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Negative k-subdecision numbers in graphs

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Abstract

Let G be a simple connected graph without isolated vertex with vertex set $V(G)$ and edge set $E(G)$. A function $f : V(G) \to \{-1, 1\}$ is said to be a *negative k-subdecision function* of G if $\sum_{x\in N_G(v)} f(x) \le 1$ for at least k vertices v of G. The value $\max \sum_{x\in V(G)} f(x)$, taking over all negative k-subdecision functions f of G, is called the *negative k-subdecision number* of G and is denoted by $\beta_{kD}(G)$. In this paper we initiate the study of the negative k-subdecision numbers in graphs and present some bounds for this parameter.

Keywords: total k-subdominating function, total k-subdomination number, negative decision function, negative decision number, negative k-subdecision function, negative k-subdecision number

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [6] for terminology and notation which are not defined here. The open neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $\deg_G(u)$ (briefly $N(u)$ and $\deg(u)$ when

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no ambiguity on the graph is possible). The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, a *support vertex* of T is a vertex adjacent to a leaf and a *strong support vertex* of T is a vertex adjacent to at least two leaves.

For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a subset S of $V(G)$ we define $f(S) = \sum_{v\in S} f(v)$. A function $f : V(G) \longrightarrow \{-1, 1\}$ is called a *total k-subdominating function* (TkSF) of G, if $\sum_{u\in N_G(v)} f(u) \geq 1$ for at least k vertices v of G. The minimum of the values $f(V(G))$, taken over all total k-subdominating functions f of G, is called the *total k-subdomination number* of G and is denoted by $\gamma^t_{ks}(G)$. The total k-subdomination number was introduced by L. Harris et al. in [1]. If $k = n$, then the total k-subdomination number is called the *signed total domination number* . The signed total domination number was introduced by B. Zelinka in [7] and has been studied by several authors (see for example [2, 3, 4]).

In this paper, we initiate the study of the negative k-subdecision numbers in graphs by changing "$\geq$" to "$\leq$" in the definition of total k-subdomination number.

A function $f : V(G) \longrightarrow \{-1, 1\}$ is called a *negative k-subdecision function* (NkSDF) of G, if $\sum_{u\in N_G(v)} f(u) \leq 1$ for at least k vertices v of G. The maximum of the values $f(V(G))$, taken over all negative k-subdecision functions f of G, is called the *negative k-subdecision number* of G and is denoted by $\beta_{kD}(G)$. The negative k-subdecision function f of G with $f(V(G)) = \beta_{kD}(G)$ is called a *$\beta_{kD}(G)$-function.* For any negative k-subdecision function f of G we define $P = \{v \in V(G) \mid f(v) = 1\}$, $M = \{v \in V(G) \mid f(v) = -1\}$ and $B_f = \{v \in V(G) \mid \sum_{u\in N_G(v)} f(u) \leq 1\}$. We use $O(B_f)$ for the set of all odd vertices in B_f.

If $k = n$, then the negative k-subdecision function is called the *negative decision function* and the negative k-subdecision number is called the *negative decision number.* The negative decision number was introduced by C. Wang in [5] and denoted by $\beta_D(G)$. Wang proves that:

Theorem A. *If G is a graph of order n with minimum degree at least 2, then*

$$\beta_D(G) \leq n + 1 - \sqrt{4n + 1}.$$

Theorem B. *If G is a graph of order n and size m with minimum degree at least 2, then*

$$\beta_D(G) \leq \frac{4m - 3n}{5}.$$

Theorem C. *For complete graphs K_n, $n \geq 3$,*

$$\beta_D(K_n) = \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 0 & \textit{if } n \textit{ is even.} \end{cases}$$

Theorem D. *For* $n \geq 2$,

$$\beta_D(P_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 2 \pmod 4 \\ 1 & \text{otherwise.} \end{cases}$$

Theorem E. *For* $n \geq 3$,

$$\beta_D(C_n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ -2 & \text{if } n \equiv 2 \pmod 4 \\ -1 & \text{otherwise.} \end{cases}$$

In this paper, we establish upper bounds on $\beta_{kD}(G)$ for a general graph. We also present a lower bound of $\beta_{kD}(T)$ for a tree, exact value of $\beta_{kD}(G)$ for some familiar graphs such as cycles, paths, complete graphs are found.

2. Upper bounds on the NkSDNs of graphs

In this section we find upper bounds for the negative k-subdecision number of a graph. We first present an upper bound for the NkSDN of a graph in terms of its order, its minimum and maximum degrees and k. Then we find an upper bound in terms of the degree sequence of the graph. We make use of the following observation in the proofs.

Observation 1. *Let* f *be an NkSDF of* G *and let* $v \in B_f$. *Then* $f(N(v)) \leq 0$ *if* $\deg(v)$ *is even and* $f(N(v)) \leq 1$ *if* $\deg(v)$ *is odd.*

Theorem 2. *Let* G *be a graph of order* n. *If* f *is a* $\beta_{kD}(G)$*-function, then*

$$\beta_{kD}(G) \leq \frac{t - k\Delta + 3n\Delta - 2n\delta}{\delta},$$

where t *is the number of vertices of odd degree in* B_f, $\Delta = \Delta(G)$ *and* $\delta = \delta(G)$. *Furthermore, the bound is sharp for cycles when* k *is even.*

Proof. Let $N = \sum_{v\in V(G)} \sum_{u\in N(v)} f(u)$. Obviously, N counts the value $f(u)$ exactly $\deg(u)$ times for each $u \in V(G)$, so $N = \sum_{u\in V(G)}(\deg(u))f(u)$. Let $O(B_f)$ denote the set of all odd vertices in B_f, hence $|O(B_f)| = t$. By Observation 1,

$$\begin{aligned} N &= \sum_{v\in O(B_f)} f(N(v)) + \sum_{v\in B_f\setminus O(B_f)} f(N(v)) + \sum_{v\notin B_f} f(N(v)) \\ &\leq t + \sum_{v\notin B_f} f(N(v)) \\ &\leq t + \sum_{v\notin B_f} \Delta = t + (n - |B_f|)\Delta \\ &\leq t + (n-k)\Delta. \end{aligned} \tag{1}$$

Let P_Δ and P_δ be the sets of all vertices of P whose degrees are Δ and δ, respectively, and let $P_\lambda = P\setminus(P_\Delta \cup P_\delta)$. Suppose that $M = M_\Delta \cup M_\delta \cup M_\lambda$, where M_Δ, M_δ and M_λ are defined similarly. Define $V_i = P_i \cup M_i$, for each $i \in \{\Delta, \delta, \lambda\}$. Therefore $n = |V_\Delta|+|V_\delta|+|V_\lambda|$. Then for each $u \in V_\lambda$, $\delta + 1 \le \deg(u) \le \Delta - 1$. Now we have,

$$\begin{aligned} N &= \textstyle\sum_{u\in P_\Delta}\Delta + \sum_{u\in P_\delta}\delta + \sum_{u\in P_\lambda}\deg(u) \\ &\quad - \textstyle\sum_{u\in M_\Delta}\Delta - \sum_{u\in M_\delta}\delta - \sum_{u\in M_\lambda}\deg(u) \\ &= \Delta|P_\Delta| + \delta|P_\delta| - \Delta|M_\Delta| - \delta|M_\delta| + \textstyle\sum_{u\in P_\lambda}\deg(u) - \sum_{u\in M_\lambda}\deg(u) \\ &\ge \Delta|P_\Delta| + \delta|P_\delta| - \Delta|M_\Delta| - \delta|M_\delta| + (\delta+1)|P_\lambda| - (\Delta-1)|M_\lambda| \\ &= \Delta|V_\Delta| + \delta|V_\delta| + (\delta+1)|V_\lambda| - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta+\delta)|M_\lambda| \\ &\ge n\delta - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta+\delta)|M_\lambda|. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned} -2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta+\delta)|M_\lambda| &= 2\Delta|P_\Delta| + 2\delta|P_\delta| + (\Delta+\delta)|P_\lambda| - \\ &\quad 2\Delta|V_\Delta| - 2\delta|V_\delta| - (\Delta+\delta)|V_\lambda|. \end{aligned} \tag{3}$$

By (2) and (3),

$$\begin{aligned} N &\ge n\delta + 2\Delta|P_\Delta| + 2\delta|P_\delta| + (\delta+\Delta)|P_\lambda| - 2\Delta|V_\Delta| - 2\delta|V_\delta| - (\delta+\Delta)|V_\lambda| \\ &\ge n\delta + 2\delta(|P_\Delta| + |P_\delta| + |P_\lambda|) - 2\Delta(|V_\Delta| + |V_\delta| + |V_\lambda|) \\ &= n\delta + 2\delta|P| - 2n\Delta. \end{aligned} \tag{4}$$

By (1) and (4),

$$2\delta|P| + (\delta - 2\Delta)n \le N \le t - k\Delta + n\Delta,$$

and so

$$2|P| \le \frac{t - k\Delta + 3n\Delta - n\delta}{\delta}.$$

Now the result follows from $\beta_{kD}(G) = 2|P| - n$. □

The next result gives an upper bound on the negative k-subdecision number of a graph in terms of its degree sequence.

Theorem 3. *Let G be a graph of order n and with degree sequence $d_1 \le d_2 \le ... \le d_n$. Let f be a $\beta_{kD}(G)$-function. Then,*

$$\beta_{kD}(G) \le \frac{t - \sum_{i=1}^{k} d_i}{d_n} + n,$$

where t is the number of vertices of odd degree in B_f. This bound is sharp for cycles if k is even.

Proof. Let $g : V(G) \to \{-1, 0\}$ be the function defined by $g(v) = \frac{f(v)-1}{2}$, for all vertices $v \in V(G)$. Let $N = \sum_{v \in B_f} \sum_{u \in N(v)} g(u)$. By Observation 1,

$$\begin{aligned} N &= \sum_{v \in B_f} \sum_{u \in N(v)} \frac{f(u)-1}{2} \\ &= \frac{1}{2}(\sum_{v \in B_f} f(N(v)) - \sum_{v \in B_f} \deg(v)) \\ &= \frac{1}{2}(\sum_{v \in O(B_f)} f(N(v)) + \sum_{v \in B_f \setminus O(B_f)} f(N(v)) - \sum_{v \in B_f} \deg(v)) \\ &\le \frac{1}{2}(t - \sum_{v \in B_f} \deg(v)) \le \frac{1}{2}(t - \sum_{i=1}^{k} d_i). \end{aligned} \tag{5}$$

Since $g(v) \le 0$ for each vertex $v \in V(G)$,

$$\begin{aligned} N &= \sum_{v \in B_f} \sum_{u \in N(v)} g(u) \ge \sum_{v \in V(G)} \sum_{u \in N(v)} g(u) \\ &= \sum_{u \in V(G)} \deg(u) g(u) \ge d_n \sum_{u \in V(G)} g(u) = d_n g(V(G)). \end{aligned} \tag{6}$$

By (5) and (6),

$$g(V(G)) \le \frac{t - \sum_{i=1}^{k} d_i}{2d_n}.$$

Now the result follows because $\beta_{kD}(G) = f(V(G)) = 2g(V(G)) + n \le \frac{t - \sum_{i=1}^{k} d_i}{d_n} + n.$ □

As an immediate consequence of Theorems 2 or 3 we have the following result.

Corollary 4. *For every r-regular ($r \ge 1$) graph G,*

$$\beta_{kD}(G) \le \begin{cases} \frac{n(r+1)}{r} - k & \text{if } r \text{ is odd} \\ n - k & \text{if } r \text{ is even.} \end{cases}$$

Corollary 5. *If G is a graph of order n and size m, then*

$$\beta_{kD}(G) \leq \frac{t-2m}{\Delta} + 2n - k,$$

where t is the number of vertices of odd degree in B_f.

Proof. Let $(d_1, d_2, ..., d_n)$ be the degree sequence of G, where $d_1 \leq d_2 \leq ... \leq d_n$. By Theorem 3,

$$\begin{aligned}\beta_{kD}(G) &\leq \frac{1}{\Delta}(t - \textstyle\sum_{i=1}^{k} d_i) + n = \frac{1}{\Delta}(t - 2m + \sum_{i=k+1}^{n} d_i) + n \\ &\leq \frac{1}{\Delta}(t - 2m + (n-k)\Delta) + n = \frac{t-2m}{\Delta} + 2n - k.\end{aligned}$$

□

3. Trees

In this section we give a lower bound for the negative k-subdecision number of a tree. For this propose first we find a lower bound for $\beta_D(T)$. It is easy to see that:

Proposition 6. *For $n \geq 2$,*

$$\beta_{kD}(K_{1,n-1}) = \begin{cases} n & \text{if } k \leq n-1 \\ 1 & \text{if } k = n \text{ and } n \text{ is odd} \\ 2 & \text{if } k = n \text{ and } n \text{ is even.} \end{cases}$$

Theorem 7. *For any tree T of order $n \geq 2$,*

$$\beta_D(T) \geq 0.$$

Furthermore, this bound is sharp for paths P_n when $n \equiv 0 \pmod 4$.

Proof. The proof is by induction on n. If $n = 3, 4$, then the result follows by Theorem D and Proposition 6. Now assume $n \geq 5$ and that the statement is true for any tree of order less than n. We consider five cases.

Case 1. T has a strong support vertex.

Let v be a strong support vertex of T and let v_1 and v_2 be two leaves adjacent to v. Assume $T' = T - \{v_1, v_2\}$ and f is a $\beta_D(T')$-function. By the inductive hypothesis, $\beta_D(T') = f(V(T')) \geq 0$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by $g(v_1) = 1, g(v_2) = -1$ and

$g(u) = f(u)$ otherwise. Obviously, g is an negative decision function (briefly NDF) of T and we have $\beta_D(T) \geq g(V(T)) = f(V(T')) \geq 0$.

Case 2. T has a vertex adjacent to at least two support vertices each of degree 2.

Suppose $v \in V(T)$ is adjacent to support vertices w_1 and w_2 with $\deg(w_1) = \deg(w_2) = 2$. Let v_i be the leaf adjacent to w_i for $i = 1, 2$. Assume $T' = T - \{v_1, w_1\}$ and f is a $\beta_D(T')$-function. By the inductive hypothesis, $\beta_D(T') = f(V(T')) \geq 0$. If $f(v) = 1$, then $f(v_2) = -1$ and the function $h : V(T) \longrightarrow \{-1, 1\}$ defined by

$$h(v_1) = h(v_2) = 1, h(v) = h(w_1) = -1 \text{ and } h(u) = f(u) \text{ if } u \in V(T) \setminus \{v, v_2, v_1, w_1\}$$

is an NDF of T and so $\beta_D(T) \geq h(V(T)) = f(V(T')) \geq 0$. If $f(v) = -1$, then the function $g : V(T) \longrightarrow \{-1, 1\}$ defined by

$$g(w_1) = -1, g(v_1) = 1 \text{ and } g(u) = f(u) \text{ if } u \in V(T')$$

is an NDF of T and $g(V(T)) = f(V(T')) \geq 0$, hence the result follows.

Case 3. T has two adjacent support vertices, one of degree 2 and one of degree 3.

Suppose that v_1 and v_2 are two adjacent support vertices, $\deg(v_1) = 3$ and $\deg(v_2) = 2$. By Case 1, we may assume T has no strong support vertices. Let w_i be the leaf adjacent to v_i for $i = 1, 2$. Assume $T' = T - \{v_1, v_2, w_1, w_2\}$ and f is a $\beta_D(T')$-function. By the inductive hypothesis, $\beta_D(T') = f(V(T')) \geq 0$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_1) = g(v_2) = -1,\ g(w_1) = g(w_2) = 1 \text{ and } g(u) = f(u) \text{ if } u \in V(T').$$

Obviously, g is an NDF of T and the result follows as before.

Case 4. T has a path $v_5v_4v_3v_2v_1$ such that $\deg(v_4) = \deg(v_3) = \deg(v_2) = 2$ and $\deg(v_1) = 1$.

If $n = 5$, then the result follows by Theorem D. Assume $n \geq 6$, $T' = T - \{v_1, v_2, v_3, v_4\}$ and f is a $\beta_D(T')$-function. By the inductive hypothesis, $\beta_D(T') = f(V(T')) \geq 0$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_1) = g(v_2) = 1, g(v_3) = g(v_4) = -1 \text{ and } g(u) = f(u) \text{ if } u \in V(T').$$

Obviously, g is an NDF of T and the result follows.

Case 5. T does not satisfy Cases 1, 2, 3 and 4.

Since $n \geq 5$, by assumptions, $d = \text{diam}(T) \geq 5$. Assume $v_1, v_2, \ldots, v_{d+1}$ is a diametral path in T. Then, by assumptions, $\deg(v_1) = 1$, $\deg(v_2) = \deg(v_3) = 2$ and $\deg(v_4) \geq 3$.

Suppose first that v_4 is a support vertex and w_1 is a leaf adjacent to v_4. Let $T' = T - \{v_1, v_2, v_3, w_1\}$ and let f be a $\beta_D(T')$-function. By the inductive hypothesis, $\beta_D(T') = f(V(T')) \geq 0$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_1) = g(w_1) = 1, g(v_2) = g(v_3) = -1 \text{ and } g(u) = f(u) \text{ if } u \in V(T').$$

Obviously, g is an NDF of T and we have $\beta_D(T) \geq g(V(T)) = f(V(T')) \geq 0$.

Now assume there is a path $v_4w_3w_2w_1$ where $w_3 \notin \{v_3, v_5\}$. Then $\deg(w_1) = 1$ and by assumptions $\deg(w_3) = \deg(w_2) = 2$. Suppose $T' = T - \{w_1, v_i \mid 1 \leq i \leq 3\}$ and f is a $\beta_D(T')$-function. If $f(v_4) = -1$, then define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_3) = g(w_1) = -1, g(v_1) = g(v_2) = 1 \text{ and } g(u) = f(u) \text{ if } u \in V(T').$$

Obviously, g is an NDF of T and by the inductive hypothesis, $\beta_D(T) \geq g(V(T)) = f(V(T')) \geq 0$. If $f(v_4) = 1$, then $f(w_2) = -1$. Now the function $h : V(T) \longrightarrow \{-1, 1\}$ defined by $h(v_1) = h(v_2) = 1$, $h(v_3) = h(v_4) = h(w_1) = -1$, $h(w_2) = 1$ and $h(u) = f(u)$ for $u \in V(T') \setminus \{v_4, w_2\}$ is an NDF of T and the result follows as before.

Finally, assume v_4 is adjacent to a support vertex w_2 and w_1 is the leaf adjacent to w_2. Assume $T' = T - \{v_1, v_2\}$ and f is a $\beta_D(T')$-function. If $f(v_3) = -1$, define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_2) = -1, g(v_1) = 1, \text{ and } g(u) = f(u) \text{ if } u \in V(T').$$

Then g is an NDF of T and by the inductive hypothesis, $\beta_D(T) \geq g(V(T)) = f(V(T')) \geq 0$. Let $f(v_3) = 1$. If $f(v_4) = -1$, then the function $g : V(T) \longrightarrow \{-1, 1\}$ defined by

$$g(v_2) = 1, g(v_1) = -1 \text{ and } g(u) = f(u) \text{ if } u \in V(T')$$

is an NDF of T and the result follows. If $f(v_4) = 1$, then $f(w_1) = -1$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by

$$g(v_1) = g(v_4) = -1, g(w_1) = g(v_2) = 1 \text{ and } g(u) = f(u) \text{ if } u \in V(T') - \{w_1, v_4\}.$$

Then g is an NDF of T and by the inductive hypothesis, $\beta_D(T) \geq g(V(T)) = f(V(T')) \geq 0$. This completes the proof. □

Next we present a lower bound on the negative k-subdecision number of a tree.

Theorem 8. *For any tree T of order $n \geq 2$ and any integer $1 \leq k \leq n$,*

$$\beta_{kD}(T) \geq n - k.$$

Proof. We proceed by induction on n. If $n \in \{2, 3, 4\}$, then the result follows by Theorems D and 11 and Proposition 6. Suppose that $n \geq 5$ and for every tree T' of order $2 \leq n' < n$ and any integer $1 \leq k' \leq n' - 1$, $\beta_{kD}(T') \geq n' - k'$. Let T be a tree of order n. If T is a star or a path, then by Theorems D or 11 or Proposition 6, $\beta_{kD}(T) \geq n - k$, for $1 \leq k \leq n$. Thus we may assume that $d = \text{diam}(T) \geq 3$ and T has at least three leaves. If $k \leq 3$, then by assigning $+1$ to all vertices the result follows. For the rest of the proof assume $k \geq 4$.

If $k = n$, by Theorem 7, $\beta_{kD}(T) \geq 0$. Hence, we may assume $4 \leq k < n$. First assume that v is a strong support vertex of T and v_1 and v_2 are two leaves adjacent to v and $T' = T - \{v_1, v_2\}$. Suppose $k' = k - 2$ and f is a $\beta_{k'D}(T')$-function. By the inductive hypothesis, $\beta_{k'D}(T') \geq (n - 2) - (k - 2) = n - k$. Define $g : V(T) \longrightarrow \{-1, 1\}$ by

$g(x) = f(x)$ if $x \in V(T')$, $g(v_1) = 1$ and $g(v_2) = -1$. Obviously, g is an NkSDF of T and we have $\beta_{kD}(T) \geq g(V(T)) \geq n - k$.

Now assume T has no strong support vertices. Let T be rooted at a leaf v_0 which is on a longest path and let v be a vertex at distance $d-1$ form v_0 on this path. By assumption, $\deg(v) = 2$. Let u be the leaf adjacent to v and let w be the parent of v. First assume $deg(w) = 2$ and $T' = T - \{u, v, w\}$. Then T' has order $n - 3$ and $2 \leq k - 2 \leq n - 3$. Let f be a $\beta_{(k-2)D}(T')$-function. Define $g : V(T) \longrightarrow \{-1, 1\}$ as follows: $g(w) = -1$, $g(v) = g(u) = 1$ and $g(x) = f(x)$ when $x \in V(T')$. Obviously, g is an NkSDF of T and by the inductive hypothesis,

$$\beta_{kD}(T) \geq g(V(T)) = f(V(T')) + 1 \geq (n-3) - (k-2) + 1 = n - k.$$

Now assume $\deg(w) \geq 3$. First let w be adjacent to a support vertex $v_1 \in T_w$ different from v. By assumption $\deg(v_1) = 2$. Assume u_1 is the leaf adjacent to v_1. Let $T' = T - \{u, v\}$. Then T' has order $n - 2$ and $2 \leq k - 2 \leq n - 3$. Let f be a $\beta_{(k-2)D}(T')$-function. Define $g : V(T) \longrightarrow \{-1, 1\}$ as follows: $g(v) = -1, g(u) = 1$ and $g(x) = f(x)$ when $x \in V(T')$. Obviously, g is an NkSDF of T and by the inductive hypothesis, $\beta_{kD}(T) \geq g(V(T)) = f(V(T')) \geq n - k$.

Finally, assume v is the only support vertex in T_w. It follows that $\deg(w) = 3$ and w is adjacent to a leaf v_1. Suppose $T' = T - \{u, v, v_1, w\}$. Since T has no strong support vertex, T' has order $n - 4 \geq 2$ and $1 \leq k - 3 \leq n - 4$. Let f be a $\beta_{(k-3)D}(T')$-function. Define $g : V(T) \longrightarrow \{-1, 1\}$ as follows: $g(w) = -1, g(v_1) = g(v) = g(u) = 1$ and $g(x) = f(x)$ when $v \in V(T')$. Obviously, g is an NkSDF of T and by the inductive hypothesis, $\beta_{kD}(T) \geq g(V(T)) = f(V(T')) + 2 \geq n - k$. This completes the proof. □

4. Complete graphs, cycles and paths

In this section we find the NkSDNs of complete graphs, cycles and paths.

Theorem 9. *For complete graphs K_n, $n \geq 3$, and any integer $1 \leq k \leq n$,*

$$\beta_{kD}(K_n) = \begin{cases} 1 & \text{if } n \text{ is odd and } 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil \\ -1 & \text{if } n \text{ is odd and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n \\ 2 & \text{if } n \text{ is even and } 1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil \\ 0 & \text{if } n \text{ is even and } \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq k \leq n. \end{cases}$$

Proof. Since $k \geq 1$, we have $f(N(v)) \leq 1$ for some vertex $v \in V(G)$. This forces $|P| \leq \left\lceil \frac{n+1}{2} \right\rceil$ and so $\beta_{kD}(K_n) \leq 2\left\lceil \frac{n+1}{2} \right\rceil - n$. If $1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil$, then assign $+1$ to $\left\lceil \frac{n+1}{2} \right\rceil$

vertices of K_n and assign -1 to the rest vertices to obtain an NkSDF for K_n of weight $2\left\lceil\frac{n+1}{2}\right\rceil - n$. Now let $\left\lceil\frac{n+1}{2}\right\rceil + 1 \le k \le n$. Then $f(N(v)) \le 1$ for some vertex v with $f(v) = -1$. This implies that $|P| \le \left\lceil\frac{n-1}{2}\right\rceil$ and so $\beta_{kD}(K_n) \le 2\left\lceil\frac{n-1}{2}\right\rceil - n$. Assign $+1$ to $\left\lceil\frac{n-1}{2}\right\rceil$ vertices of K_n and -1 to the rest vertices to obtain an NkSDF for K_n of weight $2\left\lceil\frac{n-1}{2}\right\rceil - n$. This completes the proof. □

Theorem 10. *For cycles of order $n \ge 3$ and any integer $1 \le k \le n-1$,*

$$\beta_{kD}(C_n) = n - 2\left\lceil\frac{k}{2}\right\rceil.$$

Proof. First observe that if f is an NkSDF of C_n and $v \in B_f$, then f cannot assign $+1$ to both neighbors of v. Moreover, each vertex u with $f(u) = -1$ is adjacent to at most two vertices in B_f. This forces $|M| \ge \left\lceil\frac{k}{2}\right\rceil$ and so $\beta_{kD}(C_n) = n - 2|M| \le n - 2\left\lceil\frac{k}{2}\right\rceil$. Let $v_1, v_2, ..., v_n$ be the vertices of C_n. If $k = 1, 2$, then assign -1 to v_1 and $+1$ to the remaining vertices. Let $k \ge 3$. If $k \equiv 1, 2 \pmod 4$, then assign -1 to v_1 and to the vertices v_{4i+4}, v_{4i+5} for $0 \le i \le \left\lceil\frac{k}{4}\right\rceil - 2$ and assign $+1$ to the remaining vertices. If $k \equiv 0, 3$ (mod 4), then assign -1 to the vertices v_{4i+1}, v_{4i+2} for $0 \le i \le \left\lceil\frac{k}{4}\right\rceil - 1$ and assign $+1$ to the remaining vertices. This shows that $\beta_{kD}(C_n) \ge n - 2\left\lceil\frac{k}{2}\right\rceil$, and the result follows. □

Next we establish the negative k-subdecision number of a path.

Theorem 11. *For $n \ge 2$ and $1 \le k \le n-1$,*

$$\beta_{kD}(P_n) = n - 2\left\lceil\frac{k-2}{2}\right\rceil.$$

Proof. The statement is obviously true for $2 \le n \le 4$. Let $n \ge 5$ and let $P_n = v_1, \dots, v_n$ be a path on n vertices. First observe that if f is an NkSDF of P_n, then $v_1, v_n \in B_f$ and if $v \in B_f \setminus \{v_1, v_n\}$, then f cannot assign $+1$ to both neighbors of v. Moreover, each vertex u with $f(u) = -1$ is adjacent to at most two vertices in B_f. This forces $|M| \ge \left\lceil\frac{k-2}{2}\right\rceil$ and so $\beta_{kD}(P_n) = n - 2|M| \le n - 2\left\lceil\frac{k-2}{2}\right\rceil$. If $k = 1, 2$, then assign $+1$ to each vertex of P_n. If $k = 3, 4$, then assign -1 to vertex v_3 and $+1$ to the remaining vertices. Let $k \ge 5$. If

$k \equiv 1, 2 \pmod 4$, then assign -1 to the vertices v_{4i+3}, v_{4i+4} for $0 \le i \le \left\lfloor \frac{1}{2} \left\lceil \frac{k-2}{2} \right\rceil \right\rfloor - 1$ and assign $+1$ to the remaining vertices. If $k \equiv 0, 3 \pmod 4$, then assign -1 to v_3, v_{4i+2} and v_{4i+3} for $1 \le i \le \left\lfloor \frac{1}{2} \left\lceil \frac{k-2}{2} \right\rceil \right\rfloor$ and assign $+1$ to the remaining vertices. This shows that $\beta_{kD}(P_n) \ge n - 2 \left\lceil \frac{k-2}{2} \right\rceil$, and the result follows. □

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