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A Note on 2-Distance Chromatic Numbers of Graphs

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Abstract

A 2-distance coloring of a graph G is a proper vertex coloring of G such that every two vertices at distance 2 or less are assigned different colors. The minimum integer k for which there is a k-coloring satisfying this condition is the 2-distance chromatic number $\chi_{\overline{2}}(G)$ of G. It is shown that if e is an edge and v is a vertex in a graph G, then $\chi_{\overline{2}}(G)-2 \le \chi_{\overline{2}}(G-e) \le \chi_{\overline{2}}(G)$ and $\chi_{\overline{2}}(G) - \deg v \le \chi_{\overline{2}}(G-v) \le \chi_{\overline{2}}(G)$. Furthermore, all bounds are sharp.

Keywords: proper coloring, 2-distance coloring.

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1. Introduction

A *proper coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that $c(u) \ne c(v)$ for every pair u, v of adjacent vertices of G. A *k-coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum integer k for which there is a proper k-coloring of G. For a vertex v of a graph G, let $N(v)$ denote the *neighborhood* of v and $N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. For vertices u and v in a graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest u - v path in G if u and v belong to the same component and $d(u, v) = \infty$ otherwise. In [3] an *ℓ-distance coloring* of a graph G is defined as a (proper) vertex coloring of G such that every two vertices at distance ℓ or less are assigned different colors. The minimum integer k for which there is a k-coloring satisfying this condition is the *ℓ-distance chromatic number* of G and denoted $\chi_{\overline{\ell}}(G)$. Therefore, $\chi_{\overline{1}}(G) = \chi(G)$ and, in general, $\chi_{\overline{\ell}}(G) = \chi(G^\ell)$ for each positive integer ℓ, where G^ℓ is the ℓ-th power of G. Hence, $\chi_{\overline{2}}(G) \ge 3$ if G contains a component of order at least 3.

The topic of ℓ-distance colorings has primarily been studied for the case $\ell = 2$. Wan [5] showed that if Q_n is an n-cube, then $n+1 \le \chi_{\overline{2}}(Q_n) \le 2^{\lceil \log_2(n+1) \rceil}$ and conjectured that $\chi_{\overline{2}}(Q_n) = 2^{\lceil \log_2(n+1) \rceil}$. This parameter was further studied for the case $\ell = 3$ and corresponding upper and lower bounds were given in [4], where Kim, Du, and Pardalos showed that $2n \le \chi_{\overline{3}}(Q_n) \le 2^{\lceil \log_2 n \rceil + 1}$. In [1] some general asymptotic results concerning the order of $\chi_{\overline{2}}(G)$ for any graph G in terms of its maximum degree and girth were given. Alon and Mohar [1] considered the maximum possible chromatic number of G^2, as G ranges over all graphs with maximum degree d and girth g. They found that for $3 \le g \le 6$, this maximum is $(1+o(1))d^2$ (where $o(1)$ tends to 0 as d tends to infinity); whereas for $g \ge 7$, the maximum is of order $d^2/\log d$. The result can be stated more precisely as follows. For every two integers $d \ge 2$ and $g \ge 3$, define $f_2(d,g)$ to be the maximum possible value of $\chi(G^2)$ over all graphs with maximum degree d and girth g. Then (i) there exists a function $\epsilon(d)$ that tends to 0 as d tends to infinity such that for all $g \le 6$, $(1-\epsilon(d))d^2 \le f_2(d,g) \le d^2+1$ and (ii) there are positive constants c_1 and c_2 such that for every $d \ge 2$ and $g \ge 7$, $c_1(d^2/\log d) \le f_2(d,g) \le c_2(d^2/\log d)$. In [2] the family $\mathcal{G}_d$ of d-dimensional grids was studied. In particular, it was shown that $\chi_{\overline{2}}(G) \le 2d+1$ for all $G \in \mathcal{G}_d$ with $d \ge 1$. Also, for 2-dimensional grids $G = G(n_1, n_2)$, the exact value of $\chi_{\overline{\ell}}(G)$ was determined for sufficiently large n_1 and n_2.

For a nontrivial graph G, let $e \in E(G)$ and $v \in V(G)$. It is known that both $\chi(G-e)$ and $\chi(G-v)$ are between $\chi(G)-1$ and $\chi(G)$. That is, the difference between $\chi(G)$ and each of $\chi(G-e)$ and $\chi(G-v)$ is at most 1. This, however, is not the case for the 2-distance chromatic number of a graph. In this note, we investigate how the 2-distance chromatic number of a graph is affected by deleting an edge or a vertex from the graph.

2. Edge Deletion and Vertex Deletion

We first present some elementary results. The proofs of these results are straightward and therefore are omitted. The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the greatest distance between two vertices of G. A *clique* of a graph G is a complete subgraph in G. The *clique number* $\omega(G)$ of G is the order of a largest clique in G. It is known that $\chi(G) \ge \omega(G)$ for every graph G.

Lemma 2.1. *Let G be a nontrivial graph and ℓ a positive integer.*

(a) *If H is a subgraph of G, then $\chi_{\overline{\ell}}(H) \le \chi_{\overline{\ell}}(G)$.*

(b) $\chi_{\overline{\ell}}(G)$ *equals the order of G if and only if G is connected and* $\operatorname{diam}(G) \le \ell$.

(c) $\chi_{\overline{\ell}}(G) = \chi(G^\ell) \ge \omega(G^\ell)$. *In particular,* $\chi_{\overline{\ell}}(G) \ge \omega(G^\ell) \ge \Delta(G)+1$ *for* $\ell \ge 2$.

For a positive integer k, let $\mathbb{N}_k = \{1, 2, \ldots, k\}$. We consider how the 2-distance chromatic number of a graph is affected by deleting an edge from the graph.

Theorem 2.2. *If e is an edge in a nontrivial graph G, then*

$$\chi_{\overline{2}}(G) - 2 \leq \chi_{\overline{2}}(G - e) \leq \chi_{\overline{2}}(G).$$

Furthermore, for each integer $k \geq 5$, there is a graph G with $\chi_{\overline{2}}(G) = k$ containing three edges e_0, e_1, and e_2 such that $\chi_{\overline{2}}(G - e_i) = k - i$ for $0 \leq i \leq 2$.

Proof. The inequality $\chi_{\overline{2}}(G - e) \leq \chi_{\overline{2}}(G)$ holds by Lemma 2.1(a). To show that $\chi_{\overline{2}}(G) \leq \chi_{\overline{2}}(G - e) + 2$, let $c : V(G - e) \to \mathbb{N}_k$ be a minimum 2-distance coloring of $G - e$, where $k = \chi_{\overline{2}}(G - e)$. Let $e = v_1v_2$ and consider the coloring c' of G defined by $c'(v) = c(v)$ if $v \in V(G) - \{v_1, v_2\}$ and $c'(v_i) = k + i$ for $i = 1, 2$. Then c' is a 2-distance coloring of G using at most $k + 2$ colors. Thus, $\chi_{\overline{2}}(G) \leq k + 2 = \chi_{\overline{2}}(G - e) + 2$.

For each integer $k \geq 5$, we construct a graph G_k containing edges e_0, e_1, and e_2 such that $\chi_{\overline{2}}(G_k - e_i) = k - i$ for $0 \leq i \leq 2$. Let G_5 be the graph obtained from the 6-cycle $C_6 = (v_1, v_2, \ldots, v_6, v_1)$ by adding the chord v_4v_6. Then G_5^2 is isomorphic to the complete graph of order 6 without an edge and so $\chi_{\overline{2}}(G_5) = \chi(G_5^2) = 5$. Let $e_0 = v_5v_6$, $e_1 = v_1v_6$, and $e_2 = v_4v_6$. It is then straightforward to verify that $\chi_{\overline{2}}(G_5) = \chi_{\overline{2}}(G_5 - e_i) = 5 - i$ for $0 \leq i \leq 2$.

Now let $G_6 = G_5 + v_1v_5$ and for $k \geq 7$ the graph G_k is obtained from G_6 by adding $k - 6$ new vertices $u_1, u_2, \ldots, u_{k-6}$ and joining each u_i $(1 \leq i \leq k - 6)$ to v_1 and v_4. Then $\chi_{\overline{2}}(G_k) = k$ by Lemma 2.1(b). For each $k \geq 6$, let $e_0 = v_5v_6$, $e_1 = v_1v_6$, and $e_2 = v_2v_3$ in the graph G_k. Again, one can verify that $\chi_{\overline{2}}(G_k - e_i) = k - i$ for $0 \leq i \leq 2$. □

We next consider how the 2-distance chromatic number of a graph is affected by deleting a vertex from the graph. If C_n is a cycle of order $n = 3k$ for some integer $k \geq 2$ and $v \in V(C_n)$, then $\chi_{\overline{2}}(C_n - v) = \chi_{\overline{2}}(P_{n-1}) = 3 = \chi_{\overline{2}}(C_n)$. On the other hand, $\chi_{\overline{2}}(P_n + K_1) = n + 1$ while $\chi_{\overline{2}}(P_n) = 3$ for $n \geq 3$. Therefore, for a graph G and a vertex v of G, the difference between $\chi_{\overline{2}}(G)$ and $\chi_{\overline{2}}(G - v)$ can be 0 or arbitrarily large. In fact, the following result shows that the difference is between 0 and the degree of v.

Theorem 2.3. *If v is a vertex in a nontrivial graph G, then*

$$\chi_{\overline{2}}(G) - \deg v \leq \chi_{\overline{2}}(G - v) \leq \chi_{\overline{2}}(G).$$

Proof. Since the inequality $\chi_{\overline{2}}(G - v) \leq \chi_{\overline{2}}(G)$ clearly holds by Lemma 2.1(a), it remains to show that $\chi_{\overline{2}}(G) \leq \chi_{\overline{2}}(G - v) + \deg v$. Let $c : V(G - v) \to \mathbb{N}_k$ be a minimum 2-distance coloring of $G - v$, where $k = \chi_{\overline{2}}(G - v)$. Let $N[v] = \{v = v_0, v_1, v_2, \ldots, v_d\}$ be the closed neighborhood of v, where $d = \deg v$. Then the coloring c' of G defined by $c'(u) = c(u)$ for each $u \in (V(G) - N[v]) \cup \{v_d\}$ and $c'(v_i) = k + 1 + i$ for $0 \leq i \leq d - 1$ is a 2-distance coloring of G using at most $k + d$ colors. Therefore, $\chi_{\overline{2}}(G) \leq k + d = \chi_{\overline{2}}(G - v) + \deg v$. □

We have seen that the cycle C_n with $n = 3k \geq 6$ shows that the upper bound in Theorem 2.3 is sharp. In order to verify that the lower bound is also sharp, we present additional information. For a graph G, its *corona* $\mathrm{cor}(G)$ is the graph obtained from G by adding a pendant edge at each vertex of G. Observe then that $\mathrm{diam}(\mathrm{cor}(G)) = \mathrm{diam}(G)+2$ if G is connected and $\Delta(\mathrm{cor}(G)) = \Delta(G) + 1$. It is known that the chromatic number of a nontrivial graph G equals the chromatic number of $\mathrm{cor}(G)$. This, however, is not the case for the 2-distance chromatic number. We show that $\chi_{\overline{2}}(\mathrm{cor}(G))$ is either $\chi_{\overline{2}}(G)$ or $\chi_{\overline{2}}(G) + 1$.

Theorem 2.4. *For every nontrivial graph G,*

$$\chi_{\overline{2}}(G) \le \chi_{\overline{2}}(\mathrm{cor}(G)) \le \chi_{\overline{2}}(G) + 1. \tag{1}$$

Furthermore, $\chi_{\overline{2}}(\mathrm{cor}(G)) = \chi_{\overline{2}}(G) + 1$ *if and only if* $\chi_{\overline{2}}(G) = \Delta(G) + 1$.

Proof. Since G is a subgraph of $\mathrm{cor}(G)$, it follows by Lemma 2.1(a) that $\chi_{\overline{2}}(G) \le \chi_{\overline{2}}(\mathrm{cor}(G))$. To verify the upper bound for $\chi_{\overline{2}}(\mathrm{cor}(G))$ in (1), observe that if v_1 and v_2 are distinct vertices belonging to the same component in G and u_i is the end-vertex adjacent to v_i in $\mathrm{cor}(G)$ for $i = 1, 2$, then $d(u_1, u_2) = d(v_1, v_2) + 2 \ge 3$. Therefore, if $\chi_{\overline{2}}(G) = k$ and $c : V(G) \to \mathbb{N}_k$ is a 2-distance coloring of G, then we may define a coloring $c' : V(\mathrm{cor}(G)) \to \mathbb{N}_{k+1}$ by $c'(v) = c(v)$ if $v \in V(G)$ and $c'(v) = k + 1$ if $v \in V(\mathrm{cor}(G)) - V(G)$. Since c' is a 2-distance $(k + 1)$-coloring of $\mathrm{cor}(G)$, it follows that $\chi_{\overline{2}}(\mathrm{cor}(G)) \le \chi_{\overline{2}}(G) + 1$ and so (1) holds.

Recall by Lemma 2.1(c) that $\chi_{\overline{2}}(G) \ge \Delta(G)+1$. If $\chi_{\overline{2}}(G) = \Delta(G)+1$, then $\chi_{\overline{2}}(\mathrm{cor}(G)) \ge \Delta(\mathrm{cor}(G)) + 1 = \Delta(G) + 2 = \chi_{\overline{2}}(G) + 1$ and so $\chi_{\overline{2}}(\mathrm{cor}(G)) = \chi_{\overline{2}}(G) + 1$ by (1). If $\chi_{\overline{2}}(G) = k \ge \Delta(G) + 2$, then let $c : V(G) \to \mathbb{N}_k$ be a minimum 2-distance coloring of G. For each vertex $v \in V(G)$ add an isolated vertex u_v and join u_v to v to construct $\mathrm{cor}(G)$. Since $|N_G[v]| \le \Delta(G) + 1 \le k - 1$, there exists a color in $\mathbb{N}_k$ that is not used to color the vertices in $N_G[v]$. Let $i_v \in \mathbb{N}_k$ be the smallest color that is not assigned to the vertices in $N_G[v]$ by c and consider the coloring c' of $\mathrm{cor}(G)$ such that $c'(v) = c(v)$ and $c'(u_v) = i_v$ for each $v \in V(G)$. Observe that c' is a 2-distance k-coloring of $\mathrm{cor}(G)$, that is, $\chi_{\overline{2}}(\mathrm{cor}(G)) \le k = \chi_{\overline{2}}(G)$ and so $\chi_{\overline{2}}(\mathrm{cor}(G)) = \chi_{\overline{2}}(G)$ again by (1). □

We now return to discuss the sharpness of the lower bound in Theorem 2.3. For each positive integer n, let G be the graph of order $2n + 1$ obtained from $\mathrm{cor}(K_n)$ by adding a new vertex v and joining v to each of the n end-vertices of $\mathrm{cor}(K_n)$. Hence $\deg v = n$ and $\chi_{\overline{2}}(G) = 2n+1$ since $\mathrm{diam}(G) = 2$; while $\chi_{\overline{2}}(G-v) = \chi_{\overline{2}}(\mathrm{cor}(K_n)) = n+1 = \chi_{\overline{2}}(G)-\deg v$.

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