AKCE J. Graphs. Combin., 6, No. 3 (2009), pp. 409-427

Level Hypergraphs

Hortensia Galeana-Sánchez and Martín Manrique
IMUNAM, Universidad Nacional Autónoma de México
Ciudad Universitaria, Circuito Exterior, 04510, D.F.,Mexico.
e-mail: *hgaleana@matem.unam.mx, martin@matem.unam.mx, martin.manrique@gmail.com*

Communicated by: B.D. Acharya
Received 28 May 2008; revised 2 February 2009 accepted 13 October 2009

Abstract

An often useful way of studying a mathematical object is to relate it with another one, which may be simpler, better known, or just allowing a different approach. Research on graphs and hypergraphs is not an exception, as we have incidence matrices, dual graphs and hypergraphs, line-graphs, sections and graph images of a hypergraph, product hypergraphs, and several other examples.

In the present paper we introduce a way of relating any hypergraph with a simpler one, which retains its edge-structure but has often much less vertices. This allows us to obtain results in several branches of hypergraph theory.

Keywords: hypergraph, balanced hypergraph, transversal set.

2000 Mathematics Subject Classification: 05C20, 05C65, 05C69.

1. Introduction

Given a hypergraph $H = (E_1, ..., E_m)$, its level hypergraph is the result of identifying all vertices which belong to exactly the same edges. This new hypergraph has the same edge structure as the original one, but may have less vertices. The tool makes it possible to emulate known theorems regarding bounds for important numbers (like the transversal number, the matching number, etc.) given in terms of order or rank; the new results are stated in terms of edge-structure, and often apply to different classes of hypergraphs, although there are some generalisations.

In this paper we define the tool, show some similarities and differences between a given hypergraph and its level hypergraph, and present some examples of results obtained using level hypergraphs.

For general concepts on graphs and hypergraphs we refer the reader to [4] and [5] respectively.

2. Foundations

Definition 2.1. *Given a hypergraph* $H = (E_1, ..., E_m)$, *its natural partition* $P = \{P_1, ..., P_l\}$ *is the partition of the vertex set* $V(H)$ *of* H *defined by the equivalence relation* $\approx$, *specified by the rule* $x \approx y \Leftrightarrow E_x = E_y$, *where for any vertex* w, E_w *denotes the set of edges containing* w. *The elements of* P *are called levels of* H *and for any edge* E *of* H, *the sets* $\{P_i \subset E \mid P_i \in P\}$, *called levels of* E, *form the natural partition of* E.

Definition 2.2. *Given a hypergraph* $H = (E_1, ..., E_m)$ *with natural partition* $P = \{P_1, ..., P_l\}$, *the level hypergraph* $L_H = (E'_1, ..., E'_m)$ *of* H *is the hypergraph resulting from deleting every vertex but one from each level* P_i *of* P. *In other words, we consider a set* $S = \{x_1, ..., x_l \mid x_i \in P_i \; \forall i \in \{1, ..., l\}\}$ *and take* $L_H = H[S]$, *the subhypergraph induced by* S. *It is clear that* L_H *is well defined, that is, it does not matter which vertex from each level is kept, for all of them play an equivalent role.*

The above notions were independently given by Acharya in [1] (and that of natural partition by us in [7]), where he considers hypergraphs with isolated vertices (that is, vertices which belong to no edge). Here we follow [5], so "isolated" vertices have a loop. The only difference is that in [1] all isolates belong to one level, while in this paper each "isolate" belongs to its own level. All results in this paper apply directly or are easily adapted to hypergraphs with (true) isolates.

Since every edge E of H has at least one vertex, it contains at least one level, so it induces an edge E' in L_H; then both H and L_H have the same number of edges. In the same way, E_i and E'_i contain the same number of levels and $E_i \cap E_j \neq \emptyset$ if and only if $E'_i \cap E'_j \neq \emptyset$. This implies that H is simple if and only if L_H is simple, that H has repeated edges if and only if L_H does too, and that $\nu(H) = \nu(L_H)$, where $\nu(H)$ is the maximum cardinality of a matching in H. Notice also that every hypergraph H such that every level of its natural partition has but one vertex, is a level hypergraph (of itself, to begin with).

Definition 2.3. *Given a hypergraph* H *and a vertex* $x \in V(H)$, *the edge degree* $e_H(x)$ *of* x *in* H *is the number of edges in* H *containing* x. *From now on, it will be called simply degree.*

Observe that every vertex belonging to a given level P_i of H has the same degree, as well as the vertex corresponding to that level in L_H. In particular, maximum and minimum degrees are preserved: $\Delta(H) = \Delta(L_H)$ and $\delta(H) = \delta(L_H)$.

Given an edge E of a hypergraph H, the corresponding edge of L_H will be called E' and an edge of L_H will always be written with an apostrophe. We will use the same symbol for a level of H and its corresponding level in L_H. We consider $V(L_H) = \{x_1, ..., x_l \mid x_i \in P_i \; \forall i \in \{1, ..., l\}\}$ and call x_i the *representative* of P_i.

Level hypergraphs (as a class) are easily characterised, as stated in [1]: A given hypergraph $H = (E_1, ...E_m)$ is a level hypergraph if and only if for every set $\{x, y\} \subset V(H)$, there is an edge E in H such that $x \in E$ and $y \notin E$, or $y \in E$ and $x \notin E$. Equivalently,

H is a level hypergraph if and only if for every $x \in V(H)$, the natural map $x \mapsto E_x$ is a bijection, where $E_x = (E \in H \mid x \in E)$.

Then every level hypergraph is its own level hypergraph. A (multi)graph is a level hypergraph if and only if it has no connected components isomorphic to K_2 (nor to K_2 with multiple edges). The level hypergraph of K_2 is a vertex with a loop (or multiple loops, if it has multiple edges).

Definition 2.4. *A hypergraph H is separable if for every $x \in V(H)$, $\bigcap\{E \in H \mid x \in E\} = \{x\}$.*

Notice that every separable hypergraph is a level hypergraph, but the converse is not true, as shown in Figure 1, where only $\bigcap\{E \in H \mid x_4 \in E\}$ is a singleton.

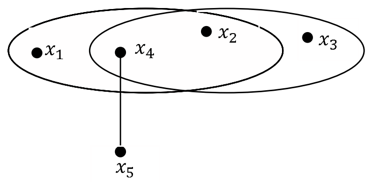

Figure 1

Proposition 2.5. *Let H be a hypergraph and $D(H)$ its dual hypergraph. Then $D(L_H)$ is like $D(H)$ without repeated edges. Moreover, a given hypergraph H is a level hypergraph if and only if $D(H)$ does not have repeated edges and for any hypergraph H, $D(H)$ is a level hypergraph if and only if H has no repeated edges.*

Proof. Indeed, $|H| = |L_H|$ and for each vertex in $V(H)$ there is a vertex in $V(L_H)$ belonging to the same edges, so $D(H)$ and $D(L_H)$ have the same structure, but $D(H)$ may have more (repeated) edges, since two vertices in the same level of H induce repeated edges in $D(H)$. On the other hand, the only way of generating repeated edges in $D(H)$ is having two vertices in the same level of H, so $D(L_H)$ has no repeated edges. Also, two vertices in the same level of $D(H)$ correspond to repeated edges in H. □

Proposition 2.6. *Let H be a hypergraph and let L_H be its level hypergraph. If L_H has a k-colouring, H does have one too, and the converse is not true. If H has a strong k-colouring, L_H has one also, and the converse is not true. If H has a good or an equitable k-colouring, L_H may have one or not, and if L_H has a good or an equitable k-colouring, H may have one or not.*

Proof. Let H be a hypergraph and let L_H be its level hypergraph. Suppose L_H has a k-colouring $(S'_1, ..., S'_k)$. Then $(S_1, ..., S_k)$ is a k-colouring of H, where every vertex in each level P_i of H belongs to the colour class S_j if and only if its representative x_i belongs to the colour class S'_j.

Now suppose H has a strong k-colouring. Since all vertices belonging to a given edge have different colours, by assigning to each vertex of L_H any of the colours present on the level of H it represents, we get a strong k-colouring of L_H. Figure 2 shows a hypergraph H such that L_H has a strong 3-colouring, which is therefore equitable and good, while H has no good 3-colouring and H has an equitable 2-colouring, while L_H does not even have a 2-colouring. □

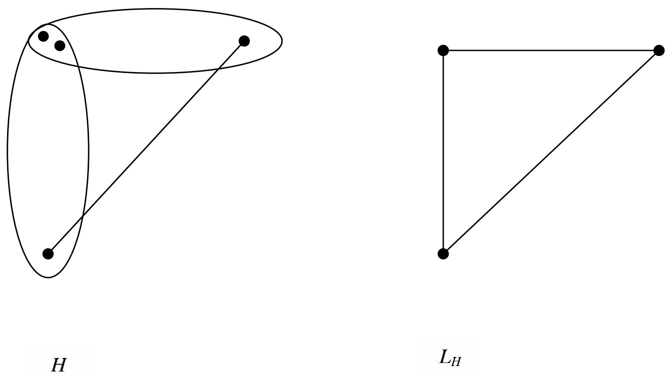

Figure 2

Definition 2.7. *Let $H = (E_1, ..., E_m)$ be a hypergraph and let $a \geq 0$ be an integer. Multiplying the edge E_i by a means to replace E_i by a identical copies of it. For an edge E, we will write aE for the result of multiplying E by a.*

Definition 2.8. *Let $H = (E_1, ..., E_m)$ be a hypergraph. H is regular if all its vertices have the same degree. H is regularisable if a regular hypergraph may be obtained by multiplying each of its edges E_i by an integer $a_i > 0$. H is quasi-regularisable if a regular hypergraph may be obtained by multiplying each of its edges E_i by an integer $a_i \geq 0$.*

Proposition 2.9. *Let H be a hypergraph and let L_H be its level hypergraph. Then:*

(i) H is regular if and only if L_H is regular.

(ii) H is regularisable if and only if L_H is regularisable.

(iii) H is quasi-regularisable if and only if L_H is quasi-regularisable.

In cases (ii) and (iii), the integer used to get the regular hypergraph is the same for an edge in H and for the edge it induces in L_H.

Proof. Let $H = (E_1, ..., E_m)$ be a hypergraph and let $L_H = (E'_1, ..., E'_m)$ be its level hypergraph. Suppose that H is regularisable (quasi-regularisable), so that $G = (a_1E_1, ..., a_mE_m)$ is regular for positive (non-negative) integers $a_1, ..., a_m$. For every vertex in H there is a vertex in L_H with the same degree, namely the representative of the level it belongs to.

Moreover, all vertices in a given level of H will be multiplied by the same integers to obtain G. This implies that $G' = (a_1E'_1, ..., a_mE'_m)$ is regular. By means of an identical reasoning, if $G' = (a_1E'_1, ..., a_mE'_m)$ is regular then $G = (a_1E_1, ..., a_mE_m)$ is regular. □

3. Balanced and Unimodular Hypergraphs

Given a hypergraph H, if L_H has a cycle of length k, then H has one too (namely, the same cycle). The converse is not true. We may have a hypergraph H with cycles of any length k such that L_H has no cycles of length k. If H has k edges, and those edges intersect in a single level with k vertices, then H has cycles of every length not larger than k, while L_H has no cycles altogether, as shown in Figure 3 for $k = 4$. However, other notions which generalise bipartite graphs to hypergraphs are preserved in level hypergraphs.

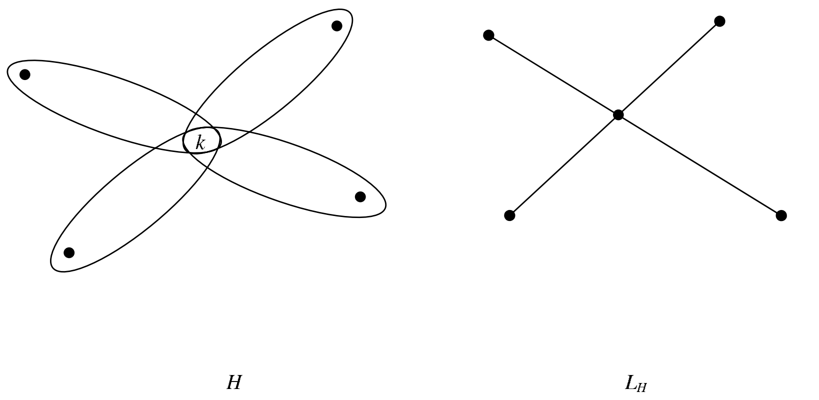

Figure 3

Definition 3.1. *A hypergraph H is balanced (respectively totally balanced) if and only if for every cycle of odd length (respectively every cycle) C there is an edge of C containing at least three vertices of C.*

Proposition 3.2. *A hypergraph H is balanced (respectively totally balanced) if and only if L_H is balanced (respectively totally balanced).*

Proof. Assume H is balanced (respectively totally balanced), and let $C' = (x_1, E'_1, ..., x_n, E'_n, x_1)$ be a cycle of odd length n (respectively of any length n) in L_H. Then the cycle $C = (y_1, E_1, ..., y_n, E_n, y_1)$, where y_i belongs to the same level as that of x_i, is a cycle of length n in H. Since H is balanced (respectively totally balanced), there is an edge E_i, $i \in \{1, ..., n\}$ with at least three vertices of C, which implies that the corresponding edge E'_i has at least three vertices of the cycle C'. Therefore L_H is balanced (respectively totally balanced).

Conversely, assume L_H is balanced (respectively totally balanced) and consider a cycle $C = (y_1, E_1, ..., y_n, E_n, y_1)$ of odd length (respectively of any length) in H. If no two vertices of $V(C)$ belong to the same level, the cycle C induces a cycle $C' = (x_1, E'_1, ..., x_n, E'_n, x_1)$ of the same length in L_H. Since L_H is balanced (respectively totally balanced), there is an edge $E'_i,\ i \in \{1, ..., n\}$ with at least three vertices of C', which implies that the corresponding edge E_i has at least three vertices of the cycle C. Therefore H is balanced (respectively totally balanced).

If there are two vertices $y_i,\ y_j,\ i < j$, which belong to the same level, then both edges E_{i-1} and E_j have at least three vertices of C: $\{y_{i-1}, y_i, y_j\} \subset E_{i-1}$ and $\{y_i, y_j, y_{j+1}\} \subset E_j$. This completes the proof. □

Definition 3.3. *A hypergraph H is strongly unimodular if and only if it is balanced and there are no cycles of odd length with one edge containing exactly three vertices of the cycle and the others containing only two.*

Proposition 3.4. *A hypergraph H is strongly unimodular if and only if L_H is strongly unimodular.*

Proof. The proof is quite similar to that of Proposition 3.2. Assume H is strongly unimodular, and let $C' = (x_1, E'_1, ..., x_n, E'_n, x_1)$ be a cycle of odd length n in L_H. Then the cycle $C = (y_1, E_1, ..., y_n, E_n, y_1)$, where y_i belongs to the same level as that of x_i, is a cycle of length n in H. Since H is strongly unimodular, there is an edge with at least four vertices of $V(C)$ or two edges with at least three vertices of $V(C)$. In any case, the corresponding edges in C' preserve those properties, which implies that L_H is strongly unimodular.

Conversely, assume L_H is strongly unimodular and consider a cycle $C = (y_1, E_1, ..., y_n, E_n, y_1)$ of odd length in H. If no two vertices of $V(C)$ belong to the same level, the cycle C induces a cycle $C' = (x_1, E'_1, ..., x_n, E'_n, x_1)$ of the same length in L_H. Since L_H is strongly unimodular, there is an edge with at least four vertices of $V(C')$, or at least two edges with three vertices of $V(C')$, which implies that the corresponding edges of C satisfy the same properties.

If there are two vertices $y_i,\ y_j,\ i < j$, which belong to the same level, then both edges E_{i-1} and E_j have at least three vertices of C: $\{y_{i-1}, y_i, y_j\} \subset E_{i-1}$ and $\{y_i, y_j, y_{j+1}\} \subset E_j$. This completes the proof. □

Definition 3.5. *A matrix $A = (a^i_j)$ is totally unimodular if and only if every square submatrix of A has a determinant equal to $0, +1$, or -1. A hypergraph H is unimodular if and only if its incidence matrix is totally unimodular.*

Definition 3.6. *Let H be a hypergraph. An equitable 2-colouring of H is a partition of $V(H)$ into two sets S_1 and S_2 such that each edge E of H satisfies $[\frac{|E|}{2}] \leq\ |E \cap S_i|\ \leq [\frac{|E|}{2}]'$, $i \in \{1, 2\}$, where $[*]$ means "largest integer not larger than $*$" and $[*]'$ means "least integer not less than $*$."*

Proposition 3.7. [5]*A hypergraph H is unimodular if and only if for every $S \subset V(H)$ the hypergraph $H[S]$ has an equitable 2-colouring.*

Note. Observe that if H is unimodular, all of its induced subhypergraphs are unimodular as well.

Proposition 3.8. *A hypergraph H is unimodular if and only if L_H is unimodular.*

Proof. Assume H is unimodular. Since L_H is an induced subhypergraph of H, we have that L_H is unimodular.

Conversely, assume L_H is unimodular. Then L_H has an equitable 2-colouring $R = \{R_1, R_2\}$. By using mathematical induction on the number of levels having more than one vertex, we will show that every induced subhypergraph of H has an equitable 2-colouring. Let $S \subset V(H)$, and let $H' = H[S]$. If every level of H' has one vertex, it is an induced subhypergraph of L_H, so it has an equitable 2-colouring.

Suppose only one level P'_i of H' has more than one vertex. If $|P'_i|$ is odd, we consider the level hypergraph $L_{H'}$ of H', which has an equitable 2-colouring $R' = \{R'_1, R'_2\}$. An even number of vertices must be added to $L_{H'}$ to get H', all of them in the same level, that is, in the same edges, so that we may add half of those vertices to R'_1 and the other half to R'_2, getting an equitable 2-colouring for H'. If $|P'_i|$ is even, we delete P'_i. Since the remaining hypergraph $H'' = H[S \setminus P'_i]$ is an induced subhypergraph of L_H, it has an equitable 2-colouring $R'' = \{R''_1, R''_2\}$. Now we add P'_i, adding half of its vertices to R''_1 and the remaining half to R''_2. Since every edge containing P'_i gets an equal number of vertices of each color class and all other edges remain unchanged, the resulting 2-colouring of H' is equitable. Thus, every induced subhypergraph of H such that only one of its levels has more than one vertex has an equitable 2-colouring.

Now consider an integer $l > 1$ and suppose that every induced subhypergraph of H with less than l levels having more than one vertex has an equitable 2-colouring, and let H' be an induced subhypergraph of H with l levels having more than one vertex. We proceed as before: Consider a level P'_i of H' with more than one vertex. If $|P'_i|$ is odd, we take from P'_i all vertices but one, say x'; then we have an induced subhypergraph of H with $l-1$ levels having more than one vertex, which has an equitable 2-colouring $R' = \{R'_1, R'_2\}$; by adding half of the vertices of $P'_i \setminus \{x'\}$ to R'_1 and the other half to R'_2 we get an equitable 2-colouring for H'. If $|P'_i|$ is even, we take all vertices from P'_i, effectively deleting that level; then we have an induced subhypergraph of H with $l-1$ levels having more than one vertex, which has an equitable 2-colouring $R'' = \{R''_1, R''_2\}$. As before, by adding half of the vertices of P'_i to one of the two color classes and the other half to the other color class we get an equitable 2-colouring for H'. Thus the proof is complete by induction. □

Theorem 3.9. [5] *A hypergraph is balanced if and only if its induced subhypergraphs are two colourable.*

The following result makes it easier to decide wether a given hypergraph is balanced or not:

Theorem 3.10. *A hypergraph H is balanced if and only if the induced subhypergraphs of its level-hypergraph L_H are two colourable.*

Proof. The theorem follows from Proposition 3.2 and Theorem 3.9. □

Theorem 3.11. [5] *A hypergraph of rank $r \leq 3$ is unimodular if and only if it is balanced.*

Theorem 3.12. *A hypergraph such that every edge has at most 3 levels is unimodular if and only if it is balanced.*

Proof. Let H be a hypergraph. If H is unimodular, then it is balanced. Now suppose H is balanced and no edge of H has more than three levels. Then L_H is balanced of rank $r \leq 3$, so from Theorem 3.11 it is unimodular. Proposition 3.8 implies that H is unimodular. □

Since $r \leq 3$ implies that every edge has at most three levels, Theorem 3.12 is a generalisation of Theorem 3.11 to a much wider class of hypergraphs.

4. Transversals and Matchings

Definition 4.1. *Let H be a hypergraph. $T \subset V(H)$ is a transversal of H if and only if it meets all its edges. A transversal T of H is minimal if and only if for every transversal T' of H, $T' \subset T \Rightarrow T = T'$. A transversal T of H is minimum if and only if for every transversal T' of H, $|T| \leq |T'|$. τ denotes the cardinality of a minimum transversal, and τ' denotes the maximum cardinality of a minimal transversal.*

Remark 4.2. *A transversal of L_H is also a transversal of H, and a transversal T of H induces a transversal of L_H by deleting all vertices but one from each level whose intersection with T has more than one element. Moreover, a minimal transversal of H has no more than one vertex per level, for two vertices in the same level belong to exactly the same edges; on the other hand, if an edge E'_i of L_H is met by a given set $S \subset V(H)$, then the corresponding edge E_i of H is met by S; it follows that every set $T \subset V(H)$ is a minimal (minimum) transversal of H if and only if T is a minimal (minimum) transversal of L_H, taking $T \cap P_i$ as x_i, the representative of P_i, whenever $T \cap P_i \neq \emptyset$. So we have $\tau'(H) = \tau'(L_H)$ and $\tau(H) = \tau(L_H)$.*

We will use this last remark to prove some results regarding bounds for τ and τ'.

Theorem 4.3. [5] *Let $H = (E_1, ..., E_m)$ be a hypergraph of order n. Let $s = \min\ |E_i|\ > 1$ and $\Delta = \Delta(H)$. Then $\tau' \leq \left[\frac{n\Delta}{\Delta+s-1}\right]$.*

Theorem 4.4. *Let $H = (E_1, ..., E_m)$ be a hypergraph with l levels and such that every edge has at least 2 levels. Let s' be the minimum number of levels contained in an edge of H, and $\Delta = \Delta(H)$. Then $\tau' \leq \left[\frac{l\Delta}{\Delta+s'-1}\right]$.*

Proof. Let $H = (E_1, ..., E_m)$ be a hypergraph satisfying the conditions in the hypothesis. Then $L_H = (E'_1, ..., E'_m)$ is a hypergraph of order l with $s' = \min\ |E'_i|\ > 1$ and $\Delta = \Delta(L_H) = \Delta(H)$. Then, by Theorem 4.3, $\tau' \leq \left[\frac{l\Delta}{\Delta+s'-1}\right]$. □

The bound given by Theorem 4.4 is not always better than the one resulting from Theorem 4.3. For example, for the hypergraph H in Figure 4 Theorem 4.4 says $\tau' \leq 5$, while Theorem 4.3 states $\tau' \leq 4$. However, for most cases Theorem 4.4 gives a better value than Theorem 4.3, but it applies to a tighter class of hypergraphs (every edge must have at least two levels, instead of at least two vertices).

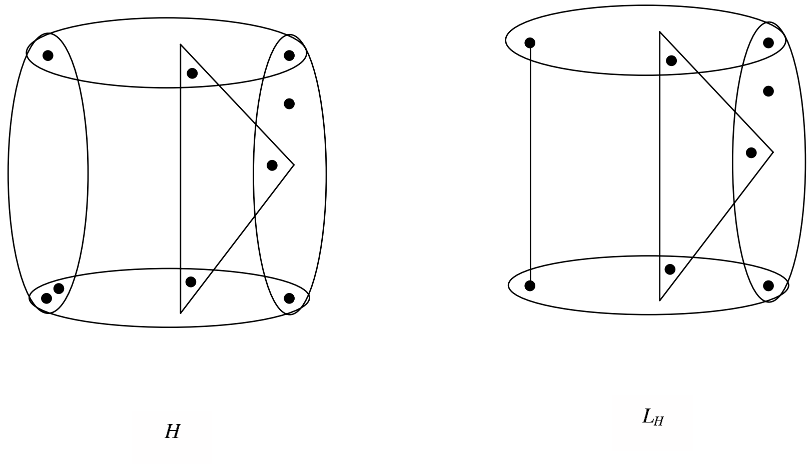

Figure 4

Theorem 4.5. [12] *Let $H = (E_1, ..., E_m)$ be a hypergraph of order n with $s = \min |E_i| > 1$, and label the vertices of $V(H)$ so that $e_H(x_1) \leq e_H(x_2) \leq ... \leq e_H(x_n)$. Then $\tau'(H) = t$ satisfies $\sum_{i=1}^{t}(e_H(x_i) + s - 1) \leq \sum_{i=1}^{n} e_H(x_i)$.*

Theorem 4.6. *Let $H = (E_1, ..., E_m)$ be a hypergraph with l levels. Let $s' > 1$ be the minimum number of levels contained in an edge of H. Let $e'_i = e_H(x),\ x \in P_i$, and label the levels of H so that*

$$e'_1 \leq e'_2 \leq ... \leq e'_l. \text{ Then } \tau'(H) = t \text{ satisfies } \sum_{i=1}^{t}(e'_i + s' - 1) \leq \sum_{i=1}^{l} e'_i.$$

Proof. Let $H = (E_1, ..., E_m)$ be a hypergraph satisfying the conditions. Then $L_H = (E'_1, ..., E'_m)$ is a hypergraph of order l with $s' = \min |E'_i|$, and such that $\{e'_1, e'_2, ..., e'_l\}$ corresponds to the set of degrees of its vertices. By applying Theorem 4.5 to L_H we get Theorem 4.6. □

Theorem 4.6 being a generalisation of Theorem 4.4, we also have that it does not always give a better bound than Theorem 4.5. It does in most cases, but it asks for $s' > 1$, while Theorem 4.5 applies whenever $s > 1$.

Definition 4.7. *Let $H = (E_1, ..., E_m)$ be a hypergraph. H is linear if and only if $|E_i \cap E_j| \leq 1$ for all $\{i, j\} \subset \{1, ..., m\}$.*

Definition 4.8. *Let $H = (E_1, ..., E_m)$ be a hypergraph. H is level-linear if and only if the intersection of any two edges consists on no more than one level.*

Theorem 4.9. [5] *Let $H = (E_1, ..., E_m)$ be a linear hypergraph of order n with $s = \min |E_i| > 2$. Then*

$$\tau'(H) \leq n + \frac{1}{2}(s^2 - 3s + 1) - \frac{1}{2}\sqrt{4n(s^2 - 3s + 2) + (s^2 - 3s + 1)^2}.$$

Theorem 4.10. *Let $H = (E_1, ..., E_m)$ be a level-linear hypergraph with l levels. Let $s' > 2$ be the minimum number of levels contained in any edge of H. Then $\tau'(H) \leq l + \frac{1}{2}(s'^2 - 3s' + 1) - \frac{1}{2}\sqrt{4l(s'^2 - 3s' + 2) + (s'^2 - 3s' + 1)^2}$.*

Proof. Let $H = (E_1, ..., E_m)$ be a hypergraph as in the hypothesis. Then $L_H = (E'_1, ..., E'_m)$ is a linear hypergraph of order l, with $s' = \min |E'_i|$. The result follows from Theorem 4.9. □

Since $s \geq s'$ and $(p^2 - 3p + 1) \geq (q^2 - 3q + 1)$ for any $p \geq q > 2$, Theorem 4.10 gives a value never larger than that given by Theorem 4.9. However, it applies to a different class of hypergraphs: Requiring H to be level-linear is less than requiring H to be linear, but $s' > 2$ is requiring more than $s > 2$.

Theorem 4.11. [5] *Let $H = (E_1, ..., E_m)$ be a linear 3-uniform hypergraph of order n. Then $\tau(H) \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$.*

Theorem 4.12. *Let $H = (E_1, ..., E_m)$ be a level-linear hypergraph with l levels such that every edge has 3 levels. Then $\tau(H) \leq l + \frac{1}{2} - \sqrt{2l + \frac{1}{4}}$.*

Proof. Let $H = (E_1, ..., E_m)$ be a hypergraph as in the hypothesis. Then $L_H = (E'_1, ..., E'_m)$ is a linear 3-uniform hypergraph of order l, and the result follows from Theorem 4.11. □

Since Theorem 4.11 is a particular case of Theorem 4.9 (thus Theorem 4.12 is a particular case of Theorem 4.10), Theorem 4.12 gives a better bound than Theorem 4.11, but it applies to a different class of hypergraphs.

Definition 4.13. *Let H be a hypergraph. A set $S \subset V(H)$ is strongly stable if and only if $|S \cap E| \leq 1$ for every edge E in H.*

A strongly stable set S in L_H is a strongly stable set in H, since L_H is an induced subhypergraph of H. Conversely, a strongly stable set S in H meets any level at most once, since it meets any edge at most once. By taking $S \cap P_i$ as x_i we have that S is a strongly stable set in L_H, for any edge $E'_i \in L_H$ and any set $A \subset V(H)$ satisfy $|E'_i \cap A| \geq 2 \Rightarrow |E_i \cap A| \geq 2$.

Notice that the concept of stability ($S \subset V(H)$ is *stable* if and only if there is no edge $E \in H$ such that $|E| > 1$ and $E \subset S$) does not translate well to level hypergraphs, since we may have $S \cap P_i \neq \emptyset$ and $(V(H) \setminus S) \cap P_i \neq \emptyset$ for a given level P_i of H.

Theorem 4.14. [5] *Let H be a balanced hypergraph. Then H has a good k-colouring for every $k \geq 2$.*

Theorem 4.15. *Let H be a balanced hypergraph such that $|E| = r$ for every edge $E \in H$. Then $V(H)$ may be partitioned into r pairwise disjoint strongly stable transversals.*

Proof. This follows from Theorem 4.14, since given a good r-colouring every color class is a strongly stable transversal. □

Theorem 4.16. *Let H be a balanced hypergraph such that every edge $E \in H$ has r' levels. Then H has r' pairwise disjoint strongly stable transversals.*

Proof. From Proposition 3.2 we have that L_H is balanced, so we may apply Theorem 4.15 to it. According to Remark 4.2, we have r' pairwise disjoint strongly stable transversals of H. □

Theorem 4.16 applies to a wider class of hypergraphs than Theorem 4.14, but it is not as strong, since the set of vertices is not necessarily partitioned.

Definition 4.17. *Let H be a hypergraph. A matching in H is a set of pairwise disjoint edges of H and $\nu(H)$ denotes the maximum cardinality of a matching in H.*

Theorem 4.18. [13] *Let $H = (E_1, ..., E_m)$ be a linear hypergraph without repeated loops and let $n = |V(H)|$. Then $\nu(H) \geq \frac{m}{n}$.*

Theorem 4.19. *Let $H = (E_1, ..., E_m)$ be a level-linear hypergraph without repeated one-level edges. Let l be the total number of levels in H. Then $\nu(H) \geq \frac{m}{l}$.*

Proof. Let H be as in the hypothesis. Then L_H is a linear hypergraph without repeated loops with l vertices and m edges. The result follows from Theorem 4.18. □

Theorem 4.19 gives a better value than Theorem 4.18, since $l \leq n$. It applies to a wider class of hypergraphs: the condition of linearity is requiring more than one-level intersections; no repeated loops seems to be less than no repeated one-level edges, but in this case both are equivalent to requiring no repeated edges altogether.

Definition 4.20. *Let H be a hypergraph. A fractional transversal of H is a function $p : V(H) \to \mathbb{R}$ such that $0 \leq p(x) \leq 1$ for every $x \in V(H)$ and $\sum_{x \in E} p(x) \geq 1$ for every edge $E \in H$. The value of a fractional transversal p of H is $\sum_{x \in V(H)} p(x)$. The symbol $\tau^*(H)$ denotes the minimum value of the fractional transversals of H.*

Proposition 4.21. *Let H be a hypergraph and let L_H be its level hypergraph. Then $\tau^*(H) = \tau^*(L_H)$.*

Proof. Let $P = \{P_1, ..., P_l\}$ be the natural partition of H and let $V(L_H) = \{x_1, ..., x_l\}$, where $x_i \in P_i$ for every $i \in \{1, ..., l\}$. Let p' be a fractional transversal of L_H whose value is $\tau^*(L_H)$. Then $p : V(H) \to \mathbb{R}$ such that $p(x) = \dfrac{p'(x_i)}{|P_i|}$ for $x \in P_i$ is a fractional transversal of H, so that $\tau^*(H) \leq \tau^*(L_H)$.

Lemma 4.22. *Let H be a hypergraph with natural partition $P = \{P_1, ..., P_l\}$ and let p be a fractional transversal of H whose value is $\tau^*(H)$. Then $\sum_{x \in P_i} p(x) \leq 1$ for every $i \in \{1, ..., l\}$.*

Proof. Let H and p be as in the hypothesis. Suppose there is a level P_j of H such that $\sum_{x \in P_j} p(x) > 1$. Let J be the set of edges of H containing P_j. Then for every $E \in J$ and every $x \in E \setminus P_j$, $p(x)$ is independent of the values $p(y)$ for $y \in \bigcup_{E \in J} E$, that is, we could have $p(y) = 0$ and $\sum_{x \in E} p(x) \geq 1$ for every edge $E \in J$ anyway. This means that $p^* : V(H) \to \mathbb{R}$ such that $p^*(x) = \dfrac{1}{|P_j|}$ for $x \in P_j$ and $p^*(x) = p(x)$ for $x \in V(H) \setminus P_j$ is a fractional transversal of H whose value is strictly less than $\tau^*(H)$. So the lemma is proven.

Now consider a fractional transversal p of H whose value is $\tau^*(H)$. Then $p' : V(L_H) \to \mathbb{R}$ such that $p'(x_i) = \sum_{x \in P_i} p(x)$ is a fractional transversal of L_H, which implies $\tau^*(H) \geq \tau^*(L_H)$. This completes the proof of Proposition 4.21. □

Theorem 4.23. [5] *Let H be an r-uniform hypergraph such that $|V(H)| = n$. Then H is quasi-regularisable if and only if $\tau^*(H) = \frac{n}{r}$.*

Theorem 4.24. *Let H be a simple hypergraph H with l levels and such that every edge has r' levels. Then H is quasi-regularisable if and only if $\tau^*(H) = \frac{l}{r'}$.*

Proof. Take H satisfying the conditions in the hypothesis. Then L_H is an r'-uniform hypergraph with $|V(L_H)| = l$, so from Theorem 4.23, Proposition 2.9, and Proposition 4.21 we have that H is quasi-regularisable if and only if $\tau^*(H) = \tau^*(L_H) = \frac{l}{r'}$. □

Theorem 4.25. [6] *Every r-uniform hypergraph H satisfies $\tau^*(H) \leq \dfrac{r^2 - r + 1}{r}\nu(H)$.*

Theorem 4.26. *Let H be a simple hypergraph such that every edge has r' levels. Then*

$$\tau^*(H) \leq \frac{(r')^2 - r' + 1}{r'}\nu(H).$$

Proof. We have already seen that $\tau^*(H) = \tau^*(L_H)$ and $\nu(H) = \nu(L_H)$ for every hypergraph H. Consider H as in the hypothesis. Then L_H is an r'-uniform hypergraph and the result follows from Theorem 4.25. □

The bound given by Theorem 4.26 is better than that given by Theorem 4.25, but it applies to a different class of hypergraphs.

Theorem 4.27. [5] *Every regular r-uniform hypergraph H such that* $|V(H)| = n$ *satisfies* $\frac{n}{r^2 - r + 1} \leq \nu(H)$.

Theorem 4.28. *Every simple hypergraph H with l levels and such that every edge has* r' *levels satisfies* $\frac{l}{(r')^2 - r' + 1} \leq \nu(H)$.

Proof. Similar to that of Theorem 4.26. □

5. Cyclomatic number and planar hypergraphs

The concept of planarity has been extended to hypergraphs by Zykov [15] and by Johnson and Pollak [8]. The concept of cyclomatic number has been extended to hypergraphs by Acharya [2]. We will now see how these concepts relate to level hypergraphs:

Definition 5.1. *A planar embedding* M *of a (planar) graph* G *with finite vertex set* $V(G)$ *partitions the plane into a finite set of faces, each bounded by a cycle of vertices. That embedding defines a hypergraph* H_M *with* $V(H_M) = V(G)$, *and such that* $E \subset V(G)$ *is an edge of* H_M *if and only if there is a face of* M *bounded by a cycle* C *such that* $V(C) = E$. *A given hypergraph* $H = (E_1, ..., E_m)$ *is Zykov-planar if and only if there exist a multigraph* G *with* $V(G) = V(H)$, *and a planar embedding* M *of* G, *such that* H *is isomorphic to* H_M.

The following characterisation, due to Jones [9], allows us to relate Zykov-planarity with level hypergraphs:

We may represent the vertices of a hypergraph H as points in the plane, and the edges of H as subsets of the plane homeomorphic to closed discs with their vertices on their boundary. It may be done in such a way that two edges intersect only in the vertices belonging to both of them if and only if H is *Zykov-planar.*

Proposition 5.2. *Let* H *be a Zykov-planar hypergraph. Then* L_H *is Zykov-planar.*

Proof. The proof is straightforward, since for any set A of edges in H we have that $\bigcap_{E\in A} E' \subset \bigcap_{E\in A} E$, where E' is the edge of L_H corresponding to E. □

The converse is not true, as shown in Figure 3 with $k \geq 3$.

Proposition 5.2 may also be proven using a result by Walsh:

Theorem 5.3. [14] *A hypergraph $H = (E_1, ..., E_m)$ is Zykov-planar if and only if the bipartite graph G_H is planar, where $V(G_H) = V(H) \cup \{E_1, ..., E_m\}$ and $\{x, E_i\}$ is an edge of G_H whenever $i \in \{1, ..., m\}$ and $x \in E_i$.*

Then the result follows from the fact that G_{L_H} is an induced subhypergraph of G_H.

Definition 5.4. *Given a hypergraph $H = (E_1, ...E_m)$, an edge-based Venn diagram representing H consists of a planar graph G, a planar embedding M of G, and a bijective function $g : H \to F$, where $F = \{F_1, ..., F_m\}$ is the set of faces of M, such that $g(E_i) = F_i$ and for every $V' \subset V(H)$, $\bigcup_{E_i \supset V'} F_i$ is a region of the plane whose interior is connected. A hypergraph $H = (E_1, ...E_m)$ is edge-planar if and only if there exists an edge-based Venn diagram representing H.*

Proposition 5.5. *A hypergraph H is edge-planar if and only if L_H is edge-planar. Moreover, every edge-based Venn diagram representing H represents L_H as well, and vice versa.*

Proof. Consider an edge-planar hypergraph H and an edge-based Venn diagram (G, M, g) representing H. Then for every $V' \subset V(H)$ and for every $P_j \in P$ such that $V' \cap P_j \neq \emptyset$, $\bigcup_{E_i \supset V'} F_i$ is the same region of the plane as $\bigcup_{E_i \supset (V' \cup P_j)} F_i$, since all vertices in P_j belong to exactly the same edges. For the same reason, $\bigcup_{E_i \supset V'} F_i$ is also the same region of the plane as $\bigcup_{E_i \supset V''} F_i$, where $V'' = (V' \setminus P_j) \cup \{x\}$, and $x \in V' \cap P_j$. □

Proposition 5.5 means that every problem on edge-planarity may be solved using level hypergraphs, which may reduce computation time considerably in some cases (and will never increase it).

Definition 5.6. *Given a hypergraph $H = (E_1, ...E_m)$, a vertex-based Venn diagram representing H consists of a planar graph G, a planar embedding M of G, and a bijective function $g : V(H) \to F$, where $F = \{F_1, ..., F_n\}$ is the set of faces of M, such that $g(v_i) = F_i$ and for every $E_j \in H$, $\bigcup_{v_i \in E_j} F_i$ is a region of the plane whose interior is connected. A hypergraph $H = (E_1, ...E_m)$ is vertex-planar if and only if there exists a vertex-based Venn diagram representing H.*

Proposition 5.7. *Let H be a hypergraph. If L_H is vertex-planar, then H is vertex-planar.*

Proof. Suppose L_H is vertex-planar, and let (G, M, g) be a vertex-based Venn diagram representing L_H. We get a vertex-based Venn diagram representing H by partitioning F_i into $|P_i|$ faces, where x_i is the representative of P_i and $g(x_i) = F_i$. □

From a vertex-planar level hypergraph we get a whole family of hypergraphs which are also vertex planar.

Conjecture 5.8. *A hypergraph H is vertex-planar if and only if L_H is vertex-planar.*

We believe that for every vertex-planar hypergraph H there exists a vertex-based Venn diagram (G, M, g) representing H such that for every level $P_j \in P$, $\bigcup_{v_i \in P_j} F_i$ is a region of the plane whose interior is connected. In such a case, we only need to erase the boundaries between the vertices of P_j (and the corresponding edges of G) to get a vertex-based Venn diagram representing L_H.

Definition 5.9. *Given a hypergraph $H = (E_1, ..., E_m)$, its intersection multigraph $G(H)$ is the graph whose set of vertices is the set of edges of H, and between any two vertices E_i and E_j, there is an edge in $G(H)$ for each vertex in $V(H)$ in $E_i \cap E_j$. Then each vertex in $V(H)$ induces a complete subgraph of $G(H)$, although not always a clique (for example, if $H = K_3$).*

Given a hypergraph H, there is at least one edge (E_i', E_j') in $G(L_H)$ if and only if there is at least one edge (E_i, E_j) in $G(H)$; however, $G(L_H)$ may have less edges altogether, for vertices in a given level P_i of H induce identical complete subgraphs in $G(H)$.

A spanning multiforest of a multigraph M is a multiforest F such that $V(F) = V(M)$. Given a hypergraph H, we will call $T(H)$ the maximum number of edges in any spanning multiforest of $G(H)$. It is clear that $T(H) \geq T(L_H)$.

Definition 5.10. *The cyclomatic number of a hypergraph H is $\mu(H) = \sum_{E \in H} |E| - |V(H)| - T(H)$.*

The concepts of intersection multigraph and cyclomatic number are not so easily related with level hypergraphs: Given a hypergraph H, its intersection multigraph shows which vertices are in the intersection of any set of edges, but there is no way to tell if those vertices belong to the same level or not. This means that two hypergraphs with different level hypergraphs may have the same intersection multigraph, as shown in Figure 5. However, some basic relations may be obtained:

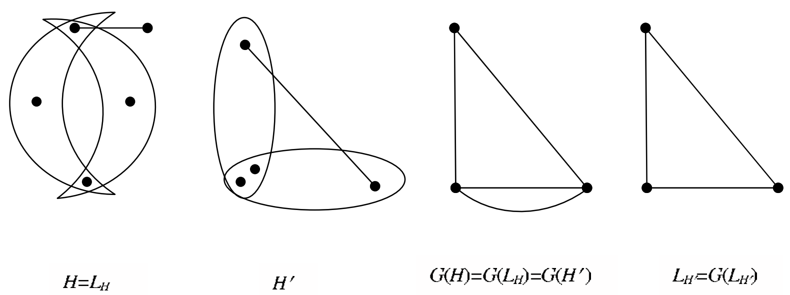

Figure 5

Proposition 5.11. *Let H be a hypergraph and let H' be an induced subhypergraph of H. Then $\mu(H) \geq \mu(H')$.*

Proof. Let H be a hypergraph, and let $H_1 = H[V(H) - x]$, the subhypergraph of H induced by $V(H) - x$, where $x \in V(H)$. Let c_x be the number of edges to which x belongs. Then $\sum_{E \in H_1} |E| = \sum_{E \in H} |E| - c_x$, $|V(H_1)| = |V(H)| - 1$, and $T(H_1) \geq T(H) - c_x + 1$, since any forest (tree) with c_x vertices has at most $c_x - 1$ edges, so x induces at most $c_x - 1$ edges in any spanning multiforest of $G(H)$. This implies that $\mu(H) \geq \mu(H_1)$. Since $V(H)$ is a finite set, the result is proven. □

Corollary 5.12. *Let H be a hypergraph. Then $\mu(H) \geq \mu(L_H)$.*

Proof. Given a hypergraph H, L_H is an induced subhypergraph of H. □

The converse is not true, as shown in Figure 6.

H $\quad\quad$ L_H

Figure 6

Proposition 5.13. *Let H be a hypergraph. Then $\mu(H) = 0$ if and only if $\mu(L_H) = 0$.*

Proof. $\mu(H) = 0 \Rightarrow \mu(L_H) = 0$ follows from Proposition 5.11. To see that $\mu(H) = 0 \Longleftarrow \mu(L_H) = 0$, we proceed by induction on the number of vertices to be removed, using a result from Acharya and Las Vergnas:

Theorem 5.14. [3] *For every hypergraph H, $\mu(H) = 0$ if and only if the 2-section H_2 of H is a triangulated graph and the set of edges of H maximal by containment is the set of cliques of H_2.*

Let H be a hypergraph and let H' be a hypergraph obtained from H by removing a vertex x from a level P_i of H such that $|P_i| \geq 2$. We will see that if H' satisfies the conditions of Theorem 5.14, then H does too. Suppose H'_2 is a triangulated graph. How could H_2 have a non-triangulated elementary cycle? Let $y \neq x$, $y \in P_i$. Since x and y belong to the same edges, we have that $N(x) \setminus \{y\} = N(y) \setminus \{x\}$, so if x belongs to an elementary cycle C_x such that $y \notin V(C_x)$, then we may take y instead of x, getting an elementary cycle C_y with the same edge structure as C_x and such that

$x \notin V(C_y)$, so that C_x is non-triangulated if and only if C_y is non-triangulated. Since C_y is a cycle of H'_2, we know it is triangulated, which implies that x may only belong to a non-triangulated elementary cycle C of H_2 such that $|V(C) \cap P_i| \geq 2$. Let C be an elementary cycle of H_2 such that $\{x, y\} \subset V(C) \cap P_i$. If x and y are adjacent in C then C is triangulated, since (x, z) is an edge of $G(H)$ if and only if (y, z) is an edge of $G(H)$, so for any vertex z adjacent to x or y in $G(H)$ we have the triangle (z, x, y). If x and y are not adjacent in $C = (y = x_0, x_1, ..., x_{j-1}, x_j = x, x_{j+1}, ..., x_k, y)$, we have the cycles $C_1 = (y = x_0, x_1, ..., x_{j-1}, y)$ and $C_2 = (y, x_{j+1}, ..., x_k, y)$, for $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. Since $x \notin V(C_1) \cup V(C_2)$, both C_1 and C_2 are cycles of H'_2, so they are triangulated. Then the cycles $C_3 = (y = x_0, x_1, ..., x_{j-1}, x, y)$ and $C_4 = (y, x_{j+1}, ..., x_k, x, y)$ are triangulated, for x and y are adjacent in both. This implies that C is triangulated too.

Now suppose the set of edges of H' maximal by containment is the set of cliques of H'_2. Let $y \neq x$, $y \in P_i$, and $E' \in H'$ such that $y \in E'$. Since E' induces a clique in H'_2, and $N(x) \setminus \{y\} = N(y) \setminus \{x\}$ in H_2, we have that $E = E' \cup \{x\}$ induces a clique in H_2.

This completes the proof of Proposition 5.13. □

6. Conclusions and Scope

As we stated in the Introduction, the objective of this work has been to introduce level hypergraphs as a tool and to show some examples of results obtained using it. At the time of this publication, we have already submitted a wider and more organised view of level hypergraphs.

The tool seems most useful for researching transversals and matchings of hypergraphs, although results in other branches of hypergraph theory are possible as well. It may be thought as a hypergraph invariant, since hypergraphs sharing a given level hypergraph behave similarly in many ways.

Other possible direction of research is the following: Characterise hypergraph properties (P) for which $L_H \in (P) \Rightarrow H \in (P)$, and those for which $H \in (P) \Rightarrow L_H \in (P)$. Clearly, every hereditary property (P) satisfies the second statement.

As a particular case of this general problem, we studied an inequality originally stated by Lovasz [10]: Let $H = (E_1, ..., E_m)$ be a hypergraph such that $|V(H)| = n$, $|E_i \cap E_j| \leq k$ for every $\{i, j\} \subset \{1, ..., m\}$, and H has p connected components. When is it true that $\sum_{i=1}^{m} (|E_i| - k) \leq n - pk$? In particular, totally balanced hypergraphs [11] and hypergraphs with cyclomatic number 0 [3] satisfy Lovasz's inequality. Unfortunately it turned to have no direct relation with level hypergrphs. Figure 2 shows an example where H satisfies Lovasz's inequality but L_H does not. The smallest hypergraph H we know such that L_H satisfies Lovasz's inequality but H does not is 4-uniform and has 10 edges.

Acknowledgements

The authors are thankful to the referee for making important remarks, as well as suggesting topics and bibliography related to this paper, which helped to improve it substantially. We are also very grateful to Dr. B. D. Acharya for letting us read his yet unpublished paper.

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