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A NOTE ON ORTHOGONAL DOUBLE COVERS OF COMPLETE BIPARTITE GRAPHS BY A SPECIAL CLASS OF SIX CATERPILLARS

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Abstract

We construct orthogonal double covers of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$, where n and m are integers, $2 \leq m \leq 10$, $m \leq n$ and $P_{m+1} \cup^* S_{n-m}$ is a tree obtained from the path P_{m+1} with m edges and a star S_{n-m} with $n-m$ edges by identifying an end-vertex of P_{m+1} with the center of S_{n-m}.

Keywords: Orthogonal double cover, symmetric starter.

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1. Introduction

For the definition of an orthogonal double cover (ODC) of the complete graph K_n by a graph G and for a survey on this topic, see [1]. In [1, 4], this concept has been generalized to ODC of any graph H by a graph G.

While in principle any regular graph is worth considering (e.g., the remarkable case of hypercubes has been investigated in [4]), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs of such graphs are of help in order to construct ODCs of K_n(see [2], p. 48).

In this paper, we assume $H = K_{n,n}$, the complete bipartite graph with part sizes n each.

An ODC of $K_{n,n}$ is a collection $\mathcal{G} = \{G_1, G_2, \ldots, G_n, F_1, F_2, \ldots, F_n\}$ of $2n$ subgraphs called *pages*) of $K_{n,n}$ such that

(i) every edge of $K_{n,n}$ is in exactly one page of $\{G_1,\ G_2,\ \ldots,\ G_n\}$ and in exactly one page of $\{F_1,\ F_2,\ \ldots,\ F_n\}$;

(ii) for $i,\ j \in \{1, 2, \ldots, n\}$ and $i \neq j$, $E(G_i) \cap E(G_j) = E(F_i) \cap E(F_j) = \emptyset$; and $|E(G_i) \cap E(F_j)| = 1$ for all $i,\ j \in \{1, 2, \ldots, n\}$.

If all the pages are isomorphic to a given graph G, then $\mathcal{G}$ is said to be an ODC of $K_{n,n}$ by G.

Denote the vertices of the partite sets of $K_{n,n}$ by $\{0_0,\ 1_0,\ \ldots,\ (n-1)_0\}$ and $\{0_1,\ 1_1,\ \ldots,\ (n-1)_1\}$. The length of an edge x_0y_1 of $K_{n,n}$ is defined to be the difference $y - x$, where $x,\ y \in \mathbb{Z}_n = \{0,\ 1,\ 2,\ \ldots,\ n-1\}$. Note that sums and differences are carried over in $\mathbb{Z}_n$ (that is, sums and differences are carried modulo n)

Let G be a subgraph of $K_{n,n}$ and $z \in \mathbb{Z}_n$. then the z-translate of G, denoted by $G + z$ is the edge-induced subgraph of $K_{n,n}$ induced by $\{(x+z)_0(y+z)_1 : x_0y_1 \in E(G)\}$.

A subgraph G of $K_{n,n}$ is called *half-starter* if $|E(G)| = n$ and the lengths of all edges in G are mutually different. We denote a half-starter G by the vector $v(G) = (v_0,\ v_1,\ \ldots,\ v_{n-1})$, where $v_0,\ v_1,\ \ldots,\ v_{n-1} \in \mathbb{Z}_n$ and v_i can be obtained from the unique edge $(v_i)_0(v_i+i)_1$ of length i in G.

Two half-starters $v(G) = (v_0,\ v_1,\ \ldots,\ v_{n-1})$ and $v(\acute{G}) = (u_0,\ u_1,\ \ldots,\ u_{n-1})$ are said to be orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

For a subgraph G of $K_{n,n}$ with n edges, the edge-induced subgraph G_s with $E(G_s) = \{y_0x_1 : x_0y_1 \in E(G)\}$ is called the *symmetric graph* of G. following three results were established in [2].

I. If G is a half-starter, then the collection of all translates of G forms an edge-decomposition of $K_{n,n}$.

II. If two half-starter $v(G)$ and $v(\acute{G})$ are orthogonal, then the union of the set of translates of G and the set of translates of $\acute{G}$ forms an ODC of $K_{n,n}$ by G.

III. G is a symmetric half-starter if and only if $\{v_i - v_{n-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

In the next section, we find an orthogonal double cover of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$, where n and m are integers, $2 \leq m \leq 10$, $m \leq n$ and $P_{m+1} \cup^* S_{n-m}$ is a tree obtained from the path P_{m+1} with m edges and a star S_{n-m} with $n-m$ edges by identifying an end-vertex of P_{m+1} with the center of S_{n-m}.

2. Main Result

Theorem 1. *let n and m be integers such that $2 \leq m \leq 10$ and $m \leq n$. Then there is an ODC of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$.*

Proof. For $m \in \{2,3,4\}$, the statement was already proved in [3]. In what follows we find a suitable symmetric starter of $\mathbb{Z}_n$ in each of the remaining six cases.

Case 1. $m = 5$.

For $n \geq 5$, define $v(G)$ by $v_i = 0$ if $i \in \{0,\ 2\}$, $v_i = 4$ if $i \in \{n-2,\ n-1\}$ and $v_i = 2$ otherwise. By the definition of $v(G)$, $E(G) = \{0_0 0_1,\ 2_0 3_1,\ 0_0 2_1\} \cup \{2_0 i_1 : i \in \{5,\ 6,\ \ldots,\ n-1\}\} \cup \{4_0 2_1,\ 4_0 3_1\}$ and hence $G \cong P_6 \cup^* S_{n-5}$. For $i \in \{3,\ 4, \ldots,\ n-3\}$, $v_i - v_{n-i} + i = i$; and for $i \in \{1,\ 2,\ n-2,\ n-1\}$, $v_i - v_{n-i} + i = n - i$ implies that $v(G)$ is a symmetric starter.

Case 2. $m = 6$.

For $n \geq 6$, define $v(G)$ by $v_0 = 2$, $v_1 = v_{n-1} = n-1$, $v_2 = v_{n-2} = 0$ and $v_i = n-i-1$ otherwise. By the definition of $v(G)$, $E(G) = \{2_0 2_1,\ (n-1)_0 0_1,\ 0_0 2_1\} \cup \{i_0 (n-1)_1 : i \in \{2,\ 3,\ \ldots,\ n-4\}\} \cup \{0_0 (n-2)_1,\ (n-1)_0 (n-2)_1\}$ and hence $G \cong P_7 \cup^* S_{n-6}$. For $i \in \{3,\ 4,\ \ldots,\ n-3\}$, $v_i - v_{n-i} + i = n - i$; and for $i \in \{1\ ,2\ ,\ n-2,\ n-1\}$, $v_i - v_{n-i} + i = i$ implies that $v(G)$ is a symmetric starter.

Case 3. $m = 7$.

For $n \geq 7$, define $v(G)$ by $v_0 = v_3 = 0$, $v_1 = v_{n-1} = 1$, $v_{n-2} = v_{n-3} = 6$ and $v_i = 2$ otherwise. By the definition of $v(G)$, $E(G) = \{0_0 0_1,\ 1_0 2_1,\ 2_0 4_1,\ 0_0 3_1\} \cup \{2_0 i_1 : i \in \{6,\ 7,\ \ldots,\ n-2\}\} \cup \{6_0 3_1,\ 6_0 4_1, 1_0 0_1\}$ and hence $G \cong P_8 \cup^* S_{n-7}$. For $i \in \{1,\ n-1\} \cup \{4,\ 5,\ \ldots,\ n-4\}$, $v_i - v_{n-i} + i = i$; and for $i \in \{2,\ 3,\ n-3,\ n-2\}$, $v_i - v_{n-i} + i = n - i$ implies that $v(G)$ is a symmetric starter.

Case 4. $m = 8$.

For $n \geq 8$, define $v(G)$ by $v_0 = v_2 = 0$, $v_1 = v_{n-2} = 4$, $v_3 = v_{n-3} = 3$, $v_{n-1} = 6$ and $v_i = 2$ otherwise. By the definition of $v(G)$, $E(G) = \{0_0 0_1,\ 4_0 5_1,\ 0_0 2_1,\ 3_0 6_1\} \cup \{2_0 i_1 : i \in \{6, 7, \ldots, n-2\}\} \cup \{3_0 0_1,\ 4_0 2_1, 6_0 5_1\}$ and hence $G \cong P_9 \cup^* S_{n-8}$. For $i \in \{3,\ 4, \ldots,\ n-3\}$, $v_i - v_{n-i} + i = i$; and for $i \in \{1,\ 2,\ n-2,\ n-1\}$, $v_i - v_{n-i} + i = n - i$ implies that $v(G)$ is a symmetric starter.

Case 5. $m = 9$.

For $n \geq 9$, define $v(G)$ by $v_0 = v_4 = 0$, $v_1 = v_{n-1} = 1$, $v_2 = v_{n-2} = 4$, $v_3 = v_{n-3} = 8$ and $v_i = 2$ otherwise. By the definition of $v(G)$, $E(G) = \{0_0 0_1,\ 1_0 2_1,\ 4_0 6_1,\ 2_0 5_1,\ 0_0 4_1\} \cup \{2_0 i_1 : i \in \{7,\ 8,\ \ldots,\ n-3\}\} \cup \{8_0 4_1,\ 8_0 5_1, 4_0 2_1,\ 1_0 0_1\}$ and hence $G \cong P_{10} \cup^* S_{n-9}$. For $i \in \{1,\ 2,\ n-2,\ n-1\} \cup \{5,\ 6,\ \ldots,\ n-5\}$, $v_i - v_{n-i} + i = i$; and for $i \in \{3, 4, n-4, n-3\}$, $v_i - v_{n-i} + i = n - i$ implies that $v(G)$ is a symmetric starter.

Case 6. $m = 10$.

For $n \geq 10$, define $v(G)$ by $v_0 = v_4 = 0$, $v_1 = v_{n-1} = 1$, $v_2 = 4$, $v_3 = v_{n-3} = 5$, $v_{n-2} = v_{n-4} = 8$ and $v_i = 3$ otherwise. By the definition of $v(G)$, $E(G) = \{0_0 0_1,\ 1_0 2_1,\ 4_0 6_1,\ 5_0 8_1,\ 0_0 4_1\} \cup \{3_0 i_1 : i \in \{8, 9, \ldots, n-2\}\} \cup \{8_0 4_1,\ 5_0 2_1, 8_0 6_1,\ 1_0 0_1\}$ and hence $G \cong P_{11} \cup^* S_{n-10}$. For $i \in \{1,\ 3,\ n-3,\ n-1\} \cup \{5,\ 6,\ \ldots,\ n-5\}$, $v_i - v_{n-i} + i = i$; and for $i \in \{2,\ 4,\ n-4,\ n-2\}$, $v_i - v_{n-i} + i = n - i$ implies that $v(G)$ is a symmetric starter. $\square$

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