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Structural properties of graphs of diameter 2 and defect 2*

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Abstract

Using eigenvalue analysis, it was shown by Erdös *et al.* that with the exception of C_4, there are no graphs of diameter 2, maximum degree d and d^2 vertices. In this paper, we prove a number of structural properties of regular graphs of diameter 2, maximum degree d and order $d^2 - 1$.

Keywords: Diameter, repeat set, repeat subgraph.

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1. Introduction

There are a number of famous and difficult graph-theoretical problems that arose over the past four decades from the design of interconnection networks (such as local area networks, parallel computers, switching system architecture in VLSI technology, and many others). One of the most prominent problems is the *degree/diameter problem* which is to determine, for each d and k, the largest order $n_{d,k}$ of a graph of maximum degree d and diameter at most k. It is easy to show that $n_{d,k} \leq M_{d,k}$ where $M_{d,k}$ is the *Moore bound* given by

$$n_{d,k} \leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}.$$

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The directed version of the problem has also been studied, for example, in [1], [4] and [5]. For a survey of the degree/diameter problem, both directed and undirected, see [7].

In this paper, we shall concentrate on the case when the diameter is equal to 2. Since a graph of diameter 2 and maximum degree d may have at most d^2+1 vertices, it was asked in [2]: Given non-negative numbers d and Δ (*defect*), is there a graph of diameter 2 and maximum degree d with $d^2+1-\Delta$ vertices? It was proved in [3] that if $\Delta = 0$ then there are unique graphs corresponding to $d = 2, 3, 7$ and possibly $d = 57$. The case $\Delta = 1$ was solved by Erdös *et al.* [2]. In this paper, we give a characterization of graphs of diameter 2 and defect 2.

Let us refer to a graph of maximum degree d, diameter $k \geq 2$ and order $M_{d,k} - \Delta$ ($\Delta \geq 1$) as a (d, k, Δ)-graph. Let G be a (d, k, Δ)-graph.

Definition 1. *Let u be a vertex in G. A vertex v in G is called a repeat of u with multiplicity $m_v(u)$ ($1 \leq m_v(u) \leq \Delta$) if there are exactly $m_v(u)+1$ different paths of lengths at most k from u to v.*

It follows that

Observation 1. *A vertex u is a repeat of v with multiplicity $m_u(v)$ if and only if v is a repeat of u with the same multiplicity.*

A repeat with multiplicity 1 will be called a *single* repeat, a repeat with multiplicity 2 will be called a *double* repeat, a repeat with multiplicity Δ will be called a *maximal* repeat.

We denote by $R_s(u)$ the set of all repeats of a vertex u in G. Taking into account the multiplicities of repeats, we denote by $R_m(u)$ the multiset of all repeats of a vertex u in G, containing each repeat v of u exactly $m_v(u)$ times.

Let u be a vertex in G, we denote by $N(u)$ the set of all neighbours of u. If A is a multiset of vertices of G, then $N(A)$ denotes the multiset of all neighbours of the vertices of A. We use $R_m(A)$ to denote the multiset of all repeats of all vertices in A.

Proposition 1. *If G is regular then for all $u \in V(G)$,*

$$|R_m(u)| = \sum_{v \in R_s(u)} m_v(u) = \Delta.$$

Definition 2. *A subset S of $V(G)$ is called a closed repeat set if $R_s(S) = S$. A closed repeat set is minimal if none of its proper subsets is a closed repeat set.*

Definition 3. *A repeat subgraph H_S of a closed repeat set S of G is a multigraph whose vertex set $V(H_S) = S$ and the number of parallel edges between a vertex u and any of its repeats, say $v \in R_m(u)$, equals the multiplicity $m_v(u)$.*

We observe that

Observation 2. *If* $\Delta < 1 + (d-1) + \cdots + (d-1)^{k-1}$ *then* G *is regular.*

It is also true that

Observation 3. *If* G *is regular then the repeat graph* H_G *of* G *is* Δ*-regular.*

The results presented in this paper are of interest to the undirected version of the *degree/diameter* problem, which is to determine the possible maximum order of a graph of given maximum degree d and diameter k.

In [6], Miller *et al.* proved the following:

Theorem 1. (Neighbourhood Theorem) *Let* G *be a regular* (d, k, Δ)*-graph* $(\Delta \geq 1)$. *Then* $N(R_m(u)) = R_m(N(u))$ *for every* $u \in V(G)$.

We shall use the Neighbourhood Theorem to prove a number of structural properties of $(d, 2, 2)-$graphs in the next section.

Note that instead of writing "a vertex x is adjacent to a vertex y" we write $x \sim y$, and if x is not adjacent to y then we write $x \nsim y$. Unless explicitly shown where necessary, by u_i and u_j $(i \neq j)$ we shall mean two distinct vertices.

In Section 2, we describe several basic properties of $(d, 2, 2)$-graphs. In Section 3, we prove a number of structural properties of such graphs. The main results of the paper regarding enumeration of vertices of each repeat type and repeat cycle structures, will be presented in Section 4.

2. Basic structural properties of $(d, 2, 2)$-graphs

In this section, we consider graphs of diameter 2 with defect 2. Note that such graphs do not exist for $d \leq 2$. Let G be a $(d, 2, 2)$-graph for $d \geq 3$. Obviously,

Observation 4. *Every* $(d, 2, 2)$*-graph for* $d \geq 3$ *is regular.*

Let us consider repeat configurations in $(d, 2, 2)$-graphs. Let u be a vertex of a $(d, 2, 2)$-graph. Then there are two possiblities:

- u has two single repeats $(r_i(u),\ i = 1, 2)$.
- u has one double (maximal) repeat $(r(u) = r_1(u) = r_2(u))$ with multiplicity 2.

With respect to repeats in G, there are five possible repeat configurations, as depicted in Figure 1.

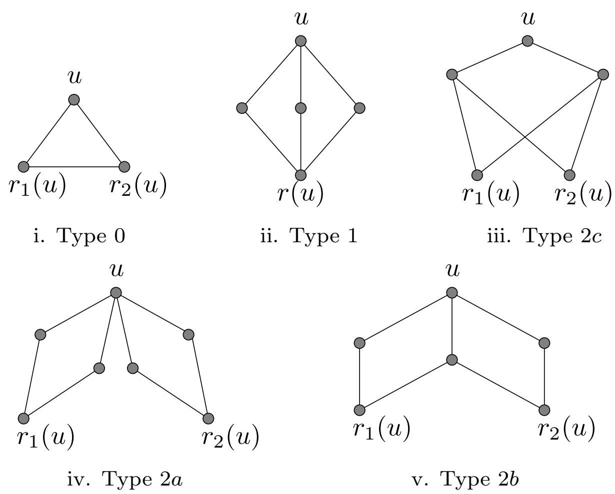


Figure 1: Possible repeat configurations for vertex u in a $(d, 2, 2)$-graph.

Figure 2 shows all the known non-isomorphic $(d, 2, 2)$-graphs.

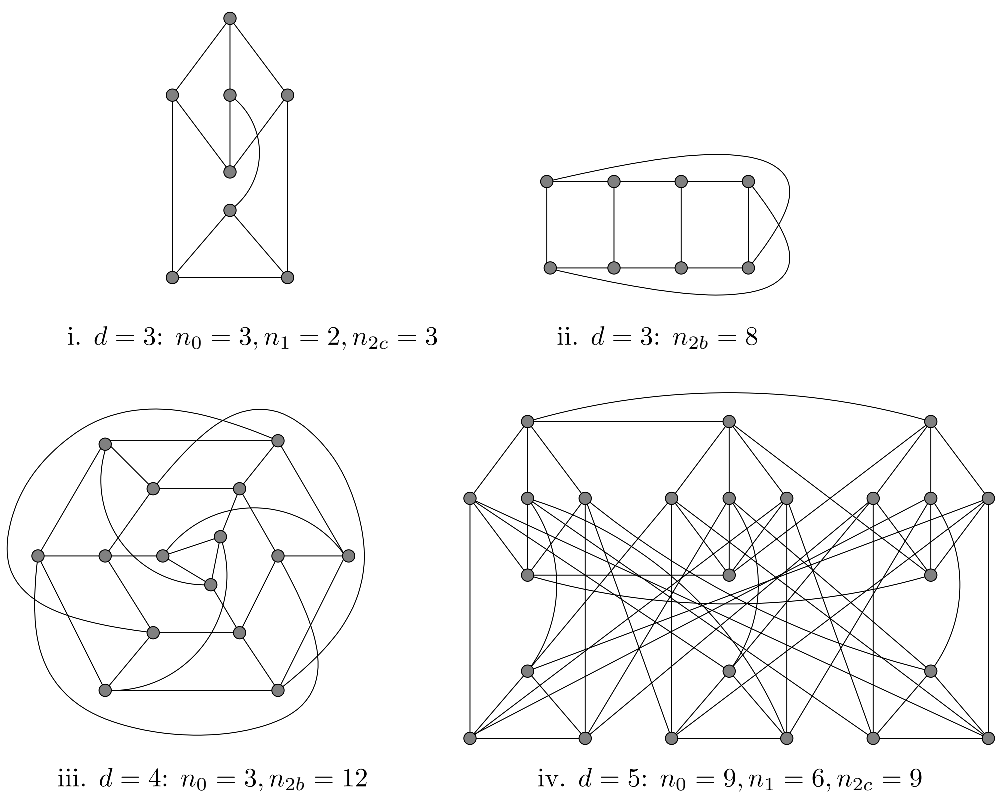


Figure 2: All known $(d, 2, 2)$-graphs.

We denote by n_0, n_1, n_{2a}, n_{2b}, n_{2c} the number of vertices of the corresponding repeat types. We observe the following.

Observation 5. $n_0 + n_1 + n_{2a} + n_{2b} + n_{2c} = d^2 - 1$.

Observation 6. *The two subgraphs in Figure* 1(*ii*) *and Figure* 1(*iii*) *are isomorphic.*

Observation 7. $n_1 \equiv 0 \pmod 2$ *and* $n_{2c} \equiv 0 \pmod 3$. *Moreover,* $n_1 = \frac{2}{3} n_{2c}$.

Observation 8. *The repeats of a vertex of type* 0 *are of type* 0.

Observation 9. *The double repeat of a vertex of type* 1 *is also of type* 1.

We shall refer to a vertex of type 1 and its double repeat as a *repeat pair* in G.

Observation 10. *The repeats of a vertex of type* $2c$ *are of type* $2c$.

The following result was proved in [8].

Theorem 2. $(n_1, n_{2c}) = (0, 0)$, $(2, 3)$, *or* $(6, 9)$.

In the case when $(n_1, n_{2c}) = (6, 9)$, the six vertices of type 1 must form a cycle as shown in Figure 3.

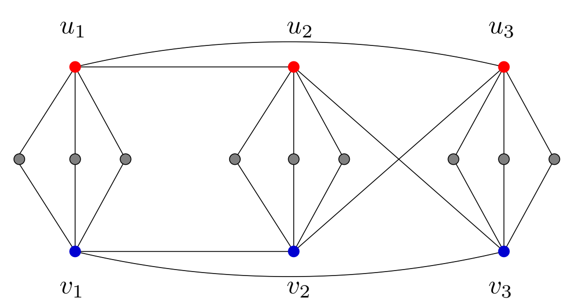


Figure 3: The structure of vertices of types 1 and $2c$ in a $(d, 2, 2)$-graph.

For the purpose of this paper, we shall consider each pair of parallel edges in H_G as a cycle of length 2.

Observation 11. *The graph* H_G *is the union of cycles of lengths* ≥ 2, *each cycle a minimal closed repeat set of* G.

From now on, each cycle in H_G will be called a *repeat cycle.*

The following Corollary follows by applying the Neighbourhood Theorem at u and at v.

Corollary 1. *Let uv be an edge of G such that both u and v are of type $2b$ and let u_1, u_2 and v_1, v_2 be the repeats of u and v, respectively. Then either $u_1 \sim v_1$ and $u_2 \sim v_2$, or $u_1 \sim v_2$ and $u_2 \sim v_1$.*

3. Towards a general classification of $(d, 2, 2)$-graphs

In this section, we shall prove further structural properties of G which will enable us to enumerate vertices of each repeat type in G.

Lemma 1. *A vertex of type 1 is adjacent to exactly three vertices of type $2c$ plus*

- *$d - 3$ vertices of type $2a$, or*
- *two vertices of type 1 and $d - 5$ vertices of type $2a$.*

Proof. Let vertex u be a vertex of type 1 and let v be the double repeat of u. Let w be a neighbour of u such that $w \notin N(v)$ and w is not of type 1. Let w_1, w_2 be the two single repeats of w. By the Neighbourhood Theorem, both w_1 and w_2 are adjacent to v. Therefore, w must be of type $2a$, since otherwise we would have $|R_m(v)| > 2$. The rest of the proof then follows from Theorem 2. □

Lemma 2. *The repeats of a vertex of type $2a$ are of type $2a$.*

Proof. Let u_0 be a vertex of type $2a$ and let u_1, u_2 be its two repeats. Obviously, u_1 and u_2 must be of type $2a$ or $2b$. Let u_3, u_4 be the two common neighbours of u_0 and u_1. Let u_5, u_6 be the two common neighbours of u_0 and u_2.

Suppose that u_1 is of type $2b$. Then either u_3 or u_4 is also of type $2b$. Let us suppose, without loss of generality, that u_3 is of type $2b$. Let u_7 be the second repeat of u_1, so that $u_3 \sim u_7$. Let u_8 be the second repeat of u_3, so that $u_1 \sim u_8$.

Since u_0 and u_1 cannot lie on another C_4, then u_4 must be of type $2a$. Let u_9 be the second repeat of u_4. By the Neighbourhood Theorem at u_1, $u_9 \in N(u_0) \bigcup N(u_7)$. So that u_9 lies on a C_4 that has precisely u_4 in common out of $u_0, \ldots, u_8$, it is clear that $u_9 \notin N(u_0)$, so $u_9 \sim u_7$. On the other hand, by the Neighbourhood Theorem at u_0, we deduce that $u_9 \sim u_2$. Thus $u_9 \in N(u_7) \bigcap N(u_2)$.

By the Neighbourhood Theorem at u_3, $R_m(u_0) \subseteq N(u_4) \bigcup N(u_8)$. Therefore, $u_2 \sim u_8$. This means that u_7 is the second repeat of u_2, so that u_2 is of type $2a$. This also means that both u_5 and u_6 are of type $2a$.

Let u_{10} be the second repeat of u_5. By the Neighbourhood Theorem at u_0, $u_{10} \sim u_1$. On the other hand, by Theorem 1 at u_2, it follows that $u_{10} \sim u_7$. This leads to a contradiction since u_{10} must differ from both u_3 and u_8.

Therefore, u_1 is of type $2a$. By a similar argument, u_2 is also of type $2a$. □

From Observations 8, 9, 10 and Lemma 2, it is clear that a vertex of type $2b$ has both repeats of type $2b$.

Therefore, we have

Theorem 3. *In a $(d, 2, 2)$-graph, a vertex and its repeats are all of the same type.*

Corollary 2. *A minimal closed repeat set in G contains all vertices of the same type.*

Corollary 3. $n_{2b} \equiv 0 \pmod 2$.

Observation 12. *The two repeats of a vertex of type $2b$ cannot be a repeat of each other*

Lemma 3. *There is no closed repeat set of 3 vertices of type $2a$.*

Proof. Let $\{u_0, u_1, u_2\}$ be a closed repeat set of type $2a$ in G. Let u_3, u_4 be the two common neighbours of u_0 and u_1; let u_5, u_6 be the two common neighbours of u_0 and u_2; and let u_7, u_8 be the two common neighbours of u_1 and u_2.

It can be seen that each u_i ($3 \leq i \leq 8$) is also of type $2a$. Since $u_0 \nsim u_7$ they must have exactly one common neighbour, say u_9. Let $S_1 = N(u_7) \setminus \{u_1, u_2, u_9\}$ and let $S_2 = N(u_0) \setminus \{u_3, u_4, u_5, u_6, u_9\}$. In order to reach all $(d-3)$ vertices of S_1 from u_0 we deduce, by pigeonhole principle, that $R_s(u_7) \subseteq S_2$. But, since $u_8 \notin S_2$, this is impossible. □

Lemma 4. *Every minimal closed repeat set of vertices of type $2a$ has cardinality 4.*

Proof. Let u_0 be a vertex of type $2a$ and let u_1, u_2 be its two repeats. Let u_4, u_5 be the two common neighbours of u_0 and u_1. Let u_6, u_7 be the two common neighbours of u_0 and u_2. By Lemma 3, u_2 cannot be the second repeat of u_1.

It can be shown that neither u_6 nor u_7 is in $R_s(u_1)$. Indeed, if, say u_7, is the second repeat of u_1 then by the Neighbourhood Theorem at u_7, $R_m(u_0) \subseteq N(u_1) \bigcup N(u_6)$. This means $u_1 \sim u_6$, an impossibility.

Let u_3 be the second repeat of u_1. It remains to show that u_3 is also the second repeat of u_2.

Since u_4 and u_5 cannot be of type $2b$, they are both of type $2a$. Let u_{10} be the second repeat of u_4 and let u_{11} be the second repeat of u_5. By Lemma 3, $u_{10} \neq u_{11}$. By the Neighbourhood Theorem at u_0 we deduce that $u_{10} \sim u_2$. On the other hand, by the Neighbourhood Theorem at u_1, it follows that $u_{10} \sim u_3$.

Similarly, u_{11} is adjacent to both u_2 and u_3. Thus u_3 is the second repeat of u_2. □

Corollary 4. $n_{2a} \equiv 0 \pmod 4$.

Lemma 5. *A vertex of type $2a$ cannot be adjacent to a vertex of type either 0 or $2c$.*

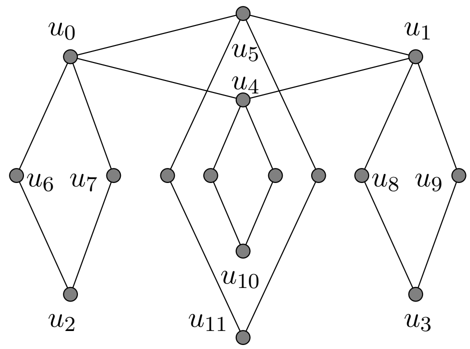


Figure 4: A closed repeat set of vertices of type $2a$.

Proof. Let $\{u_0, u_1, u_2, u_3\}$ be a minimal closed repeat set in G with all vertices of type $2a$ as shown in Figure 4. Let $\{v_0, v_1, v_2\}$ be a closed repeat set of cardinality 3. By Observation 12 and Lemma 3, v_0, v_1 and v_2 are of type either 0 or $2c$. Suppose $u_0 \sim v_0$. By the Neighbourhood Theorem, either $u_1 \sim v_1$ and $u_2 \sim v_2$, or $u_1 \sim v_2$ and $u_2 \sim v_1$. In both cases we must have, also by the Neighbourhood Theorem, $v_0 \sim u_3$. This is impossible since v_0 is not of type 1. □

Lemma 6. *For $d \geq 6$, let $\{u, v, z, w\}$ be a minimal closed repeat set of vertices of type $2a$ in G such that u and z are not repeats of each other and the distance between u and z is two. Let x be the common neighbour of u and z and let y be the common neighbour of v and w. Then (x, y) is a repeat pair in G.*

Proof. Let us consider the vertex v. Let v_1, v_2 be the two common neighbours of v and u. Let v_3, v_4 be the two common neighbours of v and z. Since the distance between v and x must be two, let v_5 be the common neighbour of v and x.

From Lemma 5, neither x nor v_5 is of type 0. Thus, in order for v to reach every vertex in $N(x) \setminus \{u, z, v_5\}$ in at most two steps, v must use $N(v) \setminus \{v_1, \ldots, v_5\}$. By the pigeonhole principle, we deduce that $R_m(x) \subseteq N(v)$.

By applying a similar argument at w we also have $R_m(x) \subseteq N(w)$. Consequently, $R_m(x) \subseteq (N(v) \bigcap N(w)) = \{y\}$. □

Lemma 7. *For $d \geq 4$, if $n_{2a} \geq 1$, then $n_1 = 6$.*

Proof. We shall show that for all $d \geq 4$, $(n_1, n_{2c}) \neq (2, 3)$. Suppose there exists $d \geq 4$ such that $(n_1, n_{2c}) \neq (0, 0)$ in G. Then, by Lemma 1 and Corollary 4, there exist in G vertices of type $2a$ and, by Lemma 4, there exists a closed repeat set $\{u_0, \ldots, u_3\}$ as shown in Figure 4. As observed earlier, $\{u_4, u_5, u_{10}, u_{11}\}$ is also a closed repeat set. Without loss of generality, let u_{10} be the second repeat of u_4 and let u_{11} be the second repeat of u_5.

Since the diameter of G is 2, the vertex u_0 must reach u_3 in at most 2 steps. We distinguish two cases:

Case 1. $u_0 \sim u_3$.

By the Neighbourhood Theorem at u_0, $R_m(u_3) \subseteq N(u_1) \bigcup N(u_2)$. This means that $u_1 \sim u_2$. We can see that $u_4 \nsim u_{11}$ since otherwise u_{11} would be another repeat of u_0. Let u_{12} be the common neighbour of u_4 and u_{11}. Similarly, $u_5 \nsim u_{10}$ and let u_{13} be the vertex such that $u_{13} \sim u_5$ and $u_{13} \sim u_{10}$.

It follows from Lemma 6 that $\{u_{12}, u_{13}\}$ is a repeat pair in G. By a similar argument with the closed repeat set $\{u_6, u_7, u_8, u_9\}$, we deduce that there must be another repeat pair, different from $\{u_{12}, u_{13}\}$, in G. Therefore, by Corollary 2, $(n_1, n_{2c}) = (6, 9)$.

Case 2. $u_0 \nsim u_3$.

Let u_{12} be the common neighbour of u_0 and u_3. By symmetry in Case 1, it follows that there must exist a unique vertex u_{13} in G such that $u_{13} \sim u_1$ and $u_{13} \sim u_2$. In other words, $\{u_{12}, u_{13}\}$ forms a repeat pair in G.

We shall next show that $u_4 \sim u_{11}$. Indeed, if $u_4 \nsim u_{11}$ then $u_5 \nsim u_{10}$. Let u_{14} be the common neighbour of u_4 and u_{11}; u_{15} be the common neighbour of u_5 and u_{10}. It follows from the Neighbourhood Theorem that $\{u_{14}, u_{15}\}$ must form a repeat pair in G. However, by Theorem 2 in [8] and as shown in Figure 3, either $u_{14} \sim u_{12}$ or $u_{14} \sim u_{13}$ which leads to a contradiction. Therefore, $u_4 \sim u_{11}$ and, by the Neighbourhood Theorem, $u_5 \sim u_{10}$.

It remains for us to apply the argument used in Case 1 to the closed repeat set $\{u_4, u_5, u_{10}, u_{11}\}$ to show that also in this case $(n_1, n_{2c}) = (6, 9)$.

In both of the above cases the existence of a vertex of type $2a$ implies the existence of at least two repeat pairs in G. Thus the proof follows from Lemma 1 and Corollaries 2 and 4. □

From Corollary 4, and Lemmas 1 and 7, it follows that

Corollary 5. *For $d \geq 6$, $(n_1, n_{2c}, n_{2a}) = (0, 0, 0)$ or $(6, 9, 4\alpha)$ for some $\alpha \geq 1$.*

Using the result of Lemma 6, we can can extend Corollary 1 to vertices of types other than 1 as follows.

Lemma 8. *Let uv be an edge of G such that neither u nor v is of type 1 and let u_1, u_2 and v_1, v_2 be the repeats of u and v, respectively. Then either $u_1 \sim v_1$ and $u_2 \sim v_2$ or $u_1 \sim v_2$ and $u_2 \sim v_1$.*

Proof. For a contradiction, let us suppose otherwise that u is not of type $2b$ and, without loss of generality, let $u_1 \sim v_1$ and $u_1 \sim v_2$. By the repeat configurations in Figure 1 and Lemma 6, u_1 is of either type 0 or type 1. It can be verified, by Theorem 3, that this is impossible. By Corollary 1, the proof then follows. □

Theorem 4. *For even* $d \geq 6$, $(n_1, n_{2c}, n_{2a}) = (0, 0, 0)$.

Proof. Suppose otherwise that $(n_1, n_{2c}, n_{2a}) \neq (0, 0, 0)$, then by Corollary 5, $(n_1, n_{2c}, n_{2a}) = (6, 9, 4\alpha)$, for some $\alpha \geq 1$.

Let $\{u, v\}$ be a repeat pair in G. By Corollary 5, there exists a vertex w_1 of type $2a$ such that $w_1 \sim u$. Let w_2, w_3 be the two single repeats of w_1. By the Neighbourhoood Theorem at u, $w_2 \sim v$ and $w_3 \sim v$. It then follows, from Lemma 4, that there must exist a unique vertex w_4 in G such that $w_4 \in R_s(w_2) \bigcap R_s(w_3)$. By the Neighbourhood Theorem, $w_4 \sim u$. In other words, w_1, w_2, w_3 and w_4 form a closed repeat set.

Consequently, the $d-5$ neighbours of type $2a$ of u can be grouped into pairs, each of which belongs to a separate closed repeat set of size 4. Since the six vertices of type 1 form a cycle, as shown in Figure 3, and d is even, it follows that there exists a vertex w_i of type $2a$ in $N(u)$ that cannot be paired with any other vertex in $N(u)$ in that specific manner. This leads to a contradiction because then w_j would be of type 1. □

Lemma 9. *If* $n_0 \geq 6$, *then* $(n_0, n_1, n_{2c}) = (9, 6, 9)$.

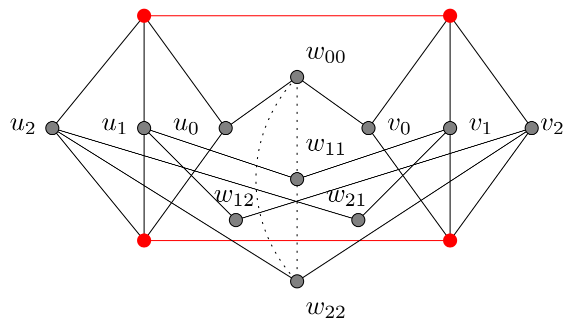
Proof. Let $S_u = \{u_0, u_1, u_2\}$ and $S_v = \{v_0, v_1, v_2\}$ be two closed repeat sets such that all vertices of S_u and S_v are of a same type in G. By Lemma 3, the vertices of S_u and S_v are either only of type 0 or only of type $2c$.

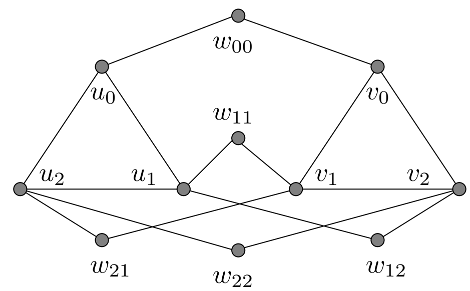
It is not difficult to see that each vertex of S_u is at distance 2 from each vertex of S_v. Let w_{00} be the common neighbour of u_0 and v_0 and let us determine the two repeats of w_{00}.

Suppose, without loss of generality, that w_{11}, which is the common neighbour of u_1 and v_1, is a repeat of w_{00}. Then the vertex w_{22}, which is the common neighbour of u_2 and v_2, must be the second repeat of both w_{00} and w_{11}. In other words, $\{w_{00}, w_{11}, w_{22}\}$ is a closed repeat set.

Using an analogous argument for the two pairs $\{u_0, v_1\}$ and $\{u_0, v_2\}$, we deduce that there must exist in G at least three closed repeat sets other than S_u and S_v. Moreover, it is not difficult to see that the vertices of those three sets must be of type different from the types of vertices of S_u and S_v (see illustration in Figure 5).

Therefore, by Corollary 2 and Lemma 3, the proof follows. □





(a) Vertices w_{00}, w_{11} and w_{22} form a triangle in G; (b) Closed repeat set $\{w_{00}, w_{11}, w_{22}\}$ of type $2c$.

Figure 5: Illustration of Lemma 9.

From Corollary 5, and Lemmas 5 and 9, we have

Corollary 6. *A vertex of type* 0 *is adjacent to two vertices of type* 0 *and*

- $d-2$ *vertices of type* $2b$ *for even* d, *or*
- *three vertices of type* $2c$ *and* $d-5$ *vertices of type* $2b$, *for odd* $d \geq 5$.

Corollary 7. *For* $d \geq 6$ *and* $n_0 \geq 6$, $(n_0, n_1, n_{2c}, n_{2a}) = (0, 0, 0, 0)$ *or* $(9, 6, 9, 4\alpha)$ *for some* $\alpha \geq 1$.

4. Main result

In this section, we count the number of vertices of each repeat type in G and prove several repeat cycle properties.

Theorem 5. *Let* C_l *and* C_h *be two repeat cycles in* H_G *each of a type other than* 1 *such that there exists in* G *an edge between a vertex on* C_l *and a vertex on* C_h. *Then* $\max(l, h) \equiv 0 \pmod{\min(l, h)}$.

Proof. Suppose that $l > h$ and h does not divide l. Let $u_0, u_1, \ldots, u_{l-1}$ be the vertices of C_l such that $R_s(u_i) = \{u_{(i-1)(\bmod\ l)}, u_{(i+1)(\bmod\ l)}\}$ $(0 \leq i \leq l-1)$. Similarly, let $v_0, v_1, \ldots, v_{h-1}$ be the vertices of C_h such that $R_s(v_j) = \{v_{(j-1)(\bmod\ h)}, v_{(j+1)(\bmod\ h)}\}$ $(0 \leq j \leq h-1)$.

Let u_0v_0 be an edge in G. By Lemma 8, either $u_1 \sim v_1$ and $u_{l-1} \sim v_{h-1}$ or $u_1 \sim v_{h-1}$ and $u_{l-1} \sim v_1$. Without loss of generality, let us suppose that $u_1 \sim v_1$ and $u_{l-1} \sim v_{h-1}$. It then follows, from Corollary 8, that $u_i \sim v_i$ $(2 \leq i \leq h-1)$, and $u_{h(\bmod\ l)} \sim v_0$, and so forth.

Let $r = \mathrm{lcm}(l,h)/l$ and $s = l/\gcd(l,h)$. Because $\gcd(l,h) < h$, it follows that $s \geq 3$. Since $r \geq 2$, there must exist at least two non-adjacent vertices on C_h (in H_G) that have exactly s common neighbours in G, all of which lie on C_l (in H_G). For example, v_0 and $v_{gcd(l,h)}$ have s common neighbours, namely $u_0, u_{\gcd(l,h)}, \ldots, u_{(s-1)\gcd(l,h)}$. However, this means that $v_{\gcd(l,h)}$ is a repeat of v_1 with multiplicity $s-1$, which is impossible. Therefore, h must divide l. □

Let the vertices u_0, u_1, u_2 form a triangle in G, denoted by T, and let Υ_{2b} be the subset of all vertices of type $2b$ in $N(u_0)\bigcup N(u_1)\bigcup N(u_2)$. Then Υ_{2b} is a minimal closed repeat set. We shall call Υ_{2b} the *outer repeat cycle* of T in H_G.

Figure 6 illustrates a labeled partial structure of G, in the case when d is even, which shows the cycle $u_0u_1u_2$ and its outer repeat cycle. Since Υ_{2b} contains all vertices of type $2b$, by Observation 3, there exists in H_G another cycle Υ'_{2b}, also of the same size as Υ_{2b}, which is $3(d-2)$. Note that, in Figure 6, $u_3 \sim u_{3d-4}$ and $u_{3d-3} \sim u_{9d-16}$.

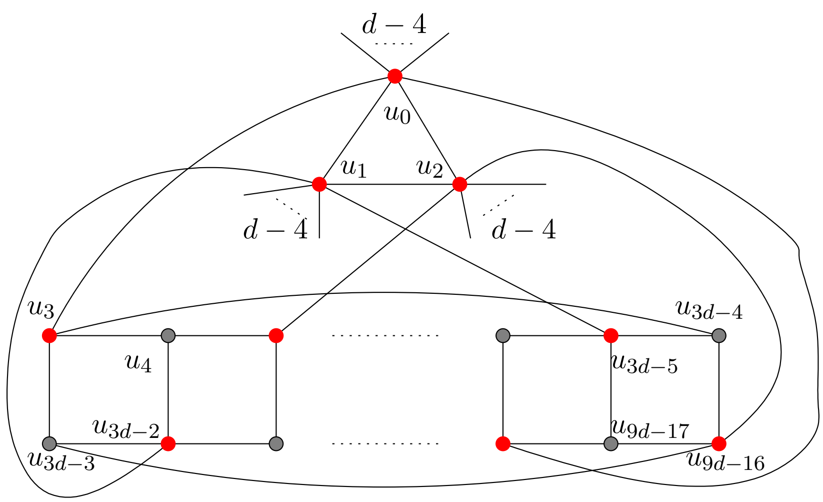


Figure 6: The triangle and its neighbourhood in G for even d.

Lemma 10. *Let T be a triangle in G and let Υ_{2b} be the outer repeat cycle of T in H_G. Let C_t be any repeat cycle in H_G of length $t \geq 4$ such that there exists in G an edge between a vertex on Υ_{2b} and a vertex on C_t. Then either $t = \frac{1}{3}|\Upsilon_{2b}|$ or $t \equiv 0 (\mathrm{mod}\ |\Upsilon_{2b}|)$.*

Proof. Let T contain three vertices u_1, u_2, u_3. We label the vertices of Υ_{2b} as $v_1, \ldots, v_{|\Upsilon_{2b}|}$ and we label the vertices of C_t as $w_1, \ldots, w_t$. Let $u_1 \sim v_1$ and $v_1 \sim w_1$ in G. By Corollary 8, without loss of generality, suppose that in G, $u_2 \sim v_2$ and $v_2 \sim w_2$. We consider three cases:

Case 1. $t < \frac{1}{3}|\Upsilon_{2b}|$.

We deduce from the Neighbourhood Theorem and Lemma 8, by considering the edges in G between C_t and Υ_{2b}, that $w_{3t} \sim v_{3t}$. But then $v_{3t+1} \sim w_1$ and $v_{3t+1} \sim u_1$. However, this means that u_1 is a repeat of w_1, a contradiction.

Case 2. $\frac{1}{3}|\Upsilon_{2b}| \leq t < |\Upsilon_{2b}|$.

By Theorem 5, t must divide $|\Upsilon_{2b}|$ and $\frac{1}{3}|\Upsilon_{2b}|$ must divide t. Thus, $t = \frac{1}{3}|\Upsilon_{2b}|$.

Case 3. $t \geq |\Upsilon_{2b}|$.

This case immediately follows from Theorem 5. □

Lemma 11. *For even $d \geq 6$, every other cycle than the triangle in H_G has length $3k(d-2)$ for some $k \geq 1$.*

Proof. As demonstrated in Section 2, H_G contains one cycle of length 3 and at least two cycles of length $3(d-2)$. Let $u_1u_2u_3$ be the triangle T of G and let $v_1 \dots v_{3(d-2)}$ be the outer repeat cycle Υ_{2b} of T in H_G such that the repeats of v_j $(1 \leq j \leq 3(d-2))$ are $v_{(j-1)(\bmod 3(d-2))}$ and $v_{(j+1)(\bmod 3(d-2))}$. Without loss of generality, let us suppose that $u_1 \sim v_1$ and $u_2 \sim v_2$ in G.

Let the a_i, $i = 1, \dots, b$, be the lengths of the cycles in H_G and let $a_1 = 3$, $a_2 = a_2 = 3(d-2)$ correspond to T, Υ_{2b} and Υ'_{2b}, respectively. Thus, $f = \sum_{i=4}^{b} a_i = (d-2)(d-4)$.

Let C_{a_j} be an arbitrary cycle in H_G $(j \neq 1, 2, 3)$. Then, by Lemma 10, either $a_j = d-2$, or $a_j \equiv 0 \pmod{3(d-2)}$. Suppose that $a_j = d-2$. Denote by $w_1, \dots, w_{d-2}$ the vertices of C_{a_j} such that the repeats of w_k $(1 \leq k \leq d-2)$ are $w_{(k-1) \pmod{(d-2)}}$ and $w_{(k+1) \pmod{(d-2)}}$.

We know that the vertices of C_{a_j} must reach the vertices of T through the vertices of Υ_{2b}. Without loss of generality, suppose that $w_1 \sim v_1$ and $w_2 \sim v_2$. However, since $(d-2)$ is not divisible by 3 when d is even, by the Neighbourhood Theorem and Lemma 8, u_1 and w_1 would then have at least three common neighbours, namely v_1, v_{d-1} and v_{2d-3}. This is clearly impossible.

Therefore, each of a_i $(4 \leq i \leq b)$ must be a multiple of $3(d-2)$. □

We shall now count the number of vertices of each repeat type in G.

Theorem 6. *(i) If d is even then $n_0 = 3$ and $n_{2b} = d^2 - 4$;*

(ii) If d is odd then

- *$d = 3$ and G is the graph in Figure 2(i) or 2(ii)*
- *$d = 5$ and G is the graph in Figure 2(iv)*
- *$d \geq 7$ and $n_{2b} = n = d^2 - 1$.*

Proof. Let us consider two cases, depending on the parity of d.

(i) d **even.**

Since n is odd for even d, by Corollary 3, Lemma 9 and Theorem 4, G must contain exactly one triangle and $n-3$ vertices of type $2b$.

(ii) d **odd.**

Let us suppose that $n_{2b} \neq n$. Since n is even, by Corollary 7 and the proof of Lemma 9, G must contain the graph in Figure 2(iv) as an induced subgraph in the case $d \geq 5$. It then follows immediately that the graph in Figure 2(iv) is the unique $(5,2,2)$-graph.

For odd $d \geq 7$, we denote by W the vertex set of this induced subgraph. Let $w \in W$ and $v \in V(G) \setminus W$ such that $v \sim w$. Because the graph in Figure 2(iv) is a $(5,2,2)$-graph, in order to reach within at most two steps, the 18 vertices of W, each of which is at distance 2 from w, we must have $|N(v)| \geq 18$. Thus, $d \geq 19$.

Let us now consider a triangle T_1 in G and let Υ_{2b} be the outer repeat cycle of T_1 in H_G. Figure 6 can also be used to illustrate the structure of a triangle and its outer repeat cycle in the case when d is odd. The only difference is that both Υ_{2b} and Υ'_{2b} are now of size $3(d-5)$.

Let z be a vertex of type $2a$ in G. In order to reach the three vertices of T_1 in at most two steps from z, the vertex z must be adjacent to exactly three vertices of Υ_{2b}. By Lemma 4, z belongs to some repeat cycle of length 4 in H_G. But then it follows from Lemma 10 that $4 = \frac{1}{3}|\Upsilon_{2b}| = d-5$, i.e., $d = 9$. This contradicts the above argument that $d \geq 19$.

Therefore, for odd $d \geq 7$, $n_{2b} = d^2 - 1$. □

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