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The Chromatic Uniqueness of a Family of 6-Bridge Graphs

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Abstract

Let $P(G, \lambda)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique, written $\chi-$unique, if for any graph H, $G \sim H$ implies that G is isomorphic with H. In this paper we prove the chromatic uniqueness of a new family of 6-bridge graphs.

Keywords: Chromatic polynomials, chromaticity, 6-bridge graphs.

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1. Introduction

All graphs considered here are finite, undirected and simple. For a graph G let $V(G)$, $E(G)$, $v(G)$, $e(G)$, $g(G)$, $P(G, \lambda)$ respectively be the vertex set, edge set, order, size, girth and chromatic polynomial of G. Two graphs G and H are said to be chromatically equivalent, and we write $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi - unique$) if $G \cong H$ for any graph H such that $G \sim H$.

By a subdivision we mean an operation of replacing an edge of a graph by a path. If a graph H can be derived from G by a sequence of subdivisions, we say H is a subdivision of G. For each positive integer h, the graph $G(h)$ obtained from G by replacing each edge of G with a path of length h is called the h-uniform subdivision of G.

A chain in a graph G is a path in G in which every internal vertex has degree 2 in G. The operation that replaces a $u-v$ chain by a an edge uv is called a chain-contraction. By contracting all maximal chains of a graph G, we arrive at multigraph $M(G)$. Two graphs G and H are homeomorphic if $M(G) = M(H)$. If G is homeomorphic to H, we also say G is a H-homeomorph.

For each integer $k \geq 2$, let θ_k be the multigraph with two vertices and k edges. Any subdivision of θ_k is called a multi-bridge graph or k-bridge graph. We denote $\theta(a_1, a_2, ..., a_k)$ where $a_1, a_2, ..., a_k \in \mathbb{N}$ and $a_1 \leq a_2 \leq ... \leq a_k$ to be the graph obtained by replacing the edges of θ_k by paths of length $a_1, a_2, ..., a_k$ respectively.

Given positive integers $a_1, a_2, ..., a_k$, where $k \geq 2$, what is the necessary and sufficient condition on $a_1, a_2, ..., a_k$ for $\theta(a_1, a_2, ..., a_k)$ to be chromatically unique? Many papers have been published on this problem, but it is still far from being completely solved ([6], [8], [9]). Khalaf and Peng [7] proved that the graph $\theta(a_1, a_2, ..., a_6)$ is $\chi-$unique for exactly two distinct values of $a_1, a_2, ..., a_6$. In this paper we prove the chromatic uniqueness of a new family of 6-bridge graphs.

2. Auxiliary Results

In this section we cite some results to be used in the sequel.

A 2-bridge graph is simply a cycle, which is $\chi-$unique. Chao and Whitehead Jr. [2], showed that every 3-bridge graph $\theta(1, a_2, a_3)$ called a theta graph is $\chi-$unique. Loerinc [11] extended the above result to all 3-bridge graphs also called generalized $\theta - graph$. Chen et al. [3] proved that the 4-bridge graph $\theta(a_1, a_2, a_3, a_4)$ is $\chi-$unique if and only if $(a_1, a_2, a_3, a_4) \neq (2, c, c+1, c+2)$ for any $c \geq 2$. Bao and Chen [1] showed that every 5-bridge graph is $\chi-$unique if its shortest maximal chains are of length greater than 3. The above result is a special case of the following result due to Xu et al. [12].

Theorem 1. [12] *For $k \geq 4$, $\theta(a_1, a_2, ..., a_k)$ is $\chi-$unique if $k - 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.*

Li and Wei [10] established that the 5-bridge graph $\theta(2, 2, 2, a, b)$ is $\chi-$unique if and only if $(a, b) \neq (3, 4)$. Ye [13] extended the above result to any k-bridge graph $\theta(2, 2, \cdots, 2, a, b)$ with $b \geq a \geq 3$ and $k \geq 5$. Xu et al. [12] showed that any h-uniform subdivision of θ_k is $\chi-$unique, as stated in the following theorem:

Theorem 2. [12] *For $k \geq 2$, the graph $\theta_k(h)$ is $\chi-$unique.*

The above result was proved independently by Dong [4] and Koh and Teo [8]. Dong et al. [6] proved the following theorem.

Theorem 3. [6] *If $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k < a_1 + a_2$, where $k \geq 3$, then the graph $\theta(a_1, a_2, ..., a_k)$ is $\chi-$unique.*

Let $k, a_1, a_2, ..., a_k \in \mathbb{N}$, and $G = \theta(a_1, a_2, ..., a_k)$. Then (see [5])

$$P(G, \lambda) = \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}} \prod_{i=1}^{k} \Big((\lambda-1)^{a_i+1} + (-1)^{a_i+1}(\lambda-1) \Big) + \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k} \Big((\lambda-1)^{a_i} + (-1)^{a_i}(\lambda-1) \Big).$$

Let $\lambda = 1 - x$. Then

$$P(G, 1-x) = \frac{(-1)^{a_1+a_2+...+a_k+1}}{(1-x)^{k-1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\Big)$$
$$= \frac{(-1)^{e(G)+1}}{(1-x)^{e(G)-v(G)+1}}\Big(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\Big)$$

where $e(G) = \sum_{i=1}^{k} a_i$ and $v(G) = \sum_{i=1}^{k} a_i - k + 2$. Also, define $Q(G, x)$ for any graph G and real number x as:

$$Q(G, x) = (-1)^{e(G)+1}(1-x)^{e(G)-v(G)+1}P(G, 1-x).$$

Then we have

Theorem 4. [6] *For any $k, a_1, a_2, ..., a_k \in \mathbb{N}$,*

$$Q(\theta(a_1, a_2, ..., a_k), x) = x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x).$$

Theorem 5. [6] *For any graphs G and H,*

(i) if $H \sim G$, then $Q(H, x) = Q(G, x)$;

(ii) if $Q(H, x) = Q(G, x)$ and $v(H) = v(G)$, then $H \sim G$.

Lemma 1. [6] *Suppose that $\theta(a_1, a_2, ..., a_k) \sim \theta(b_1, b_2, ..., b_k)$, where $k \geq 3, 2 \leq a_1 \leq a_2 \leq ... \leq a_k$ and $2 \leq b_1 \leq b_2 \leq ... \leq b_k$. Then $a_i = b_i$ for all $i = 1, 2, ..., k$.*

Let $g_e(G_1, G_2, ..., G_k)$ be the collection of all edge-gluings of all $G_1, G_2, ..., G_k$, where $k \geq 2$ and $e(G_i) \geq 1$ for all i. We also have:

Lemma 2. [6] *Let $H \sim \theta(a_1, a_2, ..., a_k)$, where $k \geq 3$ and $a_i \geq 2$ for all i. Then one of the following is true:*

(i) $H \cong \theta(a_1, a_2, ..., a_k)$;

(ii) $H \in g_e(\theta(b_1, b_2, ..., b_t), C_{b_{t+1}+1}, ..., C_{b_k+1})$, where $3 \leq t \leq k-1$ and $b_i \geq 2$ for all $i = 1, 2, ..., k$.

Theorem 6. [6] *Let $k, t, b_1, b_2, ..., b_k \in \mathbb{N}$ with $3 \leq t \leq k-1$ and $b_i \geq 2$ for all $i = 1, 2, ..., k$. If $H \in g_e(\theta(b_1, b_2, ..., b_t), C_{b_{t+1}+1}, ..., C_{b_k+1})$, then*

$$Q(H, x) = x\prod_{i=1}^{k}(x^{b_i}-1) - \prod_{i=1}^{t}(x^{b_i}-x)\prod_{i=t+1}^{k}(x^{b_i}-1).$$

The following result is well known (see [8]).

Lemma 3. *If* $G \sim H$, *then*

(i) $v(G) = v(H)$;

(ii) $e(G) = e(H)$;

(iii) $g(G) = g(H)$ *and*

(iv) G *and* H *have the same number of shortest cycles.*

Khalaf and Peng [7] proved the following theorem:

Theorem 7. *A 6-bridge graph* $\theta(a_1, a_2, ..., a_6)$ *is* $\chi-$*unique if the positive integers* $a_1, a_2, ..., a_6$ *assume exactly two distinct values.*

3. Main Result

In this section we provide a new famille of chromatically unique 6-bridge graphs.

Theorem 8. *The 6-bridge graph* $\theta(3, 3, 3, 3, b, c)$, *where* $3 \leq b \leq c$, *is* $\chi-$*unique.*

Proof. Let $G = \theta(3, 3, 3, 3, b, c)$, where $3 \leq b \leq c$. If $c < 6$, then by Theorem 3, G is $\chi-$unique. Suppose that $H \sim G$ and $c \geq 6$. Then by Lemmas 1 and 2, we need only to consider three cases:

Case 1. $H \in g_e(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1})$, where $2 \leq b_1 \leq b_2 \leq b_3$, $2 \leq b_4, b_5, b_6$ and $12 + b + c = b_1 + b_2 + \cdots + b_6$. By Lemma 3, $g(G) = g(H) = 6$. Also by Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order six, H must have six cycles of the same order also. Therefore $b_1 = b_2 = b_3 = 3$, $b_4 = b_5 = b_6 = 5$. Since $b + c = 12$, we have four subcases:

Subcase 1.1. $b = 3, c = 9$. By Theorem 7, G is $\chi-$ unique.

Subcase 1.2. $b = 4, c = 8$. We have

$$\begin{aligned} Q(G, x) =& x + x^2 + x^3 - 3\,x^4 - 4\,x^5 - 5\,x^6 + x^7 + 9\,x^8 + 9\,x^9 - x^{10} - 11\,x^{11} - 3\,x^{12} \\ &+ 6\,x^{13} + 9\,x^{14} - 5\,x^{15} - 10\,x^{16} - 4\,x^{17} + 4\,x^{18} + 6\,x^{19} - x^{24}, \\ Q(H, x) =& x + x^2 - 3\,x^4 - 3\,x^6 - 3\,x^7 + 10\,x^9 + 3\,x^{11} + 3\,x^{12} - 12\,x^{14} \\ &- x^{16} - x^{17} + 6\,x^{19} - x^{24}. \end{aligned}$$

Clearly, $Q(G, x) \neq Q(H, x)$ which contradicts Theorem 5.

Subcase 1.3. $b = 5, c = 7$. We have

$$Q_1(G,x) = x + x^2 + x^3 - 3x^4 - 3x^5 - 5x^6 + x^7 + 4x^8 + 8x^9 - x^{10} + 3x^{11} - 2x^{12} - 5x^{14} - x^{15} - 4x^{16} + 6x^{19} - x^{24},$$

$$Q_1(H,x) = x + x^2 - 3x^4 - 3x^6 - 3x^7 + 10x^9 + 3x^{11} + 3x^{12} - 12x^{14} - x^{16} - x^{17} + 6x^{19} - x^{24}.$$

Clearly, $Q_1(G,x) \neq Q_1(H,x)$ which contradicts Theorem 5.

Subcase 1.4. $b = 6, c = 6$. By Theorem 2, G is $\chi-$ unique.

Case 2. $H \in g_e(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1})$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4$, $2 \leq b_5, b_6$ and $12 + b + c = b_1 + b_2 + \cdots + b_6$. We consider three subcases.

Subcase 2.1. $b_5 = b_6 = 5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order six, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for $1 \leq i < j \leq 4$ and $(i,j) \neq (i,4)$ where $i = 2,3$. Since $b_1 + b_i = 6$ for i=2,3,4, we have $b_2 = b_3 = b_4$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4$. Note that

$$Q(G,x) = x + x^2 + x^3 - 3x^4 - 3x^5 - 4x^6 + 2x^7 + 6x^8 + 6x^9 - 4x^{10} - 4x^{11} + x^{13} - x^{2+c} + 3x^{4+b} + 3x^{4+c} - x^{2+b} - x^{3+c} - x^{1+c} - x^{1+b} - x^{3+b} - 6x^{7+c} - 6x^{8+c} + 4x^{6+c} + 4x^{6+b} + 4x^{5+b} - 6x^{8+b} + 4x^{10+c} + 4x^{5+c} - 6x^{7+b} + 4x^{10+b},$$

$$Q(H,x) = x + x^2 + x^3 - 4x^4 - 4x^5 - 2x^6 + 4x^7 - 2x^8 + 8x^9 + 8x^{10} - 13x^{12} + 2x^{17}.$$

Clearly, $Q(G,x) \neq Q(H,x)$ which contradicts Theorem 5.

Subcase 2.2. $b_5 = 5$ and $b_6 \neq 5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order 6, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for $1 \leq i < j \leq 4$ and $(i,j) \neq (3,4)$. Since $b_1 + b_i = 6$ for i=2,3,4, we have $b_2 = b_3 = b_4$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4$. There are 7 cycles of order six in H and only 6 cycles of order six in G. This is a contradiction by Lemma 3.

Subcase 2.3. $b_5 \neq 5$ and $b_6 \neq 5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order 6, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for i=1,2 and j=2,3,4 and $i < j$. Since $b_1 + b_i = 6$ for i=2,3,4, we have $b_2 = b_3 = b_4$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4$. But $12+b+c = b_1+b_2+\cdots+b_6$, which implies that $b_5+b_6 = b+c$. Since $G \sim H$,

$Q(G,x) = Q(H,x)$. After cancelling the equal terms, we have $Q_1(G,x) = Q_1(H,x)$ where

$$\begin{aligned}Q_1(G,x) = &- 3\,x^6 + 2\,x^9 - 4\,x^{11} + x^{13} - x^{2+c} + 3\,x^{4+b} + 3\,x^{4+c} - x^{2+b} - x^{3+c} - x^{1+c}\\ &- x^{1+b} - x^{3+b} - 6\,x^{7+c} - 6\,x^{8+c} + 4\,x^{6+c} + 4\,x^{6+b} + 4\,x^{5+b} - 6\,x^{8+b} + 4\,x^{10+c}\\ &+ 4\,x^{5+c} - 6\,x^{7+b} + 4\,x^{10+b},\end{aligned}$$

$$\begin{aligned}Q_1(H,x) = &- x^4 - x^5 + 3\,x^7 - 7\,x^8 + 8\,x^{10} - 7\,x^{12} + x^{17} - x^{2+b_6} - x^{3+b_6} - x^{1+b_6}\\ &- 6\,x^{7+b_6} + 4\,x^{4+b_6} + x^{12+b_6} + 4\,x^{5+b_6}.\end{aligned}$$

The term $-x^4$ in $Q_1(H,x)$ can not be cancelled in $Q_1(H,x)$. It must be cancelled in $Q_1(G,x)$. Since $b \neq 2$ and $c \neq 2$, we have either $b = 3$ or $c = 3$. If $b = 3$, then by Theorem 7, G is $\chi-$ unique. If $c = 3$ and since $3 \leq b \leq c$, then $b = c = 3$. By Theorem 2, G is $\chi-$ unique.

Case 3. $H \in g_e(\theta(b_1,b_2,b_3,b_4,b_5), C_{b_6+1})$, where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$, $2 \leq b_6$ and $12 + b + c = b_1 + b_2 + \cdots + b_6$. We have to consider two subcases:

Subcase 3.1. $b_6 = 5$. We have three subcases:

Subcase 3.1.1. $b_1 + b_5 < b_2 + b_4$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order 6, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for i=1,2 and j=2,3,4,5 and $i < j$. Since $b_1 + b_i = 6$ for i=2,3,4,5, we have $b_2 = b_3 = b_4 = b_5$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4,5$. There are 11 cycles of order six in H and only 6 cycles of order six in G. This is a contradiction by Lemma 3.

Subcase 3.1.2. $b_2 + b_4 < b_1 + b_5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order six, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for $1 \leq i < j \leq 4$. Since $b_1 + b_i = 6$ for i=2,3,4, we have $b_2 = b_3 = b_4$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4$. There are 7 cycles of order six in H and only 6 cycles of order six in G. This is a contradiction by Lemma 3.

Subcase 3.1.3. $b_2 + b_4 = b_1 + b_5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order 6, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for i=1,2 and j=2,3,4,5 and $i < j$. Since $b_1 + b_i = 6$ for i=2,3,4,5, we have $b_2 = b_3 = b_4 = b_5$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4,5$. There are 11 cycles of order six in H and only 6 cycles of order six in G. This is a contradiction by Lemma 3.

Subcase 3.2. $b_6 \neq 5$. We have three subcases:

Subcase 3.2.1. $b_1 + b_5 < b_3 + b_4$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order six, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for i=1,2 and j=2,3,4,5 and $i < j$. Since $b_1 + b_i = 6$ for i=2,3,4,5, we have $b_2 = b_3 = b_4 = b_5$. Since $b_2 + b_3 = 6$, $b_2 = b_3 = 3$. Hence we have $b_i = 3$, for each $i = 1,2,3,4,5$. There are 10 cycles of order six in H and only 6 cycles of order six in G. This is a contradiction by Lemma 3.

Subcase 3.2.2. $b_3 + b_4 < b_1 + b_5$. By Lemma 3, G and H have the same number of shortest cycles. Since G has 6 cycles of order six, H must have 6 cycles of the same order also. Therefore $b_i + b_j = 6$, for $1 \le i < j \le 4$. Since $b_1 + b_i = 6$ for i=2,3,4, we have $b_2 = b_3 = b_4$. Since $b_2 + b_3 = 2a$, $b_2 = b_3 = a$. Hence we have $b_i = 3$, for each $i = 1, 2, 3, 4$. But $12 + b + c = b_1 + b_2 + \cdots + b_6$, which implies that $b_5 + b_6 = b + c$. By Theorem 5, $Q(G, x) = Q(H, x)$. By using Theorems 4 and 6 and after cancelling the equal terms, we have $Q_1(G, x) = Q_1(H, x)$ where

$$\begin{aligned} Q_1(G,x) =& x^3 - 2x^5 - 4x^6 + 6x^8 + 5x^9 - 4x^{10} - 4x^{11} + x^{13} - x^{2+c} + 3x^{4+b} \\ &+ 3x^{4+c} - x^{2+b} - x^{3+c} - x^{1+c} - x^{1+b} - x^{3+b} - 6x^{7+c} - 6x^{8+c} \\ &+ 4x^{6+c} + 4x^{6+b} + 4x^{5+b} - 6x^{8+b} + 4x^{10+c} + 4x^{5+c} - 6x^{7+b} + 4x^{10+b}, \\ Q_1(H,x) =& -x^4 + x^7 - x^{12} - x^{2+b_6} - x^{2+b_5} - x^{1+b_5} - x^{1+b_6} - 3x^{7+b_5} - 3x^{7+b_6} + x^{12+b_6} \\ &+ 4x^{4+b_6} + 4x^{4+b_5} - x^{9+b_5} - x^{9+b_6} + x^{5+b_6} + x^{5+b_5} + x^{12+b_5}. \end{aligned}$$

Since $b \ge 3$ and $c \ge 6$, the term x^3 in $Q_1(G, x)$ can not be cancelled in $Q_1(G, x)$. It must be cancelled in $Q_1(H, x)$. Since $b_5 \ge 3$ and $b_6 > 6$, the term x^3 can not be cancelled in $Q_1(H, x)$ also. The term x^3 appears in $Q_1(G, x)$ but it is not found in $Q_1(H, x)$. This is a contradiction by Theorem 5.

The proof of the Theorem is now complete. □

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