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Kernels in infinite digraphs

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Abstract

Let D be a digraph, possibly infinite, $V(D)$ and $A(D)$ denote the sets of vertices and arcs of D, respectively.

A kernel N of D is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from w to N. A set $S \subseteq V(D)$ is a semikernel of D if it is independent and $(u, v) \in A(D)$ with $u \in S$ implies that there is an arc from v to some vertex in S. In this paper we introduce sufficient conditions for the existence of kernels in some kinds of infinite digraphs, such as transitive digraphs, symmetric digraphs, acyclic digraphs and digraphs without odd cycles. We use strongly the concept of semikernel.

Keywords: Kernel, semikernel, digraph, infinite digraph.

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1. Introduction

For general concepts we refer the reader to [1]. In this paper D will denote a possibly infinite digraph with $V(D)$ and $A(D)$ being the sets of vertices and arcs of D, respectively.

From now on we will say only walk, path and cycle to refer a directed walk, directed path and directed cycle, respectively. If P is a finite walk we will denote its length by $l(P)$. If D is infinite, an infinite outward path is an infinite sequence $(x_1, x_2, \dots)$ of distinct vertices of D such that $(x_i, x_{i+1}) \in A(D)$ for each i. Let S_1 and S_2 be subsets of $V(D)$, a finite path $(x_1, x_2, \ldots, x_n)$ will be called an S_1S_2-directed path whenever $x_1 \in S_1$ and $x_n \in S_2$, in particular when the directed path is an arc, we will call it an S_1S_2-arc.

An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. symmetrical), which is denoted by $Asym(D)$ (resp. $Sym(D)$), is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph with set of vertices S and whose arcs are those of D which join vertices of S.

A digraph D is strong if for every pair of vertices x and y there is an xy-path in D. If D is not strong, a subdigraph H of D is called a strong component of D if it is strong and it is maximal with respect this property. A strong component H is called terminal if there is not an arc from some vertex in H to another strong component.

A digraph D is said to be a transitive digraph when $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$. The following concepts are generalizations of transitive digraph. A digraph D is called right-pretransitive (resp. left-pretransitive) when $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, v) \in A(D)$ (resp. $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(v, u) \in A(D)$). A digraph D is a quasitransitive digraph if $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, u) \in A(D)$.

A digraph D is called symmetrical if $Sym(D) = D$.

A digraph D is said to be an acyclic digraph if D contains no cycle.

A digraph D is a bipartite digraph if there is a partition (X, Y) of $V(D)$ such that every arc of D is an arc with and endpoint in X and the other endpoint in Y.

Definition 1.1. [11] *A set $I \subseteq V(D)$ is independent if $A(D[I]) = \phi$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D.*

Definition 1.2. [3] *A digraph D is called a kernel-perfect digraph when every induced subdigraph of D has a kernel.*

The concept of semikernel was introduced by Victor Neumann-Lara. This concept has been very important to the development of Kernel Theory. In particular it was used ([10]) to prove in a shorter way the Richardson's Theorem (Theorem 1.7) and also in the proof of the existence of kernel in a bipartite digraph (Theorem 1.8).

Definition 1.3. [10] *A semikernel S of D is an independent set of vertices such that for each $z \in V(D) - N$ if there is a Nz-arc in D then there exists a zN-arc in D.*

The following theorems are called the classical results in Kernel Theory, all of these are about finite digraphs except Theorem 1.8.

Theorem 1.4. *If D is a symmetrical digraph then D has a kernel. In fact, a subset N of vertices of D is a kernel if and only if N is a maximal independent set.*

Theorem 1.5. [9] *Let D be a transitive digraph. Then D has a kernel. Furthermore every kernel is obtained by choosing just one vertex of each terminal strong component. So, every kernel has the same number of elements and every minimal absorbing set is a kernel.*

Theorem 1.6. [11] *Every acyclic digraph has a kernel.*

Theorem 1.7. [12, 10] *If D is a digraph without odd cycles then D has a kernel.*

Theorem 1.8. [10] *Every bipartite digraph, finite or infinite, has a kernel.*

One of the most important theorems about kernels, just for its applicability, is the following.

Theorem 1.9. [2] *Let D be a digraph. If every cycle has a symmetrical arc then D is a kernel-perfect digraph.*

There are some results about the existence of kernels in infinite digraphs.

Theorem 1.10. [4] *Let D be an outwardly finite digraph (only a finite number of successors for every vertex). D is a kernel-perfect digraph if and only if every finite induced subdigraph has a kernel.*

Theorem 1.11. [4] *Assume that every odd directed cycle C of an outwardly finite digraph D has the following property: if all arcs of C are incident to a subset T of vertices of C, then some chord of C has its head in T. Then D is a kernel-perfect digraph.*

Theorem 1.12. [7] *Let D be a digraph. Suppose that there exists two subdigraphs of D say D_1 and D_2 such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_1 is a right-pretransitive digraph, D_2 is a left-pretransitive digraph and D_i contains no infinite outward path for $i \in \{1, 2\}$. Then D is a kernel-perfect digraph.*

Corollary 1.13. *Let D be a right-pretransitive or a left-pretransitive digraph. If D contains no infinite outward path then D is a kernel-perfect digraph.*

Corollary 1.14. *Let D be a digraph. Suppose that there exists two subdigraphs of D say D_1 and D_2 such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_i is a transitive digraph and D_i contains no infinite outward path for $i \in \{1, 2\}$. Then D is a kernel-perfect digraph.*

Theorem 1.15. [8] *Let D be a digraph such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_i is a quasi-transitive subdigraph of D which contains no asymmetrical (in D) infinite outward path for $i \in \{1, 2\}$. If every triangle contained in D has at least two symmetrical arcs, then D is a kernel-perfect digraph.*

Theorem 1.16. [9] *For every digraph, finite or infinite, every semikernel is contained in a maximal semikernel.*

Theorem 1.17. [9] *Let D be a finite or infinite digraph. If every induced subdigraph of D has a nonempty semikernel then D is a kernel-perfect digraph.*

Other results about the existence of kernels in infinite digraphs can be found in [5].

In this paper we generalize part of Theorem 1.5, Theorems 1.4, 1.6, 1.7 and 1.9, and Corollary 1.13 for infinite digraphs.

2. Semikernels in Infinite Digraphs

We first prove the following result, which is used in the proofs of subsequent theorems.

Theorem 2.1. *Let D be a digraph possibly infinite. If every cycle and every infinite outward path has a symmetrical arc then there exists $u \in V(D)$ such that $\{u\}$ is a semikernel of D.*

Proof. Since $\{u\}$ is an independent set for every vertex u, we only have to prove that there is a vertex $u \in V(D)$ such that for all $z \in (V(D) - \{u\})$, $(u, z) \in A(D)$ implies $(z, u) \in A(D)$.

Suppose, for a contradiction that for every vertex u there is a vertex v such that $(u, v) \in A(D)$ and $(v, u) \notin A(D)$. Consider some $x_1 \in V(D)$. Then there exists x_2 such that $(x_1, x_2) \in A(D)$ and $(x_2, x_1) \notin A(D)$. So for each $n \in \mathbb{N}$, given $x_n \in V(D)$ there exists x_{n+1} such that $(x_n, x_{n+1}) \in A(D)$ and $(x_{n+1}, x_n) \notin A(D)$. If $x_i \neq x_j$ for all $i \neq j$ then $(x_n)_{n\in\mathbb{N}}$ is an asymmetrical infinite outward path in D, a contradiction. So, there are i, j such that $i \neq j$ and $x_i = x_j$. Suppose w.l.o.g. that $i < j$, then $(x_i, x_{i+1}, \ldots, x_j = x_i)$ is an asymmetrical closed walk in D and it contains an asymmetrical cycle C, a contradiction. □

Corollary 2.2. *Let D be a digraph possibly infinite. If every cycle and every infinite outward path has a symmetrical arc then D is a kernel-perfect digraph.*

Proof. It follows from Theorem 2.1 that every induced subdigraph of D has a nonempty semikernel. Then Theorem 1.17 implies that D is a kernel-perfect digraph. □

Corollary 2.3. *If D is a symmetrical digraph, finite or infinite, then D is a kernel-perfect digraph.*

Proof. It follows immediately from Corollary 2.2. □

Let $\mathcal{I}$ be the family of independent sets of a digraph D, which may be possibly infinite. Then $(\mathcal{I}, \subseteq)$ is a poset and it follows from Zorn's Lemma that $(\mathcal{I}, \subseteq)$ has at least one maximal element.

Observation 2.4. *Let D be a symmetrical digraph finite or infinite. A subset N of vertices of D is a kernel if and only if N is a maximal independent set.*

3. Kernels in Infinite Transitive and Pretransitive Digraphs

Lemma 3.1. [7] *Let D be a right-pretransitive or a left-pretransitive digraph. If $(x_1, x_2, \ldots, x_n)$ is a sequence of vertices such that $(x_i, x_{i+1}) \in A(D)$ and $(x_{i+1}, x_i) \notin A(D)$, then the sequence is a path and for each $i \in \{i, \ldots, n-1\}$, $(x_i, x_j) \in A(D)$ and $(x_j, x_i) \notin A(D)$, for every $j \in \{i+1, \ldots, n\}$.*

Theorem 3.2. *Let D be a digraph possibly infinite. Suppose that D is a right-pretransitive or a left-pretransitive digraph such that every infinite outward path has a symmetrical arc. Then D is a kernel-perfect digraph.*

Proof. From the hypothesis every infinite outward path has a symmetrical arc. Now, let $C = (x_1, x_2, \ldots, x_n = x_1)$ be a cycle. We claim that C has at least one symmetrical arc. Assume, for a contradiction that C has no symmetrical arc. Then $(x_1, x_2, \ldots, x_n)$ is a sequence that satisfies the hypothesis of Lemma 3.1 and hence C is a path, which is a contradiction since $x_n = x_1$. It follows from Corollary 2.2 that D is a kernel-perfect digraph. □

Corollary 3.3. *Let D be a digraph possibly infinite. Suppose that D is a transitive digraph such that every infinite outward path has a symmetrical arc. Then D is a kernel-perfect digraph.*

In order to obtain a result, in the infinite case, that generalizes the statement of Theorem 1.5 about the existence of a kernel in a transitive digraph considering the terminal strong components, we observe that under some conditions an infinite transitive digraph has at least one terminal strong component.

Observation 3.4. *Let D be a digraph, possibly infinite. Let $\mathcal{S}$ be the family of all strong subdigraphs of D. We define a relation $\leq$ on $\mathcal{S}$ by $S_1 \leq S_2$ if S_1 is a subdigraph of S_2. Then $(\mathcal{S}, \leq)$ is a poset and it follows from Zorn's Lemma that $(\mathcal{S}, \leq)$ has a maximal element. Hence D has at least one strong component.*

Now, let $\mathcal{C}$ be the family of strong components of D. If D_1 and D_2 are elements of $\mathcal{C}$ we define $D_1 \preccurlyeq D_2$ if there exists a $V(D_1)V(D_2)$-path in D. It is not difficult to see that $(\mathcal{C}, \preccurlyeq)$ is a partial ordered set, and the maximal elements of $(\mathcal{C}, \preccurlyeq)$, if they exist, are the terminal strong components of D.

Theorem 3.5. *Let D be a digraph possibly infinite. Suppose that D is a transitive digraph such that every infinite outward path has at least one symmetrical arc. Then $(\mathcal{C}, \preccurlyeq)$, with the partial order defined above, has at least one maximal element, i.e. D has at least one terminal strong component. Furthermore, for every strong component C of D there is a terminal strong component C' of D such that $C \preccurlyeq C'$.*

Proof. First, observe that if C_1 and C_2 are different strong components of D such that $C_1 \preccurlyeq C_2$, since D is a transitive digraph it follows that for every vertex $u \in V(C_1)$ and every vertex $v \in V(C_2)$, $(u, v) \in A(D)$; furthermore $(v, u) \notin A(D)$.

Now we proceed by contradiction. Suppose that for every $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}$, $C' \neq C$, such that $C \preccurlyeq C'$. Let $C_1 \in \mathcal{C}$ and let $u_1 \in V(C_1)$ then there exists $C_2 \in \mathcal{C}$, $C_1 \neq C_2$, such that $C_1 \preccurlyeq C_2$. Let $u_2 \in V(C_2)$. Then $(u_1, u_2) \in A(D)$ and $(u_2, u_1) \notin A(D)$. So, for each $n \in \mathbb{N}$, given $C_n \in \mathcal{C}$ and $u_n \in V(C_n)$ there exists $C_{n+1} \in \mathcal{C}$, $C_{n+1} \neq C_n$, and there exists $u_{n+1} \in V(C_{n+1})$ such that $(u_n, u_{n+1}) \in A(D)$ and $(u_{n+1}, u_n) \notin A(D)$. It follows from Lemma 3.1 that $(u_1, u_2, \ldots, u_{n+1})$ is an asymmetrical path in D. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is an asymmetrical infinite outward path in D, a contradiction. □

Theorem 3.6. *Let D be a digraph possibly infinite. Suppose that D is a transitive digraph such that every infinite outward path has at least one symmetrical arc. Then D has a kernel. Moreover, every kernel is obtained by choosing one vertex in each terminal strong component of D. So, all the kernels of D have the same cardinality.*

Proof. Using Theorem 3.5 and the proof technique similar to that of König [9], we get the result. □

4. Kernels in Infinite Acyclic Digraphs and Infinite Digraphs without Odd Cycles

It follows from Corollary 2.2 that every acyclic digraph without infinite outward path is kernel-perfect. We now proceed to obtain a similar result for infinite digraphs without odd cycles.

Definition 4.1. *Let D be a digraph and let $\mathcal{C}$ be the family of strong components of D. The condensation D^* of D is the digraph such that $V(D^*) = \mathcal{C}$ and $(C_i, C_j) \in A(D^*)$ if and only if there is a $V(C_i)V(C_j)$-arc in D.*

It is known that the condensation of any digraph is acyclic.

Theorem 4.2. *Let D be a digraph possibly infinite. Suppose that D contains no infinite outward path. Then D^* contains no infinite outward path.*

Proof. We will prove the contrapositive statement. Suppose that the sequence $(C_i)_{i\in\mathbb{N}}$ is an infinite outward path in D^*. Consider $c_1 \in V(C_1)$. Since $(C_1, C_2) \in A(D^*)$ and C_i is a strong component of D for each $i \in \mathbb{N}$, then there exists a c_1c_2-path in D, say T_1, such that $(V(T_1) - c_2) \subseteq V(C_1)$ and $c_2 \in V(C_2)$. Similarly, since $(C_2, C_3) \in A(D^*)$ there is a c_2c_3-path T_2 in D, such that $(V(T_2) - \{c_3\}) \subseteq V(C_2)$ and $c_3 \in V(C_3)$. So, for each $i \in \mathbb{N}$ there exists a c_ic_{i+1}-path T_i in D such that $(V(T_i) - c_{i+1}) \subseteq V(C_i)$ and $c_{i+1} \in V(C_{i+1})$. Let $T = \cup_{i\in\mathbb{N}} T_i$ and suppose that $T = (c_1 = u_1, u_2, \ldots, u_m = c_2, u_{m+1}, \ldots)$. We claim that $u_i \neq u_j$ if $i \neq j$. This is obvious if both u_i and u_j lie in the same path T_k. Suppose $u_i \in V(T_k)$ and $u_j \in V(T_l)$ and $k \neq l$. Since C_k and C_l are two different strong components, $V(T_k) - \{c_{k+1}\} \subseteq V(C_k)$ and $V(T_l) - \{c_{l+1}\} \subseteq V(C_l)$), it follows that $u_i \neq u_j$. Thus T is an infinite outward path in D. □

Theorem 4.3. *Let D be an acyclic digraph possibly infinite. Suppose that D has no infinite outward path. Then there exists a vertex u in D such that $odeg(u) = 0$ ($odeg(u)$ denotes the outdegree of u).*

Proof. Suppose that for every $u \in V(D)$, $odeg(u) \geq 1$. Let $x_1 \in V(D)$. Then there exists x_2 such that $(x_1, x_2) \in A(D)$. So, for each $n \in \mathbb{N}$, given $x_n \in V(D)$ there exists x_{n+1} such that $(x_n, x_{n+1}) \in A(D)$. Since D is acyclic, it follows that $(x_n)_{n\in\mathbb{N}}$ is an infinite outward path in D, which is a contradiction. □

Theorem 4.4. *Let D be a digraph possibly infinite. If D contains no infinite outward path then D contains at least one terminal strong component.*

Proof. Theorem 4.2 implies that D^* contains no infinite outward path. Since D^* is acyclic, it follows from Theorem 4.3 that D^* has a vertex with outdegree 0. This vertex corresponds to a terminal strong component of D. □

Theorem 4.5. *Let D be a digraph possibly infinite. If D contains no infinite outward path and contains no odd cycle then D has a nonempty semikernel.*

Proof. From Theorem 4.4, D contains a terminal strong component, namely C. Let $v_0 \in V(C)$ be a fixed vertex. We define the following sets: $S = \{v \in V(C) \mid$ there is in D a v_0v-walk of even length$\}$ and $I = \{v \in V(C) \mid$ there is in D a v_0v-walk of odd length$\}$.

Since $v_0 \in S$, S is nonempty. The proof that S is a semikernel of D is same as the proof in the finite case, given in [10] and we omit the details. □

Theorem 4.6. *Let D be a digraph possibly infinite. If D contains no infinite outward path and contains no odd cycle then D is a kernel-perfect digraph.*

Proof. Follows from Theorems 4.5 and 1.17. □

Observation 4.7. *In all previous theorems the hypothesis regarding the infinite outward path cannot be deleted as shown by the following example.*

Let $D^\sharp$ be the digraph defined as follows: $V(D^\sharp) = \mathbb{N}$ and $(i, j) \in A(D^\sharp)$ if and only if $i < j$. $D^\sharp$ satisfies:

(i) $D^\sharp$ is a transitive digraph.

(ii) $D^\sharp$ is an acyclic digraph.

(iii) $D^\sharp$ contains no odd cycles and

(iv) $D^\sharp$ contains no terminal strong component.

(v) $D^\sharp$ contains no minimal absorbing set.

(vi) $D^\sharp$ contains no vertex with null outdegree.

(vii) $D^\sharp$ *contains no kernel.*

Neumann-Lara and Galeana-Sánchez [6] have introduced the concept of critical kernel imperfect digraph.

Definition 4.8. *A digraph D is said to be a critical kernel imperfect digraph if D contains no kernel but every proper induced subdigraph of D contains a kernel.*

Theorem 4.9. *Let D be a finite digraph that contains no kernel. Then there exists an induced subdigraph of D which is a critical kernel imperfect digraph.*

Theorem 4.9 can not be applied to infinite digraphs. $D^\sharp$ has no kernel and contains no critical kernel imperfect subdigraph because:

(i) Every finite induced subdigraph of $D^\sharp$ is transitive, so it has a kernel.

(ii) Every infinite induced subdigraph of $D^\sharp$ is isomorphic to $D^\sharp$, hence it has no kernel.

(iii) If $i \in V(D^\sharp)$ then $D^\sharp - i$ is an infinite induced subdigraph of $D^\sharp$.

It is not known if there is an infinite digraph that is critical kernel imperfect. Alternatively, we give the concept of finitely critical kernel imperfect digraph and prove that every infinite digraph without kernel contains such a subdigraph.

Definition 4.10. *An infinite digraph D is said to be a finitely critical kernel imperfect digraph if D contains no kernel but every finite induced subdigraph of D contains a kernel.*

Theorem 4.11. *Let D be an infinite digraph that contains no kernel. Then there exists an induced subdigraph of D which is critical kernel imperfect or is finitely critical kernel imperfect.*

Proof. If every finite induced subdigraph of D has a kernel then D is a finitely critical kernel imperfect digraph. Now, suppose that there is a finite induced subdigraph of D, namely H, such that H contains no kernel. Then Theorem 4.9 implies that H contains a subdigraph which is critical kernel imperfect. □

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