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The gamma graph of a graph

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Abstract

The gamma graph of a graph G has its γ-sets as vertices and any two vertices are adjacent if the corresponding γ-sets differ exactly by one vertex. We obtain the collection of all forbidden subgraphs on five vertices of the gamma graph. The closure property of the gamma graphs under various graph products are studied. The structural property of the gamma graph of a cograph is also studied. The relationship between the clique number and the independence number of a graph and its gamma graph is also discussed.

Keywords: Gamma graph, forbidden subgraph, cograph.

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1. Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

The union of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The join of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. The tensor product of G_1 and G_2, denoted by $G_1 \times G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_i v_i \in E_i$ for $i = 1, 2$. The cartesian product of G_1 and G_2, denoted by $G_1 \Box G_2$ has vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if

$u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_1v_1 \in E_1$ and $u_2 = v_2$. The strong product of G_1 and G_2, denoted by $G_1 * G_2$ has vertex set $V(G_1 * G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_1v_1 \in E_1$ and $u_2 = v_2$ or $u_iv_i \in E_i$ for $i = 1, 2$.

The distance between any two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v. The eccentricity of a vertex $u \in V(G)$, $e(u) = \max\{d(u, v) : v \in V(G)\}$. The diameter of a graph $d(G) = \max\{e(u) : u \in V(G)\}$. If G is a graph on n vertices, a vertex of degree $n - 1$ is called a universal vertex.

Cographs are graphs that can be reduced to edgeless graphs by taking complements within components [2]. Cographs can be recursively obtained from K_1 by the operations of union and join [6]. Therefore, if G is a connected cograph then $G = G_1 \vee G_2$, where G_1 and G_2 are both cographs.

A maximal complete subgraph of a graph G is called a clique. The clique number of a graph G, denoted by $\omega(G)$ is the maximum of the number of vertices in a clique in G. In a graph G, two vertices are independent if there is no edge joining these two vertices. A collection of pair wise independent vertices is called an independent set. The maximum cardinality of an independent set in a graph G is called its independence number, denoted by $\alpha(G)$.

A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is minimal if no proper subset of S is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a set in $MDS(G)$, or equivalently, the minimum cardinality of a dominating set in G. For a detailed literature on domination theory, the reader may refer to [3] and [4].

In a graph G, a dominating set of cardinality $\gamma(G)$ is called a γ-set. Let $\mathcal{D}$ be the collection of all γ-sets in G. The gamma graph of G, denoted by $\gamma.G$, is the graph with vertex set $\mathcal{D}$ and any two vertices D_1 and D_2 are adjacent if $|D_1 \cap D_2| = \gamma(G) - 1$ [9]. A graph H is called a gamma graph (or γ-graph) if there exists a graph G, such that $H = \gamma.G$.

It may be noted that G is connected need not imply $\gamma.G$ is connected and viceversa, for $\gamma.(2K_2) = C_4$ and $\gamma.C_6 = 3K_1$. A graph G is γ-connected, if $\gamma.G$ is connected [9] and γ-fixed if $\gamma.G = G$. Clearly, complete graphs are γ-fixed and hence are γ-graphs. In [9], the gamma graphs of paths, cycles and Harary graph are studied. In [7], it is proved that trees and unicyclic graphs are gamma graphs. Also, trees are γ-connected [8]. A forbidden subgraph for the gamma graph is also given in [7].

In [7], the following problems were listed.

Problem 1.1. *Find forbidden subgraphs for γ-graphs.*

Problem 1.2. *Characterize gamma graphs.*

In this paper, we obtain the collection of all forbidden subgraphs on five vertices of the gamma graph. The closure property of the gamma graphs under various graph products are studied. The structural property of the gamma graph of a cograph is also studied. The relationship between the clique number and the independence number of a graph and its gamma graph is also discussed.

All graph theoretic terminology and notations not mentioned here are from [1].

2. Forbidden subgraphs

In this section, Problem 1.1 listed above is considered. We identify all forbidden subgraphs on five vertices for the gamma graphs. We need only consider connected graphs, for if $H_1 = \gamma.G_1$ and $H_2 = \gamma.G_2$, then $\gamma.(G_1 \cup G_2 \cup K_2) = H_1 \square H_2 \square K_2$, (where $\square$ denotes the cartesian product), which contains $H_1 \cup H_2$ as an induced subgraph.

Theorem 2.3. *If H is a gamma graph, then H does not contain*

(a) $K_{3,2}$

(b) $P_3 \vee K_2$

(c) $(K_2 \cup K_1) \vee 2K_1$

as an induced subgraph.

Proof. Let $H = \gamma.G$.

(a) Suppose that H contains $K_{3,2}$ as an induced subgraph. Let $\{U_1, U_2, U_3\}$ and $\{V_1, V_2\}$ be the partition of $K_{3,2}$. Let $U_1 = \{u_1, u_2, ..., u_k\}$. Since V_1 is adjacent to U_1, $V_1 = \{v_1, u_2, u_3, ..., u_k\}$. Again since V_2 is adjacent to U_1 and is not adjacent to V_1, $V_2 = \{u_1, v_2, u_3, ..., u_k\}$, where $v_1 \neq v_2$. Now, since U_2 is adjacent to V_1 and V_2 and is not adjacent to U_1, $U_2 = \{v_1, v_2, u_3, ..., u_k\}$. But, now we cannot choose U_3 different from U_1 and U_2 such that U_3 is adjacent to V_1 and V_2 and is not adjacent to U_1 and U_2. Hence $K_{3,2}$ is not an induced subgraph.

(b) Let $U_1U_2U_3$ be an induced P_3 in H. Let $U_1 = \{u_1, u_2, ..., u_k\}$. Since, U_1 adjacent to U_2, $U_2 = \{v_1, u_2, ..., u_k\}$. Again, since U_3 is adjacent to U_2 and not to U_1, $U_3 = \{v_1, v_2, u_3, ..., u_k\}$. Let U_4 be adjacent to U_1, U_2 and U_3. If $u_3 \notin U_4$, then since it is adjacent to U_1, it must contain u_1 and u_2. But, then U_4 cannot be adjacent to U_3. Similarly, $u_4, ..., u_k \in U_4$. Since, U_4 is a γ-set, there are two more vertices in U_4, which can be either u_1 and v_1 or u_2 and v_2. Therefore, there are two possible choices of a dominating set, the vertex corresponding to which are adjacent to each U_1, U_2 and U_3, namely, $U_4 = \{u_1, v_1, u_3, ..., u_k\}$ and $U_5 = \{u_2, v_2, u_3, ..., u_k\}$. But, in that case U_4 is not adjacent to U_5. Therefore, H cannot have $P_3 \vee K_2$ as an induced subgraph.

(c) Consider the induced subgraph $K_2 \vee 2K_1$ of $(K_2 \cup K_1) \vee 2K_1$. It is isomorphic to $P_3 \vee K_1$. Therefore, using the arguments in (b) above, we may assume that the vertices corresponding to $K_2 \vee 2K_1$ are $U_1 = \{u_1, u_2, ..., u_k\}$, $U_2 = \{v_1, u_2, ..., u_k\}$, $U_3 = \{v_1, v_2, u_3, ..., u_k\}$ and

$U_4 = \{u_1, v_1, u_3, ..., u_k\}$ (or $U_4 = \{u_2, v_2, u_3, ..., u_k\}$), where U_1 and U_3 correspond to $2K_1$ and U_2 and U_4 correspond to K_2. Let U_5 be the vertex which is adjacent to U_1 and U_3, but not to U_2 and U_4. If $u_3 \notin U_5$, then since U_5 is adjacent to U_1, it must contain u_1 and u_2 which contradicts U_5 adjacent to U_3. Similarly for $u_4, ..., u_k$. Now, there are two more vertices in U_5, one of which must be u_1 or u_2 and the other must be v_1 or v_2. Whichever pair we choose, U_5 becomes adjacent to or identical with U_2 or U_4. Therefore, H cannot have $(K_2 \cup K_1) \vee 2K_1$ as an induced subgraph. □

It is known [7] that, $K_2 \vee 3K_1$ is a forbidden subgraph for gamma graphs. In the above theorem, we have proved that three more graphs on five vertices are forbidden for gamma graphs. In fact, these are the only connected graphs on five vertices which are forbidden for gamma graphs. The complete graph, trees and unicyclic graphs are proved [7] to be gamma graphs and there are only seven more connected graphs on five vertices which can be easily verified to be gamma graphs.

3. Graph Products

In this section we discuss the closure property of gamma graphs under the graph products [5] - the tensor product, the cartesian product and the strong product. The gamma graphs are not closed under the tensor product, for both K_5 and K_2 are gamma graphs, $K_5 \times K_2$ is $K_{5,5}$ minus a perfect matching, which contains $K_{3,2}$ as an induced subgraph, which is forbidden for gamma graphs by Theorem 2.1(a).

Theorem 3.4. *The collection of all gamma graphs is closed under the cartesian product.*

Proof. Let $H_1 = \gamma.G_1$ and $H_2 = \gamma.G_2$. Let $G = G_1 \cup G_2$. A γ-set of G is the union of a γ-set of G_1 and a γ-set of G_2. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be any two vertices of $\gamma.G$, where A_1 and B_1 are γ-sets of G_1 and A_2 and B_2 are γ-sets of G_2. A is adjacent to B if and only if $|A \cap B| = \gamma(G) - 1$. Therefore, either $|A_1 \cap B_1| = \gamma(G_1) - 1$ and $A_2 = B_2$ or $A_1 = A_2$ and $|A_2 \cap B_2| = \gamma(G_2) - 1$. That is, A_1 is adjacent to B_1 in $\gamma.G_1$ and $A_2 = B_2$ or $A_1 = B_1$ and A_2 is adjacent to B_2 in $\gamma.G_2$. Therefore, $\gamma.G = H_1 \Box H_2$. Hence the theorem. □

The gamma graphs are not closed under the strong product, for both K_3 and P_3 are gamma graphs, but $K_3 * P_3$ contains $P_3 \vee K_2$ as an induced subgraph, which is forbidden for gamma graphs by Theorem 2.1(b).

4. Cographs

Theorem 4.5. *If G is a cograph then the vertex set of $\gamma.G$ can be partitioned into sets $V_1, V_2, ..., V_k$ such that each V_i induces $K_{p_i} \Box K_{q_i}$ for $i = 1, 2, ..., k$.*

Proof. If G is disconnected, then let $G = G_1 \cup G_2$. Therefore, $\gamma.G = (\gamma.G_1)\Box(\gamma.G_2)$ by Theorem 3.1. Hence, the vertex set can be partitioned to sets, each of which induces $\gamma.G_1$ (or $\gamma.G_2$). Therefore, it is enough if we prove the result for connected cographs.

Let G be a connected cograph. The proof is by induction on the number of vertices of G. For small values of n the theorem can be easily verified. Let G be a connected cograph with n vertices. Therefore, $G = G_1 \vee G_2$ and $\gamma(G) \leq 2$. If $\gamma(G) = 1$ then $\gamma.G = K_p$ where p is the number of universal vertices in G. But, $K_p = K_p \Box K_1$.

Let $\gamma(G) = 2$. Any vertex in G_1 together with any vertex in G_2 is a γ-set of G. The vertices of $\gamma.G$ corresponding to these γ-sets induces $K_{n_1}\Box K_{n_2}$, where n_1 and n_2 are the number of vertices in G_1 and G_2 respectively. If both $\gamma(G_1)$ and $\gamma(G_2)$ are greater than two, then these are the only γ-sets of G and hence the theorem follows. Also, both $\gamma(G_1)$ and $\gamma(G_2)$ must be greater than one, since otherwise, $\gamma(G) = 1$. If $\gamma(G_1) = 2$ (or $\gamma(G_2) = 2$), then every γ-set of G_1 (or G_2) is also a γ-set of G. But, since the number of vertices of G_1 (or G_2) is less than n, by induction the vertices corresponding to the γ-sets of G_1 (or G_2) can be partitioned to vertex sets which induces $K_p \Box K_q$.

Hence the theorem. □

Theorem 4.6. *If G is a connected cograph then $d(\gamma.G) \leq 2$. Moreover, $d(\gamma.G) = 1$ if and only if G has a universal vertex.*

Proof. The proof is by induction on the number of vertices of G. For small values of n the theorem can be easily verified. Let $G = G_1 \vee G_2$ be a connected cograph with n vertices. If $\gamma(G) = 1$, then $\gamma.G = K_p$, where p is the number of universal vertices of G and $d(K_p) = 1$.

Let $\gamma(G) = 2$. Let D_1 and D_2 be any two γ-sets of G. If $D_1 = \{u_1, v_1\}$ and $D_2 = \{u_2, v_2\}$, where $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$, then $\{u_1, v_2\}$ is also a γ-set in G which is adjacent to both D_1 and D_2. Hence $d(D_1, D_2) \leq 2$. If $D_1 = \{u_1, v_1\}$ and $D_2 = \{u_2, u_3\}$, where $u_1, u_2, u_3 \in V(G_1)$ and $v_1 \in V(G_2)$, then $\{u_2, v_1\}$ is also a γ-set in G which is adjacent to both D_1 and D_2. Hence $d(D_1, D_2) \leq 2$. If $D_1 = \{u_1, u_2\}$ and $D_2 = \{u_3, u_4\}$, where $u_1, u_2, u_3, u_4 \in V(G_1)$ then two cases arises - either G_1 is connected or G_1 has exactly two components and u_i's are universal vertices of these components.

If G_1 is connected then since it is having less than n vertices, by induction hypothesis $d_{\gamma.G_1}(D_1, D_2) \leq 2$. Since $\gamma.G_1$ is an induced subgraph of $\gamma.G$, $d_{\gamma.G}(D_1, D_2) \leq 2$.

If G has exactly two components, then let u_1 and u_3 be the universal vertices in the first component and let u_2 and u_4 be the universal vertices in the second component. Therefore, $\{u_1, u_4\}$ is also a γ-set in G, which is adjacent to both D_1 and D_2 and hence $d(D_1, D_2) \leq 2$.

Since D_1 and D_2 were arbitrary, $d(\gamma.G) \leq 2$.

If G has a universal vertex then $\gamma.G = K_p$ and $d(K_p) = 1$. If G does not have a universal vertex then $G = G_1 \vee G_2$, where both G_1 and G_2 have at least one pair of non-adjacent vertices $u_1, u_2 \in V(G_1)$ and $v_1, v_2 \in V(G_2)$. In that case, $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are two

γ-sets in G, the vertices corresponding to which in $\gamma.G$ are non-adjacent. Therefore, $d(\gamma.G) \neq 1$. □

If G is a disconnected cograph then Theorem 4.2 is not true. For, if $G = C_4 \cup K_2$ then $\gamma.G = H \Box K_2$, where H is K_6 minus a perfect matching and $d(\gamma.G) = 3$.

5. Clique number and independence number

If G is a graph with $\omega(G) = 1$, then G is the trivial graph whose only dominating set is the vertex set itself. Therefore, in that case, $\gamma.G = K_1$ and hence $\omega(\gamma.G) = 1$.

Theorem 5.7. *If a and b are any two positive integers, $a > 1$, then there exists a graph G such that $\omega(G) = a$ and $\omega(\gamma.G) = b$.*

Proof. **Case 1.** $b = 1$.

Let G be the graph obtained as follows. Let u be a vertex in a complete graph K_a. Attach b pendent vertices $u_1, u_2, ..., u_b$ to u. In this case, $\gamma(G) = 1$ and the only dominating set is $\{u\}$ so that $\gamma.G = K_1$. Therefore, $\omega(\gamma.G) = 1 = b$.

Case 2. $b > 1$.

Let G be the graph obtained as follows. Let u be a vertex in a complete graph K_a. Attach b vertices $u_1, u_2, ..., u_b$ to u. Introduce a new vertex v which is adjacent to $u_1, u_2, ..., u_{b-1}$. Then, $\gamma(G) = 2$ and the $\gamma-$sets are $\{u, u_1\}, \{u, u_2\}, ..., \{u, u_{b-1}\}$ and $\{u, v\}$. Therefore, $\gamma.G = K_b$. Hence, the theorem. □

If G is a graph with $\alpha(G) = 1$, then G is K_n and hence $\gamma.G = K_n$, so that $\alpha(\gamma.G) = 1$.

Theorem 5.8. *If a and b are any two positive integers, $a > 1$, then there exists a graph G such that $\alpha(G) = a$ and $\alpha(\gamma.G) = b$.*

Proof. Let G be defined as follows. Consider $K_{2b} - M$ where M is a perfect matching $\{u_1v_1, u_2v_2, ..., u_bv_b\}$. Join every vertex $\{u_1, u_2, ..., u_b\}$ to each of the vertex of a newly introduced independent set $\{w_1, w_2, ...w_{a-1}\}$. Clearly, $\alpha(G) = a$ and $\gamma(G) = 2$. Since, $\gamma(G) = 2$, two vertices are not adjacent if and only if the corresponding γ-sets are disjoint.

Case 1. $a = 2$.

The γ-sets are two element subsets of $\{u_1, u_2, ..., u_b, v_1, v_2, ...v_b\}$ and sets of the form $\{w_1, v_i\}$ for $i = 1, 2, ..., k$. $\{\{u_1, v_1\}, \{u_2, v_2\}, ..., \{u_b, v_b\}\}$ corresponds to a collection of b independent vertices in $\gamma.G$. Since, $|V(G)| = 2b + 1$, the number of disjoint sets of cardinality 2, cannot exceed $\lfloor \frac{2b+1}{2} \rfloor = b$. Therefore, $\alpha(\gamma.G) = b$.

Case 2. $a > 2$.

The γ-sets are two element subsets of $\{u_1, u_2, ..., u_b, v_1, v_2, ...v_b\}$. Again, $\{\{u_1, v_1\}, \{u_2, v_2\}, ..., \{u_b, v_b\}\}$ corresponds to a collection of b independent vertices in $\gamma.G$ and the number of elements in the disjoint collection of two element subsets of a set of cardinality $2b$ cannot exceed b. Therefore, $\alpha(\gamma.G) = b$. □

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