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Grundy functions in the Cartesian Product

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Abstract

Let D be a digraph, $\alpha = (\alpha_v)_{v \in V(D)}$ a family where the α_v are mutually disjoint digraphs. The Cartesian product of α over D, denoted by $\sigma(D, \alpha)$ is defined as follows:

(i) $V(\sigma(D,\alpha)) = \bigcup_{v \in V(D)} V(\alpha_v)$,

(ii) $A(\sigma(D,\alpha)) = \bigcup_{v \in V(D)} A(\alpha_v) \cup \{(x,y) | x \in \alpha_u, y \in \alpha_v \text{ and } (u,v) \in A(D)\}$.

A non-negative integer function $g(x)$ is called a Grundy function on G if for every vertex x, $g(x)$ is the smallest non-negative integer which does not belong to the set $\{g(y)|y \in \Gamma^+(x)\}$. This concept was originated by Grundy in 1939, for digraphs without directed cycles. It was extended by C. Berge and Shützenberger in 1956. Also it was difused by C. Berge in 1956.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc. A digraph D is a kernel-perfect digraph whenever each one of its induced subdigraphs has a kernel.

The concepts of kernel of a digraph and Grundy function of a digraph are nearly related.

In this paper we prove sufficient and necessary conditions for the Cartesian Product $\sigma(D,\alpha)$ of a family of digraphs $\alpha = (\alpha_v)_{v \in V(D)}$ over a digraph D to have a Grundy function in terms of the existence of Grundy function or kernel in D and in each α_v. Also it is shown a relationship between the size of the Grundy function obtained for $\sigma(D,\alpha)$ and the size of the Grundy functions of the factors α_v.

Keywords: Grundy function; Cartesian product; kernel; kernel-perfect digraph.

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1. Introduction

For general concepts we refer the reader to [4].

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. Let S_1, S_2 be subsets of $V(D)$. The arc (u_1, u_2) will be called an S_1S_2 arc whenever $u_1 \in S_1$ and $u_2 \in S_2$. And $D[S_1]$ will denote the subdigraph of D induced by S_1.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc. A digraph D is a kernel-perfect digraph whenever each one of its induced subdigraphs has a kernel.

The concept of kernel was introduced by Von Neumann and Morgenstern [14] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors, see for example ([5], [6], [7], [10], [15]). This concept has found many applications, for example in Game Theory, Set Theory, Logic, Computational Complexity, Artifitial Intelligence etc.

A non-negative integer function $g(x)$ is called a Grundy function on G if for every vertex x, $g(x)$ is the smallest non-negative integer which does not belong to the set $\{g(y)|y \in \Gamma^+(x)\}$. This concept was originated by Grundy in 1939 [12] for digraphs without directed cycles. It was extended by C. Berge and Shützenberger in 1956 [2]. Also it was difused by C. Berge in 1956 [1] and later in [3] and [4].

The Grundy function can also be defined as a function $g(x)$ such that

(1) $g(x) = k > 0$ implies that for each $j < k$, there exists a $y \in \Gamma^+(x)$ with $g(y) = j$.

(2) $g(x) = k$ implies that each $y \in \Gamma^+(x)$ satisfies $g(y) \neq k$.

Grundy functions have found many applications in Game Theory see by example ([4], [8], [9], [13]).

The concepts of kernel of a digraph and Grundy function of a digraph are nearly related as we can see in the two following results:

Theorem 1.1. [4] *If D has a Grundy function $g(x)$, then the set $N = \{x \in V(D)|g(x) = 0\}$ is a kernel of D.*

Theorem 1.2. [4] *If D is a kernel-perfect digraph, then D possesses a Grundy function.*

In [4] C. Berge defined the cartesian sum of n digraphs $D_1, \ldots, D_n$ and proved that the cartesian sum $D_1 + D_2 + \cdots + D_n$ of digraphs having Grundy function, also has a Grundy function.

In this paper we prove sufficient and necessary conditions for the digraph $\sigma(D, \alpha)$ to have a Grundy function whenever each α_v and D to have a Grundy function or a kernel. Also we obtain relationships between the size of the obtained Grundy function for $\sigma(D, \alpha)$ and the size of the Grundy functions of the α_v and of D.

2. Grundy functions in the Cartesian product

Theorem 2.1. *Let D be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of mutually disjoint digraphs. If D is a kernel-perfect digraph and each α_v has a Grundy function, then $\sigma(D, \alpha)$ possesses a Grundy function.*

Proof. Let D and $\alpha = (\alpha_v)_{v \in V(D)}$ as in the hypothesis. We consider, for each $v \in V(D)$ any fixed Grundy function f_v of α_v. And S_0 a kernel of D.

Now we define the following sets:

$N_0 = \{x \in V(\sigma(D, \alpha)) | f_y(x) = 0 \text{ for some } y \in S_0\}$.

$M_0 = \{y \in V(D) | V(\alpha_y) \subseteq N_0\}$.

S_1 a kernel of $D_1 = D[V(D) - M_0]$ (the subdigraph of D induced by $V(D) - M_0$).

For $y \in S_1$ we denote $m(1, y) = \min\{f_y(x) | x \in (V(\alpha_y) - N_0)\}$

$N_1 = \{x \in (V(\sigma(D, \alpha)) - N_0) | f_y((x) = m(1, y) \text{ for some } y \in S_1\}$.

$M_1 = \{y \in V(D) | V(\alpha_y) \subseteq (N_0 \cup N_1)\}$.

Clearly $N_0 \cap N_1 = \emptyset$; $N_0 \neq \emptyset$ and $N_1 \neq \emptyset$

Continuing this way we define a succession of subsets of vertices of $\sigma(D, \alpha)$ of D as follows: If S_i is a kernel of D_i, D_i, N_i and M_i are defined. Then we define D_{i+1}, S_{i+1}, N_{i+1} and M_{i+1} as follows:

$D_{i+1} = D[V(D) - (M_0 \cup M_1 \cup \cdots \cup M_i)]$; S_{i+1} is a kernel of D_{i+1}; $m(i+1, y) = \min\{f_y(x) | x \in (V(\alpha_y) - N_0 \cup N_1 \cup \cdots \cup N_i)\}$ for $y \in S_{i+1}$;

$N_{i+1} = \{x \in V(\sigma(D, \alpha)) - N_0 \cup N_1 \cup \cdots \cup N_i | f_y(x) = m(i+1, y) \text{ for some } y \in S_{i+1}\}$;

$M_{i+1} = \{y \in V(D) | V(\alpha_y) \subseteq (N_0 \cup N_1 \cup \cdots \cup N_{i+1})\}$.

Clearly $N_i \cap N_j = \emptyset$ for any i, j with $i \neq j$.

This procedure finishes when we get the first natural number r such that $V(D_r) = \emptyset$. Notice that this natural number r exists as $N_i \cap N_j = \emptyset$ whenever $i \neq j$ and $N_i \neq \emptyset$ for each $0 \leq i \leq r - 1$.

Now we define the function $F : V(\sigma(D, \alpha)) \to \mathbb{N}$ as follows: for $x \in V(\sigma(D, \alpha)), F(x) = k$ if and only if $x \in N_k$. F is well defined as $N_i \cap N_j = \emptyset$ for any i, j with $i \neq j$ and $V(D_r) = \emptyset$.

Claim 1. N_k is an independent set of vertices of $\sigma(D, \alpha) - (N_0 \cup \cdots \cup N_{k-1}) = \tilde{\sigma}_k$ for each $1 \leq k \leq r - 1$. And N_0 is an independent set of $\sigma(D, \alpha)$.

Let $x, z \in N_k$ with $x \neq z$ and $0 \leq k \leq r - 1$. When $\{x, z\} \subseteq V(\alpha_y)$ for some $y \in V(D)$, we have $f_y(x) = f_y(z)$ and it follows from the definition of Grundy function that $\{(x, z), (z, x)\} \cap A(\alpha_y) = \emptyset$; and from the definition of $\sigma(D, \alpha)$ it follows that $\{(x, z), (z, x)\} \cap A(\sigma(D, \alpha)) = \emptyset$. When $x \in V(\alpha_y)$ and $z \in V(\alpha_{y'})$ with $y \neq y'$, $\{y, y'\} \subseteq V(D)$ it follows from the definition of N_k that $\{y, y'\} \subseteq S_k$ where S_k is a kernel of D_k. Thus there is no arc between y and y' in D (as S_k is independent in D_k and D_k

is an induced subdigraph of D). Hence from the definition of $\sigma(D,\alpha)$ there is no arc in $\sigma(D,\alpha)$ between $V(\alpha_y)$ and $V(\alpha_{y'})$. Thus $\{(x,z),(z,x)\} \cap A(\sigma(D,\alpha)) = \emptyset$.

Claim 2. N_k is an absorbent set of vertices of $\tilde{\sigma}_k = \sigma(D,\alpha) - (N_0 \cup N_1 \cup \cdots \cup N_{k-1})$.

Let $z \in (V(\tilde{\sigma}_k) - N_k)$ $0 \le k \le r-1$ clearly $z \in V(\alpha_y)$ for some $y \in V(D)$. We consider the two possibilities.

When $y \in S_k$, it follows from the definition of N_k that $f_y(z) > m(k,y)$ and since f_y is a Grundy function of α_y, there exists $x \in \Gamma^+_{\alpha_y}(z)$ such that $f_y(x) = m(k,y)$, and clearly $x \in N_k$.

When $y \notin S_k$, in this case and since S_k is a kernel of D_k we have that there exists $y' \in S_k$ such that $(y,y') \in A(D_k)$ and there exists $x \in V(\alpha_{y'})$ such that $f_{y'}(x) = m(k,y)$. (Notice that $y \in V(D_k)$ as $z \in (V(\alpha_y) - N_k)$). Thus $x \in N_k$ and from the definitions of $\sigma(D,\alpha)$ and $\tilde{\sigma}$ we have $(z,x) \in A(\tilde{\sigma}_k)$.

Then N_k is a kernel of $\tilde{\sigma}_k$ for each k, $0 < k \le r-1$ and N_0 is a kernel of $\sigma(D,\alpha)$. And it follows that F is a Grundy function of $\sigma(D,\alpha)$. □

As a consequence of the proof of Theorem 2.1 we obtain some bounds for the Grundy function F constructed in the proof of Theorem 2.1.

Theorem 2.2. *Let D be a digraph $\alpha = (\alpha_v)_{v\in V(D)}$ a family of mutually disjoint digraphs. If D is a kernel-perfect digraph and each α_v has a Grundy function $f_u, u \in V(D)$, then $\sigma(D,\alpha)$ possesses a Grundy function F such that $\max\{F(x)|x \in V(\sigma(D,\alpha))\} \le \sum_{u\in V(D)} m_u + |V(D)| - 1$ where $m_u = \max\{f_u(x)|x \in V(\alpha_u)\}$.*

Proof. Since D is a kernel-perfect digraph then D has a Grundy function f. Let F be the Grundy function of $\sigma(D,\alpha)$ defined in the proof of Theorem 2.1. Since $F^{-1}(i) = \cup f_v^{-1}(j)$ for some $j \in \mathbb{N}$ and $v \in V(D)$ we have that the bound reaches whenever $F^{-1}(i) = f_v^{-1}(j)$ for some $j \in \mathbb{N}$ and $v \in V(D)$. □

Remark 2.1. *Figures 1 and 2 show a digraph D and a family $\alpha = (\alpha_v)_{v\in V(D)}$ for which the bound obtained in Theorem 2.2 is reached.*

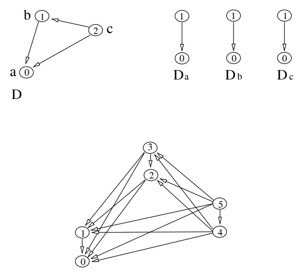

Figure 1

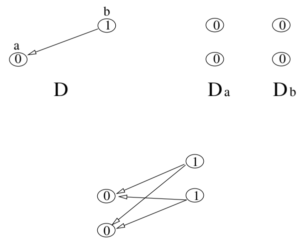

Figure 2

Remark 2.2. *Figure 3 shows a digraph D which has a Grundy function and a family $\alpha = (\alpha_v)_{v \in V(D)}$ of digraphs each one of them having a Grundy function. However $\sigma(D, \alpha)$ has no Grundy function. Thus the hypothesis that D is a kernel-perfect digraph in Theorem 2.1, cannot be dropped.*

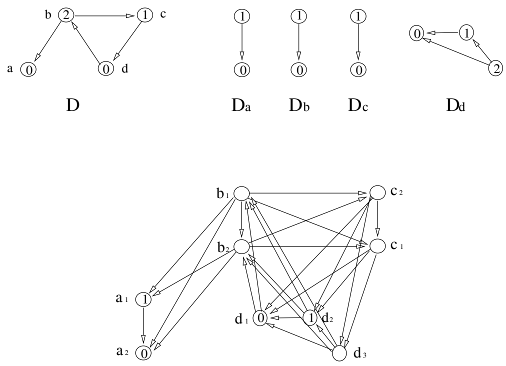

Figure 3

Theorem 2.3. *Let D be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of mutuallly disjoint digraphs. If $\sigma(D, \alpha)$ has a Grundy function, then D has a kernel and α_v has a Grundy function for each $v \in V(D)$.*

Proof. First we prove that D has a kernel. Let f be a Grundy function of $\sigma(D, \alpha)$; $S = \{x \in V(\sigma(D, \alpha)) | f(x) = 0\}$; and $A = \{u \in V(D) | V(\alpha_u) \cap S \neq \emptyset\}$.

Claim 5 A is an independent set of vertices of D.

Let $u, w \in A$ with $u \neq w$, then there exists $x \in V(\alpha_u)$ such that $f(x) = 0$ and $y \in V(\alpha_w)$ such that $f(y) = 0$. Since f is a Grundy function in $\sigma(D, \alpha)$ it follows that there is no arc in $\sigma(D, \alpha)$ between x and y. Hence from the definition of $\sigma(D, \alpha)$ it follows that there is no arc between u and w in D.

Claim 6 A is an absorbent set of vertices of D. (That means that for each $u \in (V(D) - A)$ there exists a $w \in A$ such that $(u, w) \in A(D)$).

Let $u \in (V(D) - A)$, then we have $f(x) \neq 0$ for each $x \in V(\alpha_u)$. Take $x \in V(\alpha_u)$, so from the definition of the Grundy function $f(x) > 0$. Therefore there exists $y \in \Gamma^+_{\sigma(D,\alpha)}(x)$

such that $f(y) = 0$. Now $y \in V(\alpha_w)$ for some $w \in V(D)$. From the definition of $\sigma(D, \alpha)$ we have $w \in \Gamma^+_D(u)$. And from the definition of A we have $w \in A$.

Claim 7 A is a kernel of D. Is a direct consequence of Claims 5 and 6.

Now let $u \in V(D)$ be. We will prove that α_u has a Grundy function. Let $n = \max\{f(x)|x \in V(\sigma(D, \alpha))\}$ be and $S_i = \{x \in V(\sigma(D, \alpha))|f(x) = i\}$, $0 \leq i \leq n$. Since f is a Grundy function, it is clear that S_k is a kernel of $\sigma(D, \alpha) - (\bigcup_{j=0}^{k-1} S_j) = \tilde{\sigma}_k$. And S_0 is a kernel of $\sigma(D, \alpha)$.

Let $m = \min\{f(y)|y \in V(\alpha_u)\}$ and $M = \max\{f(y)|y \in V(\alpha_u)\}$.

Now we construct a succession of subsets of $V(\alpha_u)$, $A_0, A_1, \ldots, A_j$; and a succession of natural numbers $r_0, r_1, \ldots, r_j$ as follows: $r_0 = m$ and $A_0 = S_{r_0} \cap V(\alpha_u)$; if r_i and A_i are defined and $r_i < M$ we define r_{i+1} and A_{i+1} as follows: $r_{i+1} = \min\{t \in \{r_i + 1, r_i + 2, \ldots, M\}|S_t \cap V(\alpha_u) \neq \emptyset\}$ and $A_{i+1} = S_{r_{i+1}} \cap V(\alpha_u)$. The successions finished when we get the first r_j such that $r_j = M$. Thus $r_j = M$ and $A_j = S_M \cap V(\alpha_u)$.

Claim 8 A_i is a kernel of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$ for each $1 \leq i \leq j$ and A_0 is a kernel of α_u. Let $i, 0 \leq i \leq j$.

A_i is an independent set of vertices of α_u because: $A_i \subseteq S_{r_i}$, S_{r_i} is an independent set of vertices of $\sigma(D, \alpha)$; $A_i = S_{r_i} \cap V(\alpha_u)$, and α_u is an induced subdigraph of $\sigma(D, \alpha)$. Therefore A_i is an independent set of vertices of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$.

A_i is an absorbent set of vertices of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$. Let $x \in V(\alpha_u - (\bigcup_{s=0}^{i} A_s))$. Clearly $\alpha_u - (\bigcup_{s=0}^{i-1} A_s) \subseteq \sigma(D, \alpha) - \bigcup_{t=0}^{r_i} S_t$ and S_{r_i} is a kernel of $\sigma(D, \alpha) - \bigcup_{t=0}^{r_i - 1} S_t$. Thus there exists $y \in S_{r_i}$ such that $(x, y) \in A(\sigma(D, \alpha))$. Now we have that $y \in V(\alpha_u)$. Otherwise $y \in V(\alpha_w) \cap S_{r_i}$ for some $w \neq u$, $w \in V(D)$; thus $(u, w) \in A(D)$. Let $z \in A_i$ be; from the definition of $\sigma(D, \alpha)$ we have $(z, y) \in A(D)$ with $\{z, y\} \subseteq S_{r_i}$ contradicting that S_{r_i} is a kernel of $\tilde{\sigma}_{r_i}$. Thus $y \in V(\alpha_u) \cap S_{r_i} = A_i$ and A_i is absorbent in $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$. And Claim 8 is proved.

Let $f_u : V(\alpha_u) \to \mathbb{N}$ the function defined as follows: $f_u(x) = k$ if and only if $x \in A_k$.

$f_u(x)$ is well defined as from the definition we have $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and clearly $V(\alpha_u) = \bigcup_{s=0}^{j} A_s$.

Claim 9 f_u is a Grundy function of α_u.

(1) $f_u(x) = k > 0$ implies that for each $i, 0 \leq i < k$, there exists a $y \in \Gamma^+_{\alpha_u}(x)$ with

$f_u(y) = i$.

Suppose $f_u(x) = k > 0$ and let $i, 0 \leq i < k$. From Claim 8 A_i is a kernel of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$ for $1 \leq i < k$ and A_0 is a kernel of α_u. Thus A_i is an absorbent set of vertices of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$ for $1 \leq i < k$ and A_0 is an absorbent set of vertices of α_u. Clearly $\alpha_u - (\bigcup_{s=0}^{k-1} A_s)$ is an induced subdigraph of $\alpha_u - (\bigcup_{s=0}^{i-1} A_s)$ $(A_{-1} = \emptyset)$. Since $x \in A_k$ (from the definition of f_u) and $A_k \cap (\bigcup_{s=0}^{i} A_s) = \emptyset$; there exists $y \in A_i$; such that $(x, y) \in A(\alpha_u - (\bigcup_{s=0}^{i-1} A_s)) \subseteq A(\alpha_u)$ and $f_u(y) = i$.

(2) $f_u(x) = k$ implies that each $y \in \Gamma^+_{\alpha_u}(x)$ satisfies $f_u(y) \neq k$.

Let $y \in \Gamma^+_{\alpha_u}(x)$ be and assume by contradiction that $f_u(y) = k$. Therefore $f_u(x) = f_u(y) = k$ which implies $\{x, y\} \subseteq A_k$, contradicting that A_k is independent in $\alpha_u - (\bigcup_{s=0}^{k-1} A_s)$ (as A_k is a kernel of $\alpha_u - (\bigcup_{s=0}^{k-1} A_s)$ (Claim 8)). So Claim 9 is proved and Theorem 2.3 is proved. □

Remark 2.3. *Figure 4 shows a digraph D, a family of mutually disjoint digraphs $(\alpha_v)_{v \in V(D)}$ and the sum $\sigma(D, \alpha)$ such that: $\sigma(D, \alpha)$ has a Grundy function, each α_v possesses a Grundy function $v \in V(D)$, D has a kernel but D has no Grundy function. Thus the conclusion of Theorem 2.3 is the best possible in terms of the existence of Grundy functions or kernels in D and in each α_v, $v \in V(D)$.*

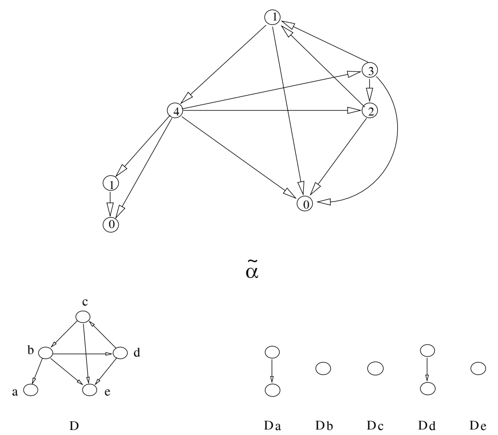

Figure 4

Theorem 2.4. *Let D be a digraph and $\alpha = (\alpha_u)_{u\in V(D)}$ a family of mutually disjoint digraphs. If D has a Grundy function f and each α_u has a Grundy function f_u $(u \in V(D))$ such that* $\max\{f_u(x)|x \in V(\alpha_u)\} = \max\{f_v(x)|x \in V(\alpha_v)\}$ *whenever* $\{u, v\} \subseteq f^{-1}(i)$ *for some* $i \in \mathbb{N}$. *Then* $\sigma(D, \alpha)$ *possesses a Grundy function F which satisfies:* $\max\{F(x)|x \in V(\sigma(D, \alpha))\} = n + \sum_{i=0}^{n} m_i$ *where* $n = \max\{f(x)|x \in V(D)\}$ *and* $m_i = \max\{f_v(x)|x \in V(\alpha_v)$ *and* $v \in f^{-1}(i)\}$, *for each* $0 \leq i \leq n$.

Proof. Define $F : V(\sigma(D, \alpha)) \to \mathbb{N}$ as follows:

Let $x \in V(\sigma(D, \alpha))$ be; we have that there exist an unique $y \in V(D)$ such that $x \in V(\alpha_y)$ and an unique $i \in \{0, 1, \ldots, n\}$ such that $y \in f^{-1}(i)$; then we define $F(x) = \sum_{j=0}^{i-1} m_j + f(y) + f_y(x)$; whenever $1 \leq i \leq n$ and $F(x) = f_y(x)$ whenever $i = 0$.

We will prove that F is a Grundy function of $\sigma(D, \alpha)$.

(1) $F(x) = k > 0$ implies that for each $s < k$, there exists a $z \in \Gamma^+_{\sigma(D,\alpha)}(x)$ with $F(z) = s$.

Let $x \in V(\sigma(D, \alpha))$ such that $F(x) = k > 0$ and s, $0 \leq s < k$. Take the unique y and i such that $x \in V(\alpha_y)$ and $y \in f^{-1}(i)$.

If $i = 0$, then $F(x) = f_y(x) = k > 0$. Since f_y is a Grundy function of α_y, there exists a $z \in \Gamma^+_{\alpha_y}(x)$ with $f_y(z) = s$. Therefore $z \in \Gamma^+_{\sigma(D,\alpha)}(x)$ and $F(z) = f_y(x) = s$ (as $i = 0$).

If $1 \leq i \leq n$, then $F(x) = \sum_{j=0}^{i-1} m_j + f(y) + f_y(x) = k$.

Let $s_0 = 0$ and $s_t = \sum_{j=0}^{t-1} m_j + t$ for each $1 \leq t \leq n$. Clearly $s_0 \leq s < s_n$ (as $0 \leq s < k$ and from the definition of F).

Let $r = \max\{a \in \{0, 1, \ldots, n\}|s \geq s_a\}$ notice that r is well defined as $s_0 \leq s < s_n$. Thus $s \geq s_r$ and $s < s_{r+1}$. Therefore $\sum_{j=0}^{r-1} m_j + r \leq s$ and $s < \sum_{j=0}^{r} m_j + r + 1$. Hence $s \leq \sum_{j=0}^{r-1} m_j + m_r + r$. It follows that $0 \leq s - \sum_{j=0}^{r-1} m_j - r \leq m_r$.

Now we prove that $r \leq i$. Assume by contradiction that $r > i$, thus $r - 1 \geq i$.

Let $b = s - \sum_{j=0}^{r-1} m_j - r$. Thus $0 \leq b \leq m_r$ and $s = b + \sum_{j=0}^{r-1} m_j + r$. Since $0 \leq s < k$ we have $s = b + \sum_{j=0}^{r-1} m_j + r < k = F(x) = \sum_{j=0}^{i-1} m_j + i + f_y(x)$. Now since $f_y(x) \leq m_i$ we have

$b+\sum_{j=0}^{r-1} m_j+r<\sum_{j=0}^{i} m_j+i$ and then $0<\sum_{j=0}^{r-1} m_j-\sum_{j=0}^{i} m_j+(r-i)<-b$ (as $r-i>0$ and $r-1\geq i$).

Since $b\geq 0$ we have $-b\leq 0$. Therefore $0<-b\leq 0$ and so $0<0$, a contradiction. Hence $r\leq i$.

Since $f(y)=i\geq r\geq 0$ and f is Grundy function of D; there exists $z\in\Gamma_D^+(y)$ such that $f(z)=r$.

From the hypothesis and the definition of m_r we have that $m_r=\max\{f_v(x)|x\in V(\alpha_v)$ and $v\in f^{-1}(r)\}$ and $\max\{f_u(x)|x\in V(\alpha_u)\}=\max\{f_v(x)|x\in V(\alpha_v)\}$ whenever $\{u,v\}\subseteq f^{-1}(r)$. Hence $m_r=\max\{f_z(x)|x\in V(\alpha_z)\}$.

Since f_z is a Grundy function on α_z and $0\leq s-\sum_{j=0}^{r-1} m_j-r\leq m_r$, then there exists $w\in V(\alpha_z)$ such that $f_z(w)=s-\sum_{j=0}^{r-1} m_j-r=s-\sum_{j=0}^{r-1} m_j-f(z)$.

This is $f_z(w)=s-\sum_{j=0}^{r-1} m_j-f(z)$.

On the other hand we have that $F(w)=\sum_{j=0}^{r-1} m_j+f(z)+f_z(w)=\sum_{j=0}^{r-1} m_j+f(z)+(s-\sum_{j=0}^{r-1} m_j-f(z))=s$. Thus $F(w)=s$ and since $w\in V(\alpha_z)$ and $z\in\Gamma_D^+(y)$ it follows from the definition of $\sigma(D,\alpha)$ that $w\in\Gamma^+_{\sigma(D,\alpha)}(x)$ (as $x\in V(\alpha_y)$). We conclude that $w\in\Gamma^+_{\sigma(D,\alpha)}(x)$ with $F(w)=s$, as required.

(2) $F(x)=k$ implies that each $y\in\Gamma^+_{\sigma(D,\alpha)}(x)$ satisfies $F(y)\neq k$.

Suppose $F(x)=k$ and assume by contradiction that there exists $y\in\Gamma^+_{\sigma(D,\alpha)}(x)$ such that $F(y)=k=F(x)$. Let $z,w\in V(D)$ and $r,s\in\{0,\ldots,n\}$ be such that $x\in V(\alpha_z)$, $f(z)=r$, $y\in V(\alpha_w)$, $f(w)=s$.

Here we consider the two possible cases:

Case 1. $w=z$.

In this case $r=s$. If $r=s=0$ then $F(x)=f_z(x)=F(y)=f_w(y)=f_z(y)$ with $\{x,y\}\subseteq V(\alpha_z)$. Since $F(x)=k=f_z(x)$, $y\in\Gamma^+_{\alpha_z}(x)$ (as $y\in\Gamma^+_{\sigma(D,\alpha)}(x)$ and $\{x,y\}\subseteq V(\alpha_z)$) and f_z is a Grundy function of α_z it follows that $f_z(y)\neq k$, a contradiction.

If $r=s>0$, then $F(x)=\sum_{j=0}^{r-1} m_j+r+f_z(x)=F(y)=\sum_{j=0}^{r-1} m_j+r+f_z(y)$. Thus $f_z(x)=f_z(y)$ with $y\in\Gamma^+_{\alpha_z}(x)$, a contradiction as f_z is a Grundy function of α_z.

Case 2 $w \neq z$.

From the definition of $\sigma(D,\alpha)$ and since $(x,y) \in A(\sigma(D,\alpha))$; it follows that $(z,w) \in A(D)$. Since f is a Grundy function of D it follows that $f(z) \neq f(w)$. Assume without loss of generality that $f(z) < f(w)$. When $f(z) = 0 = r < s = f(w)$ we have $F(x) = f_z(x)$ and $F(y) = \sum_{j=0}^{s-1} m_j + s + f_w(y)$. From our assumption we obtain $f_z(x) = \sum_{j=0}^{s-1} m_j + s + f_w(y)$. Since $f(z) = 0$ from the definition of m_0 we have $f_z(x) \leq m_0$ and $s \geq 1$. Thus $m_0 \geq f_z(x) = \sum_{j=0}^{s-1} m_j + s + f_w(y) \geq m_0 + 1 + f_w(y)$. So $1 + f_w(y) \leq 0$, a contradiction as $f_w(y) \geq 0$.

So we may assume $0 < f(z) = r < s = f(w)$. From our assumption we have $F(x) = F(y)$. Thus $\sum_{j=0}^{r-1} m_j + r + f_z(x) = \sum_{j=0}^{s-1} m_j + s + f_w(y)$. Clearly $s-1 \geq r$. Since $f(z) = r$, $m_r = \max\{f_v(x) | x \in V(\alpha_v) \text{ and } v \in f^{-1}(r)\}$, and from the hypothesis $\max\{f_u(x)|x \in V(\alpha_u)\} = \max\{f_v(x)|x \in V(\alpha_v)\}$ for any $u, v \in f^{-1}(r)$; it follows that $f_z(x) \leq m_r$. Therefore $\sum_{j=0}^{s-1} m_j + s + f_w(y) = \sum_{j=0}^{r-1} m_j + r + f_z(x) \leq \sum_{j=0}^{r} m_j + r$. Then $\sum_{j=0}^{s-1} m_j - \sum_{j=0}^{r} m_j + (s-r) \leq -f_w(y)$. But $s - r > 0$ (as $s > r$); $\sum_{j=0}^{s-1} m_j - \sum_{j=0}^{r} m_j \geq 0$ (as $s - 1 \geq r$) and $f_w(y) \geq 0$ (as f_w is a Grundy function of α_w). Thus $0 < \sum_{j=0}^{s-1} m_j - \sum_{j=0}^{r} m_j + (s-r) \leq -f_w(y) \leq 0$. A contradiction. We conclude that F is a Grundy function of $\sigma(D,\alpha)$.

From the definition of F it is clear that $\max\{F(x)|x \in V(\sigma(D,\alpha))\} = n + \sum_{i=0}^{n} m_i$. $\square$

Definition 2.1. [11] *Let D be a digraph, $\tilde{\alpha} = (\alpha_v, S_v)_{v \in V(D)}$ a family where the α_v are mutually disjoint digraphs and S_v is a non-empty subset of $V(\alpha_v)$ for each $v \in V(D)$. We denote by $\sigma(D,\tilde{\alpha})$ the digraph defined by the following conditions:*

(i) $V(\sigma(D,\tilde{\alpha})) = \bigcup_{v \in V(D)} V(\alpha_v)$,

(ii) $A(\sigma(D,\tilde{\alpha})) = (\bigcup_{v \in V(D)} A(\alpha_v)) \cup \{(x,y)|x \in S_u, y \in S_v \text{ and } (u,v) \in A(D)\}$.

Theorem 2.5. *Let D be a digraph, $\tilde{\alpha} = (\alpha_v, S_v)_{v \in V(D)}$ a family where the α_v are mutually disjoint digraphs and S_v is a non-empty subset of $V(\alpha_v)$ for each $v \in V(D)$. If each α_v has a Grundy function f_v ($v \in V(D)$) such that $f_u(x) \neq f_v(y)$ whenever $(u,v) \in A(D)$, $x \in S_u$ and $y \in S_v$. Then $\sigma(D,\tilde{\alpha})$ possesses a Grundy function F which satisfies* $\max\{F(x)|x \in V(\sigma(D,\tilde{\alpha}))\} = \max\{f_v(x)|x \in V(\alpha_v),$ *and* $v \in V(D)\}$.

Proof. Let $F : V(\sigma(D,\tilde{\alpha})) \to \mathbb{N}$ defined as follows: If $x \in V(\sigma(D,\tilde{\alpha}))$ then there is a unique $u \in V(D)$ such that $x \in V(\alpha_u)$. And we define $F(x) = f_u(x)$.

We will prove that F is a Grundy function of $\sigma(D,\tilde{\alpha})$.

(1) $F(x) = k > 0$ implies that for each $j < k$, there exists a $y \in \Gamma^+_{\sigma(D,\alpha)}(x)$ with $F(y) = j$.

Let $x \in V(\sigma(D,\tilde{\alpha}))$ such that $F(x) = k > 0$, and let $u \in V(D)$ the vertex of D such that $x \in V(\alpha_u)$. From the definition of F we have $F(x) = f_u(x)$. Since f_u is a Grundy function of α_u, there exists $y \in \Gamma^+_{\alpha_u}(x)$ such that $f_u(y) = j$. From the definition of $\sigma(D,\tilde{\alpha})$ we have $y \in \Gamma^+(x)$ and from the definition of F we have $F(y) = j$.

(2) $F(x) = k$ implies that each $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$ satisfies $F(y) \neq k$.

Let $x \in V(\sigma(D,\tilde{\alpha}))$ such that $F(x) = k > 0$ and $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$. Also let u be the vertex of D such that $x \in V(\alpha_u)$. When $y \in V(\alpha_u)$ we have from the definition of $\sigma(D,\tilde{\alpha})$ that $y \in \Gamma^+_{\alpha_u}(x)$ (as $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$). Since f_u is a Grundy function of α_u and $f_u(x) = F(x) = k$ we obtain $f_u(y) \neq k$. Thus $F(y) = f_u(y) \neq k$. So we will assume $y \notin V(\alpha_u)$.

Let $v \in V(D)$ such that $y \in V(\alpha_v)$. Since $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$, $x \in V(\alpha_u)$, $y \in V(\alpha_v)$ and $u \neq v$, it follows from the definition of $\sigma(D,\tilde{\alpha})$ that $(u,v) \in A(D)$, $x \in S_u$ and $y \in S_v$. Thus from the hypothesis we have $f_u(x) \neq f_v(y)$. And from definition of F we obtain $F(x) = f_u(x) \neq f_v(y) = F(y)$.

We conclude that F is a Grundy function of $\sigma(D,\tilde{\alpha})$.

Clearly from the definitions of F and $\sigma(D,\tilde{\alpha})$ we have $\max\{F(x)|x \in V(\sigma(D,\tilde{\alpha})\} = \max\{f_v(x)|x \in V(\alpha_v) \text{ and } v \in V(D)\}$. □

Remark 2.4. *The hypothesis that $f_u(x) \neq f_v(y)$ whenever $(u,v) \in A(D)$, $x \in S_u$ and $y \in S_v$ in Theorem 2.5 cannot be dropped. Figure 5 shows digraphs D, D_a, D_b; $S_a = \{a_1, a_2\} \subseteq V(D_a), S_b = \{b_1, b_2\} \subseteq V(D_b)$, f_a a Grundy function of D_a, f_b a Grundy function of D_b, $f_a(a_2) = f_b(b_2) = 2$. However $\sigma(D,\tilde{\alpha})$ has no Grundy function. (Observe that if F where any Grundy function of $\sigma(D,\tilde{\alpha})$, then $F(a_1) = F(b_1) = 0$, $F(b_2) = 1$, $F(a_2) = F(b_3) = 2$ and F cannot be extended to the remaining vertices of $\sigma(D,\tilde{\alpha})$, a_3, a_4 and a_5 because the subdigraph of $\sigma(D,\tilde{\alpha})$ induced by $\{a_3, a_4, a_5\}$ is a directed cycle of length 3 which has no kernel.*

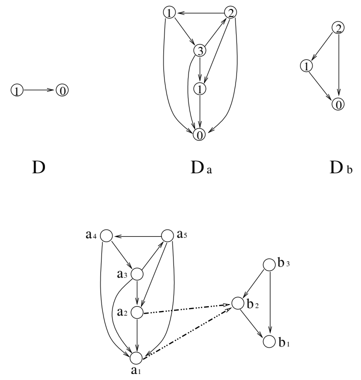

Figure 5

Theorem 2.6. *Let D be a digraph, $\tilde{\alpha} = (\alpha_v, S_v)_{v \in V(D)}$ a family where the α_v are mutually disjoint digraphs and S_v is a non-empty subset of $V(\alpha_v)$ for each $v \in V(D)$. If D is a kernel-perfect digraph and each α_v has a Grundy function f_v such that: For each $i \in \{0, 1, \ldots, n_v\}$ $f_v^{-1}(i) \cap S_v \neq \emptyset$ implies $f_v^{-1}(i) \subseteq S_v$ where $n_v = \max\{f_v(x) | x \in V(\alpha_v)\}$. Then $\sigma(D, \tilde{\alpha})$ possesses a Grundy function.*

Proof. **Claim 10.** If D and $\tilde{\alpha} = (\alpha_v, S_v)_{v \in V(D)}$ satisfy the hypothesis of Theorem 2.6 then $\sigma(D, \tilde{\alpha})$ has a kernel.

Let $A_0 = \{u \in V(D) | S_u \cap f_u^{-1}(0) \neq \emptyset\}$. If $A_0 = \emptyset$ then for each $u \in V(D)$ we have $S_u \cap f_u^{-1}(0) = \emptyset$. Hence from the definition of $\sigma(D, \tilde{\alpha})$ we have $\bigcup_{u \in V(D)} f_u^{-1}(0)$ is an independent set. Since f_u is a Grundy function of α_u, it follows from Theorem 1.1 that $f_u^{-1}(0)$ is a kernel of α_u; in particular $f_u^{-1}(0)$ is an absorbent set of vertices of α_u. Thus $\bigcup_{u \in V(D)} f_u^{-1}(0)$ is also an absorbent set of vertices of $\sigma(D, \tilde{\alpha})$. We conclude that $\bigcup_{u \in V(D)} f_u^{-1}(0)$ is a kernel of $\sigma(D, \tilde{\alpha})$, whenever $A_0 = \emptyset$.

Now we suppose that $A_0 \neq \emptyset$. Take $T_0 = \bigcup_{u \in V(D) - A_0} f_u^{-1}(0)$. Since $f_u^{-1}(0)$ is an independent set of vertices of α_u, and α_u is an induced subdigraph of $\sigma(D, \tilde{\alpha})$, we have that $f_u^{-1}(0)$ is an independent set of vertices of $\sigma(D, \tilde{\alpha})$. Notice that for each $u \in V(D) - A_0$ we have $S_u \cap f_u^{-1}(0) = \emptyset$. Thus from the definition of $\sigma(D, \tilde{\alpha})$, we have that T_0 is an independent set of vertices of $\sigma(D, \tilde{\alpha})$.

Let R_0 be a kernel of $D[A_0]$ (such a kernel exists because from the hypothesis D is a kernel-perfect digraph and $D[A_0]$ is the subdigraph of D induced by A_0). Let $T_0' = \bigcup_{u \in R_0} f_u^{-1}(0)$. Since R_0 is independent in D, it follows that T_0' is independent in $\sigma(D, \tilde{\alpha})$. Moreover from the definition of A_0 and the definition of $\sigma(D, \tilde{\alpha})$ we have that $T_0 \cup T_0'$ is an independent set of $\sigma(D, \tilde{\alpha})$. Denote by $n_u = \max\{f_u(x) | x \in V(\alpha_u)\}$ and $m_u = \min\{n \in \{0, \ldots, n_u\} | f_u^{-1}(n) \cap S_u = \emptyset\}$. (Notice that from the hypothesis we have m_u is well defined).

Let $T_0'' = \bigcup_{u \in (A_0 - R_0)} f_u^{-1}(m_u)$. Since f_u is a Grundy function of α_u we have that $f_u^{-1}(m_u)$ is an independent set of α_u (and hence independent set of $\sigma(D, \tilde{\alpha})$); moreover T_0'' is independent in $\sigma(D, \tilde{\alpha})$ because $f_u^{-1}(m_u) \cap S_u = \emptyset$. Therefore $T_0' \cup T_0''$ is independent as $f_u^{-1}(m_u) \cap S_u = \emptyset$ for each $u \in (A_0 - R_0)$ and $f_u^{-1}(0) \subseteq S_u$ for each $u \in R_0$ (as $R_0 \subseteq A_0$ from the definition of A_0 $f_u^{-1}(0) \cap S_u \neq \emptyset$ and thus from the hypothesis $f_u^{-1}(0) \subseteq S_u$ for each $u \in R_0$). Hence $T_0 \cup T_0' \cup T_0''$ is an independent set of vertices of $\sigma(D, \tilde{\alpha})$.

Now we prove that $T_0 \cup T_0' \cup T_0''$ is an absorbent set of vertices of $\sigma(D, \tilde{\alpha})$. Let $x \in V(\sigma(D, \tilde{\alpha})) - (T_0 \cup T_0' \cup T_0'')$ be. And let $z \in V(D)$ such that $x \in V(\alpha_z)$.

If $z \in (V(D) - A_0)$ then $f_z(x) \neq 0$ (because $x \notin T_0$). Since $f_z^{-1}(0)$ is a kernel of α_z (by Theorem 1.1) we have that there exists $y \in f_z^{-1}(0) \subseteq T_0$ such that $(x, y) \in A(\alpha_z) \subseteq A(\sigma(D, \tilde{\alpha}))$, and we are done.

If $z \in R_0$ then $f_z(x) \neq 0$ (because $x \notin T_0'$). Since $f_z^{-1}(0)$ is a kernel of α_z and $f_z(x) \neq 0$, then there exists $y \in f^{-1}(0) \subseteq T_0'$ such that $(x, y) \in A(\alpha_z) \subseteq A(\sigma(D, \tilde{\alpha}))$.

If $z \in (A_0 - R_0)$ then we consider the two possibilities: When $x \in S_z$. Since R_0 is a kernel of $D[A_0]$; there exists $w \in R_0$ such that $(z, w) \in A(D)$

Taking $y \in f_w^{-1}(0)$ we have $y \in f_w^{-1}(0) \subseteq T_0'$ and $(x, y) \in A(\sigma(D, \tilde{\alpha}))$ (as $(z, w) \in A(D)$, $x \in S_z$ and $y \in f_w^{-1}(0) \subseteq S_w$). When $x \notin S_z$ we will prove that $f_z(x) > m_z$; clearly $f_z(x) \neq m_z$ (because $z \in (A_0 - R_0)$ and $x \notin T_0''$). If $f_z(x) = r < m_z$ then $f_z^{-1}(r) \cap S_z = \emptyset$ otherwise $f_z^{-1}(r) \cap S_z \neq \emptyset$ and from the hypothesis we have $f_z^{-1}(r) \subseteq S_z$ but $x \in f_z^{-1}(r)$ and $x \notin S_z$); contradicting the definition of m_z. Thus $f_z(x) > m_z$ and since f_z is a Grundy function there exists $y \in \Gamma^+_{\alpha_z}(x)$ such that $f_z(y) = m_z$. Thus $(x, y) \in A(\sigma(D, \tilde{\alpha}))$ and $y \in f_z^{-1}(m_z) \subseteq T_0''$.

We conclude that $T_0 \cup T_0' \cup T_0'' = N_0$ is a kernel of $\sigma(D, \tilde{\alpha})$.

Now we will prove that $\sigma(D, \tilde{\alpha}) - N_0$ has a kernel. Let $D' = D[\{u \in V(D) | V(\alpha_u) - N_0 \neq \emptyset\}]$. For each $u \in (V(D') - A_0) \cup (R_0 \cap V(D'))$ define $f_u' : V(\alpha_u) - f_u^{-1}(0) \to \mathbb{N}$ by $f_u' = f_u - 1$. And for each $u \in (A_0 - R_0) \cap V(D')$ we define $f_u' : V(\alpha_u) - f_u^{-1}(m_u)$ by: $f_u'(x) = f_u(x)$ whenever $f_u(x) < m_u$ and $f_u'(x) = f_u(x) - 1$ whenever $f_u(x) > m_u$.

Clearly f_u' is a Grundy function of $\alpha_u' = (\alpha_u - N_0)$ which satisfies the hypothesis of Theorem 2.6 with respect to D' and $\tilde{\alpha}' = (\alpha_v', S_v')$ $(S_v' = S_v - N_0)$. Thus from Claim 10 we have that $\sigma(D, \tilde{\alpha}) - N_0$ possesses a kernel, let N_1 be such a kernel.

Arguing again as above we can prove that $\sigma(D, \tilde{\alpha}) - (N_0 \cup N_1)$ possesses a kernel and we take N_2 such a kernel. Continuing this way we can find a succession of subsets of $V(\sigma(D, \tilde{\alpha})$, $N_0, N_1, \ldots, N_p$ such that N_0 is a kernel of $\sigma(D, \tilde{\alpha})$, N_i is a kernel of $\sigma(D, \tilde{\alpha}) - \bigcup_{j=0}^{i-1} N_j$ for each $1 \leq i \leq p$ and $V(\sigma(D, \tilde{\alpha})) = \bigcup_{j=0}^{p} N_j$.

Now we define $F : V(\sigma(D, \tilde{\alpha})) \to \mathbb{N}$ as follows $F(x) = s$ if and only if $x \in N_s$.

Claim 11 F is a Grundy function of $\sigma(D, \tilde{\alpha})$.

(1) $F(x) = k > 0$ implies that for each $t < k$, there exists a $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$ with $F(y) = t$.

Since $t < k$, then $x \in (V(\sigma(D,\tilde{\alpha})) - \bigcup_{j=0}^{t} N_j)$. And since N_t is a kernel of $\sigma(D,\tilde{\alpha}) - \bigcup_{j=0}^{t-1} N_j$ whenever $t \geq 1$ and N_t is a kernel of $\sigma(D,\tilde{\alpha})$ whenever $t = 0$; we have that there exists $y \in N_t$ such that $(x,y) \in A(\sigma(D,\tilde{\alpha}))$. Thus $F(y) = t$ with $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$.

(2) $F(x) = k$ implies that each $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$. satisfies $F(y) \neq k$.

Assume by contradiction that there exists $y \in \Gamma^+_{\sigma(D,\tilde{\alpha})}(x)$ with $F(y) = k$. From the definition of F we have that $\{x,y\} \subseteq N_k$ and from the definition of N_k we have that N_k is a kernel of $\sigma(D,\tilde{\alpha}) - \bigcup_{j=0}^{k-1} N_j$.

In particular N_k is an independent set of vertices of $\sigma(D,\tilde{\alpha})$, thus $(x,y) \in A(\sigma(D,\tilde{\alpha}))$ with $\{x,y\} \subseteq N_k$ which is a contradiction.

We conclude that F is a Grundy function of $\sigma(D,\tilde{\alpha})$. □

Definition 2.2. *Let D_1 and D_2 be two mutually disjoint digraphs. A sum of D_1 and D_2 is a digraph denoted by $\sigma(D_1,D_2)$ which satisfies: $\sigma(D_1,D_2) = D_1 \cup D_2 \cup B$ where $B \subseteq \{(x,y),(y,x) | x \in V(D_1), y \in V(D_2)\}$ such that for each $x \in V(D_1)$ and $y \in V(D_2)$, $|\{(x,y),(y,x)\} \cap B| \geq 1$.*

We will say that a sum of D_1 and D_2, $\sigma(D_1,D_2)$ satisfies the t-property whenever: For any three vertices x_1, x_2, x_3 of $\sigma(D_1,D_2)$ such that ($x_1 \in V(D_1)$ and $\{x_2,x_3\} \subseteq V(D_2)$) or ($\{x_1,x_2\} \subseteq V(D_1)$ and $x_3 \in V(D_2)$), it holds $(x_1,x_2) \in A(\sigma(D_1,D_2))$ and $(x_2,x_3) \in A(\sigma(D_1,D_2))$ implies $(x_1,x_3) \in A(\sigma(D_1,D_2))$.

Theorem 2.7. *Let D_1, D_2 be mutually disjoint digraphs and $\alpha = \sigma(D_1,D_2)$ a sum of D_1 and D_2. If S_i is a kernel of D_i for $i \in \{1,2\}$ and α satisfies the t-property, then at least one: S_1 is a kernel of α or S_2 is a kernel of α.*

Proof. Suppose that S_1 is not a kernel of α. We will prove that S_2 is a kernel of α.

Since S_1 is not a kernel of α and S_1 is a kernel of D_1; there exists $y \in V(D_2)$ such that $(y,w) \notin A(\alpha)$ for each $w \in S_1$. Then, from the definition of α we have $(w,y) \in A(\alpha)$ for each $w \in S_1$.

If $y \in S_2$ then for each $x \in (S_1 \cup (V(D_2) - S_2))$ we have that there exists an xS_2-arc in α. Now let $x \in (V(D_1) - S_1)$. Since S_1 is a kernel of D_1 there exists $z \in S_1$ such that $(x,z) \in A(\alpha)$. Thus it follows that $(x,y) \in A(\alpha)$ (as α satisfies the t-property). Thus S_2 is a kernel of α.

If $y \in (V(D_2) - S_2)$ then there exists $z \in S_2$ such that $(y,z) \in A(\alpha)$ (as S_2 is a kernel of D_2). For each $x \in S_1$ we have proved $(x,y) \in A(\alpha)$, since $(y,z) \in A(\alpha)$ and α satisfies the t-property it follows $(x,z) \in A(\alpha)$.

For $x \in (V(D_1) - S_1)$, there exists $a \in S_1$ such that $(x, a) \in A(\alpha)$. We have proved $(a, z) \in A(\alpha)$. Thus, since α satisfies the t-property it follows that $(x, z) \in A(\alpha)$. Therefore S_2 is a kernel of α. □

Theorem 2.8. *Let D_1, D_2 be mutually disjoint digraphs and $\sigma(D_1, D_2)$ a sum of D_1 and D_2. If D_i has a Grundy function say f_i, $i \in \{1, 2\}$ and $\sigma(D_1, D_2)$ satisfies the t-property, then $\sigma(D_1, D_2)$ has a Grundy function F which satisfies* $\max\{F(x)|x \in V(\sigma(D_1, D_2))\} = \max\{f_1(x)|x \in V(D_1)\} + \max\{f_2(x)|x \in V(D_2)\} + 1$.

Proof. Since f_i is a Grundy function of D_i, it follows from Theorem 1.1 that $f_i^{-1}(0)$ is a kernel of D_i; for $i \in \{1, 2\}$. And from the hypothesis $\sigma(D_1, D_2)$ satisfies the t-property. Thus, from Theorem 2.7 at least one $f_1^{-1}(0)$ or $f_2^{-1}(0)$ is a kernel of $\sigma(D_1, D_2)$. Assume without loss of generality that $f_1^{-1}(0) = N_0$ is a kernel of $\sigma(D_1, D_2)$. Clearly $f_1 - 1$ is a Grundy function of $D_1 - N_0$, and $\sigma(D_1, D_2) - N_0$ is a sum of $D_1 - N_0$ and D_2. Therefore arguing as above we have that $\sigma(D_1, D_2) - N_0$ has a kernel N_1 and $\sigma(D_1, D_2) - (N_0 \cup N_1)$ is a sum of two digraphs each one having a Grundy function. Continuing this way we obtain a succession of subsets of $V(\sigma(D_1, D_2))$; $N_0, N_1, \ldots, N_p$ such that N_0 is a kernel of $\sigma(D_1, D_2)$, N_i is a kernel of $\sigma(D_1, D_2) - (\bigcup_{t=0}^{i-1} N_i)$ for each $1 \leq i \leq p$ and $\bigcup_{t=0}^{p} N_i = V(\sigma(D_1, D_2))$.

Now we define $F : V(\sigma(D_1, D_2)) \to \mathbb{N}$ as follows: $F(x) = s$ if and only if $x \in N_s$. It is easy to prove that F is a Grundy function of $\sigma(D_1, D_2)$ (the proof is the same as the proof of $F : V(\sigma(D, \tilde{\alpha})) \to \mathbb{N}$ is a Grundy function (in the final part of the proof of Theorem 2.6)).

From the definition of F it is clear that $\max\{F(x)|x \in V(\sigma(D_1, D_2))\} = \max\{f_1(x)|x \in V(D_1)\} + \max\{f_2(x)|x \in V(D_2)\} + 1$. □

Theorem 2.9. *Let D_1, D_2 be mutually disjoint digraphs and $\sigma(D_1, D_2)$ a sum of D_1 and D_2. If $\sigma(D_1, D_2)$ has a kernel then at least one: D_1 has a kernel or D_2 has a kernel.*

Proof. If S is a kernel of $\sigma(D_1, D_2)$, then from the definition of $\sigma(D_1, D_2)$ we have $S \subseteq V(D_1)$ or $S \subseteq V(D_2)$. □

Theorem 2.10. *Let D_1, D_2 be mutually disjoint digraphs and $\sigma(D_1, D_2)$ a sum of D_1 and D_2. If $\sigma(D_1, D_2)$ has a Grundy function, then each D_i possesses a Grundy function for $i \in \{1, 2\}$.*

Proof. We will prove that D_1 has a Grundy function. (The proof of that D_2 has a Grundy function is the same).

Let F be a Grundy function of $\sigma(D_1, D_2)$ and $A_1 = \{n \in N | F^{-1}(n) \cap V(D_1) \neq \emptyset\}$. We write $A_1 = \{n_0, n_1, \ldots, n_m\}$ in such a way that $n_i < n_j$ iff $i < j$.

Define $f_1 : V(D_1) \to \mathbb{N}$ as follows: $f_1(x) = r$ iff $F(x) = n_r$

We will prove that f_1 is a Grundy function of $V(D_1)$.

(1) For $x \in V(D_1)$, $f_1(x) = k > 0$ implies that for each $j, 0 \le j < k$ there exists a $y \in \Gamma^+_{D_1}(x)$ with $f_1(y) = j$.

From definition of f_1 we have $F(x) = n_k$ and $0 \le n_j < n_k$. Since F is a Grundy function of $\sigma(D_1, D_2)$, there exists $y \in \Gamma^+_{\sigma(D_1,D_2)}(x)$ with $F(y) = n_j$. Now we prove that $y \in V(D_1)$. We have $n_j \in A_1$. Thus there exists $z \in V(D_1)$ with $F(z) = n_j$. Therefore $y \in V(D_1)$ (otherwise $y \in V(D_2)$ and from the definition of $\sigma(D_1, D_2)$, there exists an arc in $\sigma(D_1, D_2)$ between y and z. But $\{y, z\} \subseteq F^{-1}(n_j)$ is an independent set, as F is a Grundy function, a contradiction).

(2) $f_1(x) = k$ implies that each $y \in \Gamma^+_{D_1}(x)$ satisfies $f_1(y) \ne k$.

Let $y \in \Gamma^+_{D_1}(x)$ be, then $y \in \Gamma^+_{\sigma(D_1,D_2)}(x)$ and $F(x) = n_k$. Since F is a Grundy function of $\sigma(D_1, D_2)$ it follows that $F(y) \ne n_k$ i.e. $f_1(y) \ne k$. We conclude that f_1 is a Grundy function of D_1. □

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