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On Chromatic Uniqueness of a Family of K_4-Homeomorphs II

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Abstract

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. A K_4-homeomorph denoted by $K_4(a, b, c, d, e, f)$ if the six edges of complete graph K_4 are replaced by the six paths of length a, b, c, d, e, f respectively. In this paper, we study the chromatically unique of such K_4-homeomorph with girth $3a + 1$, where $b = a$, $c = a + 1$ and $d, e, f \geq a$.

Keywords: Chromatic polynomial; Chromatic uniqueness; K_4-homeomorph.

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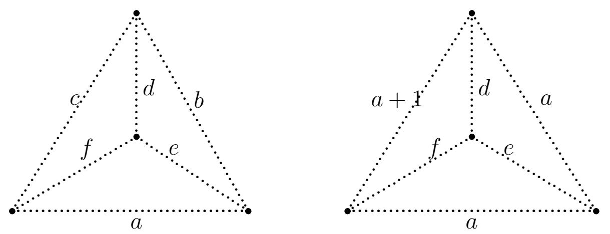
1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$, i.e, H is isomorphic to G.



(a) $K_4(a,b,c,d,e,f)$ (b) $K_4(a,a,a+1,d,e,f)$
Figure 1: K_4-Homeomorphs

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1(a). So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \le g \le 9$, has been studied by many authors (see [3, 5, 6, 7]). Recently, Y.L.Peng in [8] has studied the chromaticity of one type of K_4-homeomorphs with girth 7, namely, $K_4(1,3,3,d,e,f)$. S.Catada-Ghimire et al. in [1] studied the chromatic uniqueness of one family of K_4-homeomorphs with girth 10, namely, $K_4(3,3,4,d,e,f)$. In this paper, we discuss the chromatic uniqueness of $K_4(a,a,a+1,d,e,f)$ (as shown in Figure 1(b)) with girth $3a+1$, where $a \ge 3$ and d,e,f are at least a.

2. Preliminary results

In this section, we give some known results used in the sequel.

Lemma 2.1. *Assume that G and H are $\chi-$equivalent. Then, we have:*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [4]);*

(2) *Let $g(G)$ and $g(H)$ denote the girths of G and H, respectively. Then $g(G) = g(H)$ and G and H have the same number of cycles with length equal to their girth (see [12]);*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [2]);*

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$. Then*

  (i) *min $\{a,b,c,d,e,f\}$ = min $\{a',b',c',d',e',f'\}$ and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that this minimum occurs in the list $\{a',b',c',d',e',f'\}$ (see [11]);*

  (ii) *if $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ as multisets, then $H \cong G$ (see [5]).*

Lemma 2.2. [9] *Let $G = K_4(a,b,c,d,e,f)$ (see Figure 1(a)) where exactly three of a,b,c,d,e,f are the same. Then G is not χ-unique if and only if G is isomorphic to $K_4(r,r,r-2,1,2,r)$ or $K_4(r,r-2,r,2r-2,1,r)$ or $K_4(t,t,1,2t,t+2,t)$ or $K_4(t,t,1,2t,t-1,t)$ or $K_4(t,t+1,t,2t+1,1,t)$ or $K_4(1,t,1,t+1,3,1)$ or $K_4(1,1,t,2,t+2,1)$, where $r \ge 3$, $t \ge 2$.*

Lemma 2.3. [5, 10] *The graph $K_4(a,b,c,d,e,f)$ is $\chi-$unique if exactly four numbers among a,b,c,d,e,f are the same.*

Lemma 2.4. [11] *The graph $K_4(a,b,c,d,e,f)$ is $\chi-$unique if a,b,c,d,e,f assume no more than two distinct values.*

3. Main result

Lemma 3.1. *Let $G \cong K_4(a,a,a+1,d,e,f)$ and $H \cong K_4(a,a,a+1,d',e',f')$. Then*

(1) $P(G) = (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^{a+2} - 3s^{a+1} - 2s^a + s^2 + 3s + 2 + R(G)]$, *where* $R(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+a} + s^{f+a} + s^{e+a+1} + s^{e+2a} + s^{d+2a+1} + s^{f+2a+1} + s^{d+e+f}$, $s = 1-\lambda$, *x is the number of the edges of G.*

(2) *If $P(G) = P(H)$, then $R(G) = R(H)$.*

Proof. (1) Let $s = 1-\lambda$. From [11], the chromatic polynomial of K_4-homeomorph $K_4(a,b,c,d,e,f)$ is as follows:

$$P(K_4(a,b,c,d,e,f) = (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2)-(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+ (s^{a+d}+s^{b+f}+s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}-s^{x-1})].$$

So when $b = a$ and $c = a+1$, we have

$$\begin{aligned}P(K_4(a,a,a+1,d,e,f) &= (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2)-(s+1)(s^a+s^a+s^{a+1}\\ &\quad +s^d+s^e+s^f)+(s^{d+a}+s^{f+a}+s^{e+(a+1)}+s^{e+2a}+s^{d+(2a+1)}+\\ &\quad s^{f+(2a+1)}+s^{d+e+f}-s^{x-1})]\\ &= (-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^{a+2}-3s^{a+1}-2s^a+s^2+3s+\\ &\quad 2-s^d-s^e-s^f-s^{d+1}-s^{e+1}-s^{f+1}+s^{d+a}+s^{f+a}+s^{e+(a+1)}+\\ &\quad s^{e+2a}+s^{d+(2a+1)}+s^{f+(2a+1)}+s^{d+e+f}]\\ &= (-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^{a+2}-3s^{a+1}-2s^a+s^2+3s\\ &\quad +2+R(G)], \text{ where}\\ R(G) &= -s^d-s^e-s^f-s^{d+1}-s^{e+1}-s^{f+1}+s^{d+a}+s^{f+a}+s^{e+(a+1)}\\ &\quad +s^{e+2a}+s^{d+(2a+1)}+s^{f+(2a+1)}+s^{d+e+f} \text{ as required.}\end{aligned}$$

(2) If $P(G) = P(H)$, then we can clearly see that $R(G) = R(H)$. □

Theorem 3.2. *Any K_4-homeomorph $K_4(a,a,a+1,d,e,f)$ with girth $3a+1$, where $\min\{d,e,f\} \geq a$ and $a \geq 3$, is $\chi-$unique.*

Proof. Let $G \cong K_4(a,a,a+1,d,e,f)$ with girth (the length of the smallest cycle of G) $3a+1$, that is, $d+e \geq 2a+1$, $e+f \geq 2a+1$, where $\min\ \{d,e,f\} \geq a$. If there exists a graph H such that $P(G) = P(H)$, then from Lemma 2.1(3 and 4(ii)), H must be a K_4-homeomorph $K_4(a',b',c',d',e',f')$ with $\min\{a',b',c',d',e',f'\} = a$ and the number of times that a occurs in the list $\{a, a, a+1, d, e, f\}$ is equal to the number of times that a occurs in the list $\{a', b', c', d', e', f'\}$. By Lemma 2.1(2), H has girth 3a+1. Thus, H must be of the type $K_4(a, a, a+1, d', e', f')$, where $\min\{d', e', f'\} \geq a$ and $d' + e' \geq 2a+1$, $e' + f' \geq 2a+1$. We now solve the equation $P(G) = P(H)$. If one of $\{d', e', f'\}$ is equal to a, then by Lemma 2.2, H is $\chi-$unique. Since $G \sim H$, we have $G \cong H$. If two of $\{d', e', f'\}$ are equal to a, then by Lemma 2.3, H is $\chi-$unique. Since $G \sim H$, we have $G \cong H$. If all of $\{d', e', f'\}$ are equal to a, then by Lemma 2.4, H is $\chi-$unique. Since $G \sim H$, we have $G \cong H$. Let us find G and H which are not isomorphic, where $\min\{d', e', f'\} \geq a+1$. By symmetry, we can assume $d \leq f$ and $d' \leq f'$. Let us solve the equation $R(G) = R(H)$. From Lemma 3.1, we have

$$R(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+a} + s^{f+a} + s^{e+a+1} + s^{e+2a} + s^{d+(2a+1)} + s^{f+(2a+1)} + s^{d+e+f}$$

$$R(H) = -s^{d'} - s^{e'} - s^{f'} - s^{d'+1} - s^{e'+1} - s^{f'+1} + s^{d'+a} + s^{f'+a} + s^{e'+(a+1)} + s^{e'+2a} + s^{d'+(2a+1)} + s^{f'+(2a+1)} + s^{d'+e'+f'}.$$

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1(1), $d + e + f = d' + e' + f'$. We obtain the following after simplification: (Note that our assumption in the following steps of the proof is $R_j(G) = R_j(H)$, where $1 \leq j \leq 6$).

$$R_1(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+a} + s^{f+a} + s^{e+(a+1)} + s^{e+2a} + s^{d+(2a+1)} + s^{f+(2a+1)}$$

$$R_1(H) = -s^{d'} - s^{e'} - s^{f'} - s^{d'+1} - s^{e'+1} - s^{f'+1} + s^{d'+a} + s^{f'+a} + s^{e'+(a+1)} + s^{e'+2a} + s^{d'+(2a+1)} + s^{f'+(2a+1)}.$$

Let us consider the h.r.p. in $R_1(G)$ and the h.r.p. in $R_1(H)$. We have $\max\{e+2a, f+(2a+1), d+(2a+1)\} = \max\{e'+2a, f'+(2a+1), d'+(2a+1)\}$. Without loss of generality, we will consider only the following six cases.

Case 1. If $\max\{e+2a, f+(2a+1), d+(2a+1)\} = e+2a$ and $\max\{e'+2a, f'+(2a+1), d'+(2a+1)\} = e'+2a$, then $e = e'$. Thus, we can cancel the following pairs of terms in $R_1(G)$ and $R_1(H)$: $-s^e$ with $-s^{e'}$, $-s^{e+1}$ with $-s^{e'+1}$, $s^{e+(a+1)}$ with $s^{e'+(a+1)}$ and s^{e+2a} with $s^{e'+2a}$. Since $d \leq f$ and $d' \leq f'$, the l.r.p. in $R_1(G)$ is d and the l.r.p. in $R_1(H)$ is d'. So, $d = d'$ and we have $e = e'$. Since $d+e+f = d'+e'+f'$, $f = f'$. Thus, we have $\{d, e, f\} = \{d', e', f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 2. If $\max\{e+2a, f+(2a+1), d+(2a+1)\} = d+(2a+1)$ and $\max\{e'+2a, f'+(2a+1), d'+(2a+1)\} = d'+(2a+1)$, then $d = d'$. Since $d \leq f$ and $d' \leq f'$, this case is only possible if $d = f$ and $d' = f'$. So, we have $d = f = d' = f'$. From $d+e+f = d'+e'+f'$, we have $e = e'$. Thus, $\{d, e, f\} = \{d', e', f'\}$ as multisets. By Lemma 2.1(4(ii)), $G \cong H$.

Case 3. If $\max\{e+2a, f+(2a+1), d+(2a+1)\} = f+(2a+1)$ and $\max\{e'+2a, f'+(2a+1), d'+(2a+1)\} = f'+(2a+1)$, then $f = f'$. Thus, we can cancel the following pairs of terms in $R_1(G)$ and $R_1(H)$: $-s^f$ with $-s^{f'}$, $-s^{f+1}$ with $-s^{f'+1}$, s^{f+a} with $s^{f'+a}$

and $s^{f+(2a+1)}$ with $s^{f'+(2a+1)}$. After simplification, the l.r.p. in $R_1(G)$ is d or e and the l.r.p. in $R_1(H)$ is d' or e'. So, $d = d'$ or $d = e'$ or $e = e'$ or $e = d'$. We know that $f = f'$. Since $d + e + f = d' + e' + f'$, we have $\{d, e, f\} = \{d', e', f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 4. If max $\{e+2a, f+(2a+1), d+(2a+1)\} = e+2a$ and max $\{e'+2a, f'+(2a+1), d'+(2a+1)\} = d'+(2a+1)$, then $e = d'+1$. Since $d' \le f'$, max $\{e'+2a, f'+(2a+1), d'+(2a+1)\}$ can only be equal to $d' + (2a + 1)$ if $d' = f'$. Therefore, we have

$$d' = f' = e - 1. \tag{1}$$

As $d + e + f = d' + e' + f'$, we have

$$d + f - e = e' - 2. \tag{2}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), min $\{d, e, f\}$ = min $\{d', e', f'\}$. Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 4.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then $d = d'$. Thus, we can cancel the following pairs of terms in $R_1(G)$ and $R_1(H)$: $-s^d$ with $-s^{d'}$, $-s^{d+1}$ with $-s^{d'+1}$, s^{d+a} with $s^{d'+a}$ and $s^{d+(2a+1)}$ with $s^{d'+(2a+1)}$. Since max $\{e + 2a, f + (2a + 1), d + (2a + 1)\} = e + 2a$ we have $e + 2a \ge f + (2a + 1)$, i.e., $e > f$. After simplification, the l.r.p. in $R_1(G)$ is f and the l.r.p. in $R_1(H)$ is f' or e'. So, $f = f'$ or $f = e'$. We know that $d = d'$. Since $d + e + f = d' + e' + f'$, we have the two multisets $\{d, e, f\}$ and $\{d', e', f'\}$ the same. From Lemma 2.1 (4(ii)), $G \cong H$.

Subcase 4.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From equation (2), we have $f = e - 2$. Note that $d' = f' = e - 1 = f + 1$ from equation (1). We can write $R_1(G)$ and $R_1(H)$ as follows:

$$\begin{aligned} R_2(G) &= -s^d - s^{f+2} - s^f - s^{d+1} - s^{f+2} - s^{f+1} + s^{d+a} + s^{f+a} + s^{f+2a} + s^{f+(2a+2)} \\ &\quad + s^{d+(2a+1)} + s^{f+(2a+1)} \\ R_2(H) &= -s^{f+1} - s^d - s^{f+1} - s^{f+2} - s^{d+1} - s^{f+2} + s^{f+(a+1)} + s^{f+(a+1)} + s^{d+(a+1)} \\ &\quad + s^{d+2a} + s^{f+(2a+2)} + s^{f+(2a+2)}. \end{aligned}$$

After simplifying $R_2(G)$ and $R_2(H)$, we have

$$\begin{aligned} R_3(G) &= -s^f + s^{d+a} + s^{f+a} + s^{f+2a} + s^{d+(2a+1)} + s^{f+(2a+1)} \\ R_3(H) &= -s^{f+1} + s^{f+(a+1)} + s^{f+(a+1)} + s^{d+(a+1)} + s^{d+2a} + s^{f+(2a+2)}. \end{aligned}$$

Clearly, this contradicts $R_3(G) = R_3(H)$.

Subcase 4.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. Since $d' \le f'$, $d' = f'$. Thus, $d' = f' = d$. From Equa.(1), we have $d = e - 1$. From Equa. (2), we have $e' = f + 1$. From $R_1(G) = R_1(H)$, we have

$$R_4(G) = -s^d - s^{d+1} - s^f - s^{d+1} - s^{d+2} - s^{f+1} + s^{d+a} + s^{f+a} + s^{d+(a+2)} + s^{d+(2a+1)} + s^{d+(2a+1)} + s^{f+(2a+1)}$$

$$R_4(H) = -s^d - s^{f+1} - s^d - s^{d+1} - s^{f+2} - s^{d+1} + s^{d+a} + s^{d+a} + s^{f+(a+3)} + s^{f+(2a+1)} + s^{d+(2a+1)} + s^{d+(2a+1)}.$$

After simplifying the equations we get $d = f$. Thus, $d = d' = f' = f$. We also have $d + 1 = e = f + 1 = e'$. Therefore, $e = e'$, $d = d'$ and $f = f'$. So, $G \cong H$.

Case 5. If max $\{e+2a, f+(2a+1), d+(2a+1)\} = d+(2a+1)$ and max $\{e'+2a, f'+(2a+1), d'+(2a+1)\} = f'+(2a+1)$, then $f' = d$. Since max $\{e+2a, f+(2a+1), d+(2a+1)\} = d+(2a+1)$ and $d \le f$, this case is only possible when $d = f$. So, $d = f = f'$. Thus, we can cancel the following corresponding terms in $R_1(G)$ and $R_1(H)$: $-s^f$ and $-s^{f'}$, $-s^{f+1}$ with $-s^{f'+1}$, s^{f+a} with $s^{f'+a}$ and $s^{f+(2a+1)}$ with $s^{f'+(2a+1)}$. After simplification, the l.r.p. in $R_1(G)$ is d or e and the l.r.p. in $R_1(H)$ is d' or e'. So, $d = d'$ or $d = e'$ or $e = e'$ or $e = d'$. We know that $d = f = f'$. Since $d+e+f = d'+e'+f'$, we have the two multisets $\{d, e, f\}$ and $\{d', e', f'\}$ the same. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 6. If max $\{e+2a, f+(2a+1), d+(2a+1)\} = e+2a$ and max $\{e'+2a, f'+(2a+1), d'+(2a+1)\} = f'+(2a+1)$, then we have

$$f' = e - 1. \tag{3}$$

From $d + e + f = d' + e' + f'$, we have

$$d + f = d' + e' - 1. \tag{4}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have min $\{d, e, f\}$ = min $\{d', e', f'\}$. Without loss of generality, let min $\{d, e, f\} = d$. We consider the following subcases.

Subcase 6.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then $d = d'$. This case can be dealt with the same way as Case 4(Subcase 4.1). So, we get the same result, $G \cong H$.

Subcase 6.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From equation (4), we have $d' = f + 1$. Note that from equation(3) we have $f' = e - 1$. From $R_1(G)$ and $R_1(H)$, we have

$$R_6(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+a} + s^{f+a} + s^{e+(a+1)} + s^{e+2a} + s^{d+(2a+1)} + s^{f+(2a+1)}$$

$$R_6(H) = -s^{f+1} - s^d - s^{e-1} - s^{f+2} - s^{d+1} - s^e + s^{f+(a+1)} + s^{e+2} + s^{d+(a+1)} + s^{d+2a} + s^{f+(2a+2)} + s^{e+2a}.$$

After simplification, we have $-s^f - s^{e+1} + s^{d+a} + s^{f+a} + s^{e+(a+1)} + s^{d+(2a+1)} + s^{f+(2a+1)} = -s^{e-1} - s^{f+2} + s^{f+(a+1)} + s^{e+2} + s^{d+(a+1)} + s^{d+2a} + s^{f+(2a+2)}$.

Since max $\{e+2a, f+(2a+1), d+(2a+1)\} = e+2a$, we have $e+2a \ge f+(2a+1)$, i.e.,

$e+1 \geq f+2$ and $e+2a \geq d+(2a+1)$, i.e., $e+1 \geq d+2$. If $e+1 > f+2$, then we consider the following possibilities: If $e+1 = d+a$ then $e+(a+2) = d+(2a+1)$. Since there is no term with the power exponent $e+(a+2)$, we can assume $e+(a+2) = d+(2a+1) = f+(2a+2)$. This contradicts the assumption $d \leq f$. If $e+1 = f+a$, then $e+2 = f+(a+1)$. The terms with power exponents $e+2$ and $f+(a+1)$ cannot be cancelled since they are of the same sign belonging to the same side of the equation. This is a contradiction. If $e+1 = d+(2a+1)$, then $e+(a+1) = d+(3a+1)$. Since there is no term with the power exponent $d+(3a+1)$, we can assume $e+(a+1) = d+(3a+1) = f+(2a+2)$. Consider the term with power exponent $f+(2a+1)$. This term can only be cancelled if there exists a term with power exponent $e+a$ or $d+3a$ since $e+a = d+3a = f+(2a+1)$. But there is no such term in the equation. This is a contradiction. If $e+1 = f+(2a+1)$, then $e+(a+1) = f+(3a+1)$. Since there is no term with the power exponent $f+(3a+1)$, we can consider $e+(a+1) = f+(3a+1) = d+2a$. This contradicts the assumption $d \leq f$. Lastly, the above equation can only be satisfied if $e+1 = f+2$ and $e+(a+1) = f+(a+2) = d+(a+1)$, i.e., $e = f+1 = d$. But again this contradicts the assumption $d \leq f$.

Subcase 6.3. If min $\{d,e,f\} = d$ and min $\{d',e',f'\} = f'$, then $d = f'$. From equation(3), we have $f' = e-1 = d$. Since max $\{e'+2a, d'+(2a+1), f'+(2a+1)\} = f'+(2a+1)$ and min $\{d',e',f'\} = f'$, this case is only possible if $d' = f'$. Thus, $d = f' = d'$. We can cancel the following pairs of terms in $R_1(G)$ and $R_1(G)$: $-s^d$ with $-s^{d'}$, $-s^{d+1}$ with $-s^{d'+1}$, s^{d+a} with $s^{d'+a}$ and $s^{d+(2a+1)}$ with $s^{d'+(2a+1)}$. Since $\max\{e+2a, f+(2a+1), d+(2a+1)\} = e+2a$, we have $e+2a \geq f+(2a+1)$, i.e., $e > f$. Thus, after simplification, the l.r.p. in $R_1(G)$ is f and the l.r.p. in $R_1(H)$ is f' or e'. So, $f = f'$ or $f = e'$. We know that $d = f = f'$. Since $d+e+f = d'+e'+f'$, we have the two multisets $\{d,e,f\}$ and $\{d',e',f'\}$ the same. From Lemma 2.1 (4(ii)), $G \cong H$.

At this point, we have shown that given a K_4-homeomorph, $G = K_4(a,a,a+1,d,e,f)$ where $\{d,e,f\} \geq a$, $d+e \geq 2a+1$,$d+f \geq 2a$,$f+e \geq 2a+1$ and $a \geq 3$, whenever there exists a K_4-homeomorph, $H = K_4(a',b',c',d',e',f')$ such that $G \sim H$, $G \cong H$. Therefore, K_4-homeomorph, $G = K_4(a,a,a+1,d,e,f)$ with girth 3a+1, where d,e,f are at least a and $a \geq 3$, is χ-unique.

The proof is now complete. □

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