that each vertex in the dominating set and all of its neighbors are over-dominated.

Theorem 3. *For all $m, n \ge 3$, $\frac{\gamma_e(C_m \times C_n)}{mn} > \frac{1}{15.875}$.*

Proof. Let D be a set that exponentially dominates the graph $C_m \times C_n$, $w = (x, y) \in D$, $a = (x+1, y+1)$, and $b = (x+1, y)$. Since w contributes only $\frac{1}{2}$ to the domination of a, other members of D must also contribute at least $\frac{1}{2}$, and thus at least $\frac{1}{4}$ to the domination of b. Therefore, $\sum_{w \in D} (\frac{1}{2})^{d(b,w)-1} \geq \frac{5}{4}$. Hence, we may view the contribution of w towards the exponential domination of b as having an excess of at least $\frac{1}{4}$. Note that if $|D \cap B_1(b)| > 1$, then the contribution of each dominator in $D \cap B_1(b)$ towards the exponential domination of b has an even larger excess (at least $1 - \frac{1}{|D \cap B_1(b)|} \geq \frac{1}{2} > \frac{1}{4}$). The contribution of w towards the exponential domination of each of its other three neighbors also has an excess of at least $\frac{1}{4}$. Furthermore, other members of D must also contribute at least $\frac{1}{8}$ to the domination of w, in addition to the 2 that w contributes to its own domination. Thus, the contribution of w toward the exponential domination of the graph has an excess of at least $4(\frac{1}{4}) + \frac{9}{8} = \frac{17}{8}$ and hence $mn + \frac{17}{8}|D| = |V| + \frac{17}{8}|D| \leq \sum_{v \in V} \left[\sum_{w \in D} (\frac{1}{2})^{d(v,w)-1}\right] = \sum_{w \in D} \left[\sum_{v \in V} (\frac{1}{2})^{d(v,w)-1}\right] < 18|D|$. □

3. Values for Small Graphs and an Upper Bound

It is straightforward but tedious to determine $\gamma_e(C_m \times C_m)$ for small values of m. Table 1 lists these values for $2 \leq m \leq 8$. The third line of the table gives the ratio of the exponential domination number to the number of vertices in these graph, corresponding to the ratios given in Theorems 2 and 3.

	m						
	2	3	4	5	6	7	8
$\gamma_e\,(C_m \times C_m)$	2	2	2	3	4	5	6
$\frac{\gamma_e(C_m \times C_m)}{m^2}$	$\frac{1}{2}$	$\frac{1}{4.5}$	$\frac{1}{8}$	$\frac{1}{8.\overline{3}}$	$\frac{1}{9}$	$\frac{1}{9.8}$	$\frac{1}{10.\overline{6}}$

Table 1: Exponential Domination Numbers of Small Graphs

Figure 1 displays minimum exponential dominating sets of $C_m \times C_m$ for $2 \leq m \leq 8$, where a dominating vertex is indicated by an "X". Observe that in each of the graphs of Figure 1 the *minimum distance* of the dominating set, defined to be the minimum distance between distinct dominators, is less than 5. This doesn't have to be the case as the exponential dominating set of $C_{13} \times C_{13}$ shown in Figure 2 has minimum distance 5. The vertices in this set are the centers of 2-balls that form a partition of the set of vertices of the graph. This suggests a way of dominating large graphs where both m and n are multiples of 13, with one dominator for each 13 vertices, and leads to the following theorem.

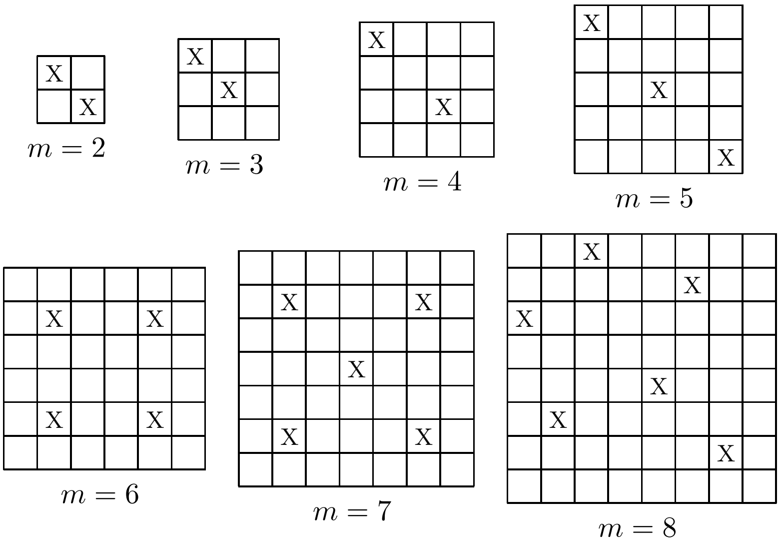


Figure 1: Minimum exponential dominating sets of $C_m \times C_m$, $2 \leq m \leq 8$

Theorem 4. *For any positive integers m and n,* $\frac{\gamma_e(C_{13m} \times C_{13n})}{(13m)(13n)} \leq \frac{1}{13}$.

Proof. In each block $\{13j, 13j+1, \ldots, 13j+12\} \times \{13k, 13k+1, \ldots, 13k+12\}$, with $0 \leq j < m$ and $0 \leq k < n$, arrange the dominators as in Figure 2. □

Theorem 4 can be extended to large graphs in a limiting sense, as shown by the following theorem.

Theorem 5. $\lim_{m,n\to\infty} \frac{\gamma_e(C_m \times C_n)}{mn} \leq \frac{1}{13}$.

Proof. Write $m = 13p + r$ and $n = 13q + s$, with $0 \leq r, s < 13$. In each block $\{13j, 13j + 1, \ldots, 13j + 12\} \times \{13k, 13k + 1, \ldots, 13k + 12\}$, with $0 \leq j < p$ and $0 \leq k < q$, arrange the dominators as in Figure 2. Let exponential dominating set D contain these vertices as well as the $13ps + 13qr + rs$ vertices not in any of these blocks. Then $\frac{\gamma_e(C_m \times C_n)}{mn} \leq \frac{13pq+13ps+13qr+rs}{(13p+r)(13q+s)} \leq \frac{13pq+13ps+13qr+rs}{13^2pq} = \frac{1}{13} + \frac{s}{13q} + \frac{r}{13p} + \frac{rs}{13^2pq}$. Since the limit of the last three terms as p and q become large is 0, we obtain the desired result. □

4. A Result for the Infinite Grid Graph

Figure 2 illustrates an exponential dominating set of $C_{13} \times C_{13}$ having minimum distance 5. Here we show that, in the infinite grid graph, any exponential dominating set of minimum distance at least 5 must, in fact, be of minimum distance exactly 5 and must be a distance-2 dominating set. Before stating and proving this theorem, we define a function and give a lemma supplying a bound for it.

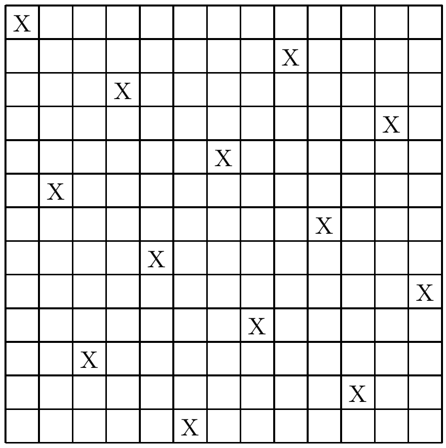


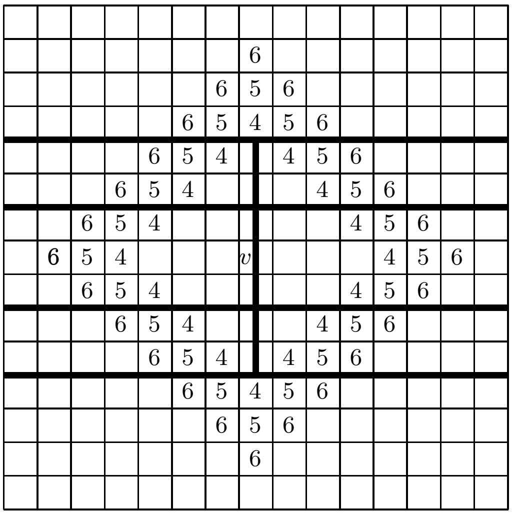
Figure 2: An exponential dominating set of $C_{13} \times C_{13}$

Definition 6. *Given the infinite grid graph G and a set D of vertices in G, define the function I_n for $n \geq 1$ by $I_n(v) = \sum_{w \in D - B_{n-1}(v)} (\frac{1}{2})^{d(v,w)-1}$ for any vertex v of G.*

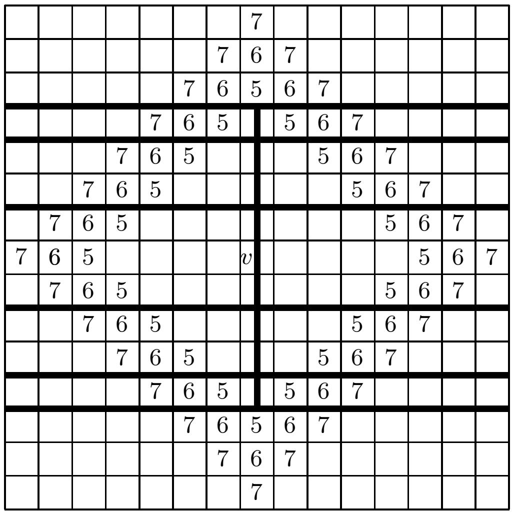
Notice that $I_n(v)$ gives that contribution to the exponential domination of v by all vertices of D of distance n or more from v.

Lemma 7. *With the notation of the preceding definition, if the minimum distance between vertices in D is at least 5, then for any vertex v, $I_n(v) \leq \frac{35n+36}{147 \cdot 2^{n-4}}$.*

Proof. Let v be a vertex in G and assume $v = (0,0)$. Consider the annulus $A_{n,n+2}(v)$. We give lines which partition the vertices in the annulus so that the maximum distance between any two vertices in a set of the partition is 4, and thus at most one vertex of each set can be in D. When n is even, we use the horizontal lines $y = \pm(1.5 + 2i)$, where $0 \leq i \leq \frac{n}{2} - 1$. If n is odd, we use the same horizontal lines along with $y = \pm\,(n - .5)$ and the vertical line segment from $(0, -(n - .5))$ to $(0, n - .5)$. Figure 3 illustrates this partitioning for $n = 4$ and $n = 5$, where the number in a vertex position refers to the distance of that vertex from v. There are $2n$ sets when n is even and $2n+2$ when n is odd. It follows that there can be at most $2n + 2$ members of D in the annulus. There are $4n$ vertices in the sphere $S_n(v)$. and at most every third one of these can be in D. If all other vertices in D are in $S_{n+1}(v)$, a situation that maximizes the contribution to the exponential domination of v by vertices in the annulus, the total exponential domination on v by the vertices of D in that annulus is at most $\left(\frac{4n}{3}\right)\left(\frac{1}{2^{n-1}}\right) + \left(2n + 2 - \frac{4n}{3}\right)\left(\frac{1}{2^n}\right) = \left(\frac{10n}{3} + 2\right)\left(\frac{1}{2^n}\right)$. Thus $I_n(v) \leq \sum_{k=0}^{\infty} \left(\frac{10(n+3k)}{3} + 2\right)\left(\frac{1}{2^{n+3k}}\right) = \frac{35n+36}{147 \cdot 2^{n-4}}$. □



Partition of $A_{4,6}$



Partition of $A_{5,7}$

Figure 3: Partitions of $A_{n,n+2}$ for $n = 4$ and $n = 5$

The following values of I_n will be required below.

Corollary 8. $I_6(v) \leq \frac{41}{98} < \frac{1}{2}$, $I_7(v) \leq \frac{281}{1176} < \frac{1}{4}$, *and* $I_8(v) \leq \frac{79}{588} < \frac{9}{64}$.

We now present the main result of this section.

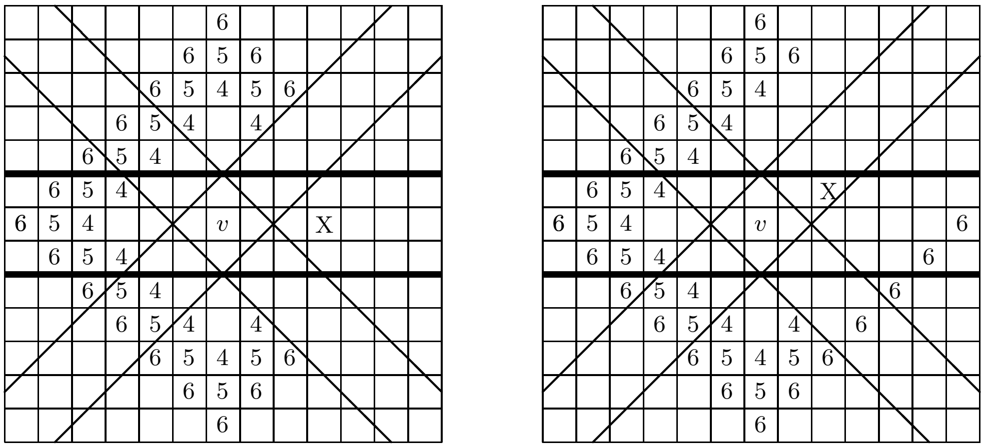
Theorem 9. *If D is a an exponential dominating set of the infinite grid graph G and the 2-balls centered on the members of D are disjoint, then these 2-balls partition the vertices of G.*

Proof. Suppose that D is an exponential dominating set of the infinite grid graph G such that the 2-balls about members of D are disjoint, and suppose these 2-balls do not partition the vertices of G. Notice that disjointness requires the distance of D is at least 5. Of all the vertices in G not within distance 2 of D, let v be one with a minimum number of dominators at distance 3. We assume that $v = (0, 0)$.

Let v have three or four vertices in D at distance 3. If all of these dominators are at corners of $S_3(v)$, then two of them must be on adjacent corners. Suppose one of these vertices, say $(-2, -1)$, is not on a corner of $S_3(v)$. By ruling out vertices within distance 4 of $(-2, -1)$, we see that the adjacent corners $(0, 3)$ and $(3, 0)$ must be in D. Therefore we can assume without loss of generality that $(0, 3)$ and $(3, 0)$ are in D. This implies that $(1, 1)$ also has distance 3 from D, but only two dominators at distance 3, a contradiction to the choice of v. Hence, v has at most two vertices of D at distance 3.

Suppose there is exactly one dominator X with distance 3 from v. By symmetry, we may assume X is either (3,0) or (2,1). Figure 4 illustrates both cases. Here the distance figures are given only for vertices of distance 4, 5, or 6 from v that also are distance at least 5 from

X, that is, those vertices of distance 6 or less from v that remain eligible to be in D. These vertices are partitioned in two ways, with the vertices within a set of either partition all within distance 4 of each other. The partition formed by the horizontal lines shows that there can be at most three vertices in D distance 4 from v. If $X = (3,0)$, the partition formed by the diagonal lines shows there can be at most five vertices with distance 4, 5, or 6 from v. Thus, the total domination of D on v is at most $\frac{1}{4}+\frac{3}{8}+\frac{2}{16}+I_7(v) < 1$. Suppose, then, that $X = (2,1)$. Again, there can be at most three vertices with distance 4 from v, and if there are, then there can be no vertices with distance 5 and at most one of distance 6. In this case, the total exponential domination of v is at most $\frac{1}{4}+\frac{3}{8}+\frac{1}{32}+I_7(v) < 1$. If there are two vertices with distance 4, there can be two of distance 5 and two of distance 6 and the total domination of v is at most $\frac{1}{4}+\frac{2}{8}+\frac{2}{16}+\frac{2}{32}+I_7(v) < 1$. If there are fewer than two vertices of distance 4, there can be at most five of distance 4 or 5 and at most seven of distance 4, 5, or 6. So the total exponential domination is at most $\frac{1}{4}+\frac{1}{8}+\frac{4}{16}+\frac{2}{32}+I_7(v) < 1$.



$X = (3,0)$ $X = (2,1)$

Figure 4: Exactly one vertex X in D with distance 3 from v

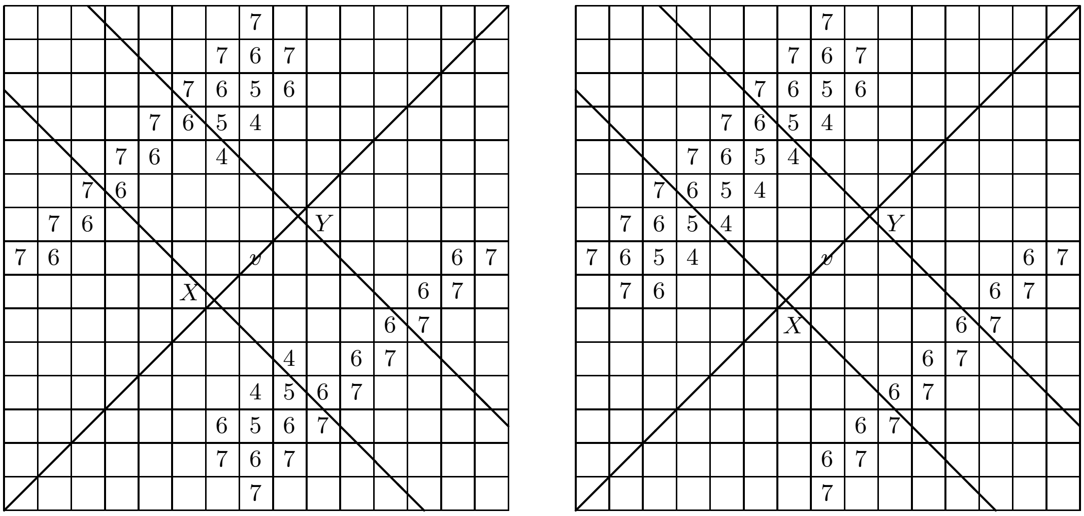
Suppose there are no vertices in D of distance 3 from v. Examining the vertices of distance 4 and 5 from v, we see that D could have (1) four vertices with distance 4 and no vertices with distance 5 from v, (2) three vertices with distance 4 and two of distance 5, or (3) fewer than three vertices of distance 4 and a total of at most six. In all three cases the total exponential domination of D on v is at most $\frac{1}{2}+I_6(v) < 1$.

The only remaining case is when every vertex v not within distance 2 of D has exactly two vertices X and Y in D at distance 3. Due to the symmetry of the graph about the vertex v, there are only six cases to consider for the pair (X,Y):

$$((-3,0),(3,0)),((-2,-1),(3,0)),$$

$$((0,-3),(3,0)),((-1,-2),(3,0)),$$
$$((-2,-1),(2,1)),((-1,-2),(2,1)).$$

Suppose $X \in \{(-3,0),(-2,-1)\}$ and $Y = (3,0)$ and consider $w = (0,1)$. Vertex w also has distance at least 3 from D, but is within distance 3 of at most one vertex in D, a contradiction to the fact that all vertices with minimum distance 3 to D have exactly two such vertices in D. Furthermore, if $(X,Y) = ((0,-3),(3,0))$, then $w = (1,-1)$ also has these same two vertices in D with distance 3 and so this case will follow from the case $(X,Y) = ((-1,-2),(2,1))$. Similarly, $(X,Y) = ((-1,-2),(3,0))$ will follow from $(X,Y) = ((-2,-1),(2,1))$ with $w = (1,-1)$. Two cases remain to be considered: $(X,Y) = ((-2,-1),(2,1))$ and $(X,Y) = ((-1,-2),(2,1))$, both of which are illustrated in Figure 5.



$X = (-2,-1)$ and $Y = (2,1)$ $X = (-1,-2)$ and $Y = (2,1)$

Figure 5: Two cases with two dominators of distance 3

Vertices within distance 7 of v that could be in D, given that X and Y are in D, are labeled in the figure with their distances from v. The lines $y = -x + 2.5$, $y = -x - 2.5$, and $y = x$ partition these vertices into six groups. In each group, all vertices are within a distance 4 of each other, and hence there can be at most six members of D selected from these vertices, one from each group.

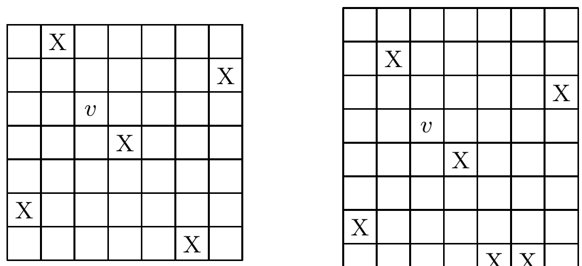
Suppose $(X,Y) = ((-2,-1),(2,1))$. Restricting our attention to the vertices (x,y) where $y > x$, we see that there can be at most one such vertex in D with distance 4 or 5 from v and that, if there is a vertex with distance 4 from v, there can be at most one vertex with distance 6 from v. Hence the maximum influence these three groups can have on v is with vertices of distances 4, 6, and 7 from v. The other three groups are symmetric to these three about v, and so the maximum exponential domination that $D \cap B_7(v)$ could have on v is $2(\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+\frac{1}{64})=\frac{54}{64}$. Therefore the maximum exponential domination all of D has on v is at most $\frac{54}{64}+I_8(v)<1$.

Finally, suppose $(X,Y)=((-1,-2),(2,1))$. Again, restricting our attention to the vertices where $y>x$, we see that there can be at most three such vertices in D with distance 4 or 5 from v, one with distance 4 and two with distance 5, or two of distance 4 and one of distance 7. Of these choices, the maximum influence these three groups can have on v is achieved with vertices of distances 4, 4, and 7 from v. The other three groups can, at best, have three vertices with distance 6 from v. Therefore, the maximum exponential domination that $D\cap B_7(v)$ could have on v is $\frac{2}{4}+\frac{2}{8}+\frac{3}{32}+\frac{1}{64}=\frac{55}{64}$, and hence the maximum exponential domination D has on v is at most $\frac{55}{64}+I_8(v)<1$.

Thus, in every case, there is a vertex that is not exponentially dominated by D. □

5. Distance-2 and Exponential Domination

The *distance-2 domination number* of G, written $\gamma_{\leq 2}(G)$, is the smallest cardinality of a distance-2 dominating set. The exponential dominating set of Figure 2 is also a distance-2 dominating set. It is not necessarily the case that a distance-2 dominating set is an exponential dominating set, or vice versa. It can be shown that every graph $C_m\times C_n$ with $m\leq 6$ and $n\geq m$ (with $n\geq 4$ when $m=1$) possesses a distance-2 dominating set that is not an exponential dominating set. Furthermore, Figure 6 shows this also is the case for $C_7\times C_7$ and $C_7\times C_8$, where v is a vertex that is not exponentially dominated. In all other cases, however, every distance-2 dominating set of $C_m\times C_n$ is also an exponential dominating set, as is shown in the following theorem.



$C_7\times C_7$ $C_7\times C_8$

Figure 6: Distance-2 but not exponential dominating sets of $C_7\times C_7$ and $C_7\times C_8$

Theorem 10. *Let $m,n\geq 8$ or $m=7$ and $n\geq 9$. Then every distance-2 dominating set of $C_m\times C_n$ is also an exponential dominating set of $C_m\times C_n$.*

Proof. Let v be an arbitrary vertex of $C_m\times C_n$, D be a distance-2 dominating set of $C_m\times C_n$, and X be a vertex in D whose distance from v is minimum. Without loss of generality, we assume $X=(0,0)$. If the distance between v and X is less than 2, then v

is exponentially dominated by D. If the distance between v and X is exactly 2, then, by symmetry of the graph, we can assume that $v \in \{(0,2),(2,0),(1,1)\}$.

Case 1. $v \in \{(0,2),(2,0)\}$.

The proof is given for $v = (0,2)$ with that for $v = (2,0)$ being virtually identical. If v is within distance 2 from a second dominator in D, then it is exponentially dominated by D. If not, the vertices $(-1,2)$ and $(1,2)$ must be within distance 2 from D and, in fact, must have distance exactly 2 from dominators Y and Z, neither of which is X. We know Y and Z are distinct since they are distance at least 3 from v and m and n are at least 7. These two dominators each have distance 3 from v, and so v is exponentially dominated by X, Y, and Z.

Case 2. $v = (1,1)$ and $m, n \geq 8$.

As in Case 1, we can assume that v is not distance 2 from a second dominator. Vertices $(1,2)$ and $(2,1)$ also must be distance 2 from D. If they have distinct dominators at distance 2, then v is exponentially dominated. Therefore, we assume there is one dominator Y with distance 2 to both. Y must be either $(2,3)$ or $(3,2)$ and, without loss of generality, we assume $Y = (3,2)$. Since m and n are both at least eight, the vertices in $R = \{(1,3),(2,-1),(-2,1)\}$ have distance at least 5 from each other. Thus they must lie in distinct 2-balls centered on distinct members Z, U, and W, respectively, of D. Furthermore, these three dominators also are distinct from X and Y since neither X nor Y is within distance 2 of R. Because the vertices in R have distances 2, 3, and 3 from v, the vertices Z, U, and W have distances of at most 4, 5, and 5 from v. It follows that the vertices in $\{X,Y,Z,U,W\}$ fully exponentially dominate v since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} = 1$.

Case 3. $v = (1,1)$, $m = 7$, and $n \geq 9$.

The vertices $(2,1)$ and $(1,2)$ must both be within distance 2 of D. If either has a distance less than 2 to D or if they have distance 2 from two distinct vertices in D, then v is exponentially dominated by D. Therefore, we assume that there is a vertex Y in D with distance 2 from both of them and distance 3 from v. Y must be either $(2,3)$ or $(3,2)$.

If $Y = (3,2)$, then, since $m = 7$ and $n \geq 9$, the vertices $(1,3)$, $(-2,1)$, and $(2,-1)$ each must have a distinct vertex, not X or Y, in D within distance 2. This means that, in addition to X and Y, there are vertices in D with distances at most 4, 5, and 5 from v and so v is fully exponentially dominated because $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} = 1$.

Suppose that $Y = (2,3)$. If there is another dominator Z in D with distance 3 from v, then X, Y, and Z fully exponentially dominate v, and so we assume that there is no such vertex. The vertices $(3,1)$ and $(1,-2)$ have distance 5 and so must have distinct vertices Z and W, respectively, in D with distance 2 or less. Also, because $n \geq 9$, the vertex $(-1,2)$ has distance 6 from $(1,-2)$ and so W is not within distance 2 of it. If Z also is not within distance 2 from $(-1,2)$, then v is fully exponentially dominated by X, Y, Z, W, and the vertex in D within distance 2 from $(-1,2)$, since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} = 1$. So, we assume that Z is within distance 2 of both $(3,1)$ and $(-1,2)$. Thus, Z is $(-3,2)$. If W is not within distance 2 of $(3,0)$, then v is dominated by X, Y, Z, W, and the vertex in D within a distance 2 of $(3,0)$, since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}=1$. Assuming $(3,0)$ is also within distance 2 of W, we see that $W=(3,-2)$. But now, vertices $(-1,3)$ and $(1,-3)$ have distance of at least 5 from each other since $n\geq 9$. It follows that X, Y, Z, W, and the two vertices in D within 2 of $(-1,3)$ and $(1,-3)$ (one for each) fully exponentially dominate v, since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}=1$. .

Since our choice of v was arbitrary, we conclude that every vertex in G is exponentially dominated by D, and therefore every distance-2 dominating set is also an exponential dominating set. □

Corollary 11. *Let $m,n\geq 8$ or $m=7$ and $n\geq 9$. Then $\gamma_e(C_m\times C_n)\leq\gamma_{\leq 2}(C_m\times C_n)$.*

6. Two Questions

Theorem 5 gives an upper bound of $\frac{1}{13}$ for the limit of $\frac{\gamma_e(C_m\times C_n)}{mn}$ as m and n get large. We think that this bound is actually a lower bound for all m and n, but have not been able to prove this. We present this as our first question in the form of a conjecture.

Conjecture 12. *For all m and n, $\frac{\gamma_e(C_m\times C_n)}{mn}\geq\frac{1}{13}$ and this bound is sharp (take $m=n=13$).*

In Theorem 10 we showed that for $C_m\times C_n$, with m and n sufficiently large, every distance-2 dominating set is also an exponential dominating set. We can easily create exponential dominating sets that are not distance-2 dominating sets. One simple way is to make every vertex a dominator except those within a 3-ball centered on some vertex. Figure 1 exhibits minimum exponential dominating sets for $C_7\times C_7$ and $C_8\times C_8$ that are not distance-2 dominating sets. However, for large such graphs we have been unable to find a minimum exponential dominating set which is not also a distance-2 dominating set, leading to the following open question.

Question 13. *Is it the case that for large values of m and n (greater than 8?), that every minimum exponential dominating set of $C_m\times C_n$ is also a distance-2 dominating set?*

References

[1] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi, and H. Swart, Domination with Exponential Decay, *Discrete Math.*, (To appear).

[2] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.