AKCE J. Graphs. Combin., 7, No. 1 (2010), pp. 61-72

Girth, minimum degree and transversals of longest paths*

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Communicated by: S. Arumugam
Received 11 November 2008; accepted 27 April 2010

Abstract

We provide conditions on the minimum degree, the girth and the length of a longest path for a digraph to have the following property: *Every longest path intersects every maximal independent set.* Digraphs with this property satisfy, in a strong sense, the Laborde-Payan-Xuong conjecture (see [10]).

Keywords: Girth, minimum degree, longest path, transversal.

2010 Mathematics Subject Classification: 05C20

1. Introduction and Main Results

Let $D = (V(D), A(D))$ be a digraph consisting of a vertex set $V(D)$ and arc set $A(D) \subset V(D)\times V(D)$ (we will write $\overrightarrow{xy} \in A(D)$ whenever $x, y \in V(D)$ are such that $(x, y) \in A(D)$). Let $\lambda(D)$ be the number of vertices in a longest path. (A *path of length* $k \geq 1$ consists of a sequence of $k + 1$ distinct vertices $P = (x_0, \ldots, x_k)$ such that $\overrightarrow{x_{i-1}x_i} \in A(D)$ for all $i = 1, \ldots, k$ and if k is maximal, i.e. $k = \lambda(D) - 1$, then P is a *longest path.*) Recall that a subset of vertices $S \subset V(D)$ is *independent* if it possesses no adjacent vertices, and is *maximal* if it is contained properly in no independent set. In [10], Laborde, Payan and Xuong stated their conjecture: *For every loopless digraph G, there exists an independent set S such that* $\lambda(G - S) < \lambda(G)$ (in other words, they conjectured that there always exists an independent set intersecting every longest path, a *transversal* of longest paths). The conjecture is known to hold for many cases. For instance, they showed that in every symmetric digraph, there exists an independent set intersecting every longest path and with the property that each of its vertices is the origin of a longest path (they also conjectured that this holds for all digraphs). In [1] it is shown that there always exist hamiltonian cycles on strongly connected and locally in-semicomplete

*This work was partly supported by grant CONACyT-CB99888.

digraphs (this implies that every independent set is transversal of longest paths), and also that a locally in-semicomplete digraph has a hamiltonian path if and only if it contains a vertex that can be reached by all other vertices by a directed path, a result that constituted a sufficient condition for any independent set to be transversal of longest paths. Several sufficient conditions for a digraph to posses an independent transversal of longest paths are provided in [8]. In [5], Galeana-Sánchez presents sufficient conditions for a digraph to have the property that each of its induced subdigraphs has a maximal independent set transversal to its *non-augmentable* paths (longest paths are particular instances of non-augmentable paths), necessary and sufficient conditions for this property to hold when the digraph is asymmetrical, and necessary and sufficient conditions for any orientation of a graph to have this property. In [9], F. Havet proved that if a digraph has stability number at most two, then there exists a stable set that intersects every longest path. Here, the stability number is the cardinality of a largest stable set, i.e., the cardinality of a largest independent set. Generalizations of tournaments are considered in [6] not only for longest paths but also for non-augmentable paths and in [7] locally semi-complete and locally transitive digraphs are considered. (There exist generalizations to the conjecture, e.g. the *directed path partition conjecture* stated in [4] and which still remains unsolved, although it is known to hold for some cases, e.g. see [2] for generalizations of tournaments.)

In this paper we present wide classes of digraphs that satisfy the conjecture in a strong sense for the conclusion is that every maximal independent set is transversal of longest paths. To state the results, recall more notation and definitions. For each $x \in V(D)$, let $d^+(x), d^-(x), N^+(x)$ and $N^-(x)$ denote its out-degree, in-degree, out-neighborhood and in-neighborhood respectively. Let $\delta(D) = \min\{\min\{d^+(x), d^-(x)\} : x \in V(D)\}$. Let $\mathcal{C}(D)$ be the *girth* of D, the minimum length of a directed cycle. The following are our main results.

Theorem 1.1. *Let $D = (V(D), A(D))$ be a digraph with $\delta(D) \geq 2$ and let $k \geq 5$ be the length of a longest path in D. If*

$$k \leq \max\left\{\frac{3\mathcal{C}(D)}{2} + \delta(D) - 4, \mathcal{C}(D) + 2\delta(D) - 6\right\}, \tag{1.1}$$

then every maximal independent set intersects every longest path.

Theorem 1.2. *Let $D = (V(D), A(D))$ be a digraph with $\delta(D) \geq 2$ and $\mathcal{C}(D) \geq 3$; and let $k \geq 1$ be the length of a longest path in D. If*

$$k \leq 2\delta(D) + 1, \tag{1.2}$$

then every maximal independent set intersects every longest path.

The following corollary is an immediate consequence of Theorem 1.2 and corresponds to Theorem 44 in [6].

Corollary 1.3. *Let D be a digraph with $\delta(D) \geq 2$ and let $k \geq 1$ be the length of a longest path in D. If $\delta(D) \geq \frac{2}{3}(k+1)$, then every maximal independent set intersects every longest path.*

For the proofs we use several lemmas which are all compiled in section §2. In section §3 we present the proofs of Theorem 1.1 and Theorem 1.2. For general concepts we refer the reader to [3].

2. Lemmas

Let $D = (V(D), A(D))$ be a digraph. Let $P = (x_0, \ldots, x_k)$ be a path in D. If P is a longest path, then $N^-(x_0), N^+(x_k) \subseteq V(P)$. If $\delta(D) \geq 2$, then we record the first and last two indices in P of the elements in $N^-(x_0)$ by letting

$$\begin{aligned} m^-_{0,P} &= \min\{i : x_i \in N^-(x_0)\} \\ M^-_{0,P} &= \max\{i : x_i \in N^-(x_0)\} \\ \widehat{M}^-_{0,P} &= \max\{i < M^-_{0,P} : x_i \in N^-(x_0)\} \end{aligned}$$

and analogously for the last and first two indices in P of the elements of $N^+(x_k)$ with

$$\begin{aligned} M^+_{k,P} &= \max\{i : x_i \in N^+(x_k)\} \\ m^+_{k,P} &= \min\{i : x_i \in N^+(x_k)\} \\ \widehat{m}^+_{k,P} &= \min\{i > m^+_{k,P} : x_i \in N^+(x_k)\} \end{aligned}$$

(we will omit the subindex P if the path is clear in the context). Observe that if we only

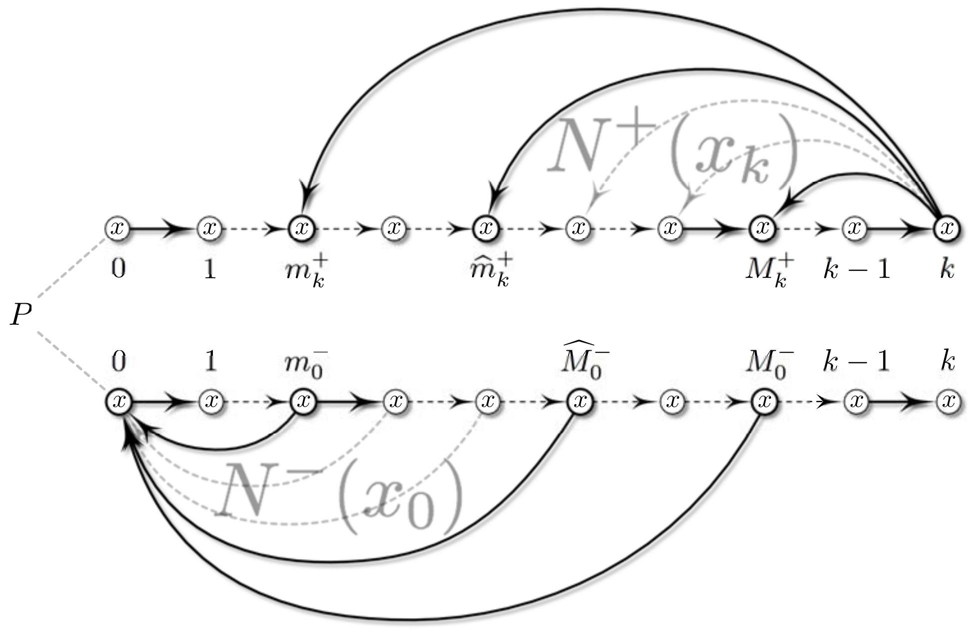

Figure 1: Distinguished elements of $N^-(x_0)$ and $N^+(x_k)$.

know $\delta(D) \geq 1$, then the quantities $\widehat{M}^-_{0,P}$ and $\widehat{m}^+_{k,P}$ are defined only if $d^-(x_0) \geq 2$ and

$d^+(x_k) \geq 2$ (see Remark 3.1). In Figure 1, the top represents the distinguished elements of $N^+(x_k)$ and those of $N^-(x_0)$ are represented on the bottom. In general, in words, there are many ways of "gluing together" these top and the bottom parts along P. Suitable conditions will yield us to a convenient form (see Figure 2), but to get there we still need some work. First observe that if there exists a longest path P not intersecting a maximal independent set, then, from Lemma 2.1 in [7], $M^-_{0,P} < k$ and $m^+_{k,P} > 0$. In fact, the proof of Theorem 1.1 will be carried out by contradiction so the existences of a longest path and a maximal independent set which do not intersect are assumed and this requires $\lambda(D) \geq 6$ (longest paths have length at least 5, see Figure 3).

Given two integers i, j such that $0 \leq i \leq j \leq k$, let

$$p_0[i,j] = |\{x_r \in N^-(x_0) : i \leq r \leq j\}|$$

and

$$p_k[i,j] = |\{x_r \in N^+(x_k) : i \leq r \leq j\}|.$$

Given two paths $P = (x_0, \ldots, x_r)$ and $Q = (y_0, \ldots, y_s)$ with $x_r = y_0$, let

$$P \bullet Q = (x_0, \ldots, x_r = y_0, \ldots, y_s).$$

Lemma 2.1. *Let D be a digraph with $\delta(D) \geq 2$ and let $P = (x_0, \ldots, x_k)$ be a longest path in D. Then $\widehat{M}^-_0 - \widehat{m}^+_k \geq 2(\mathcal{C}(D) + \delta(D) - 3) - k$.*

Proof. Just observe that $\widehat{M}^-_0 \geq \mathcal{C}(D) + \delta(D) - 3$ and $\widehat{m}^+_k \leq k - (\mathcal{C}(D) + \delta(D) - 3)$. $\square$

In what follows, let $D = (V(D), A(D))$ be a digraph (not necessarily $\delta(D) \geq 2$), let $P = (x_0, \ldots, x_k)$ be a longest path in D and suppose that there is a maximal independent set $\mathcal{I} \subseteq V(D)$ such that $V(P) \cap \mathcal{I} = \varnothing$.

Lemma 2.2. *Suppose that there exist $s > r$ such that $\overrightarrow{x_s x_0} \in A(D)$ and $\overrightarrow{x_k x_r} \in A(D)$. Then*

$$s - r \geq p_0[r, s-2] + p_k[r+2, s] + 1.$$

Proof. Suppose that for some $r \leq i \leq s-2$ we have $\overrightarrow{x_i x_0} \in A(D)$ and $\overrightarrow{x_k x_{i+2}} \in A(D)$. Then

$$P' = (x_{i+1}, \ldots, x_k, x_r, \ldots, x_i, x_0, \ldots, x_{r-1})$$

and

$$P'' = (x_{s+1}, \ldots, x_k, x_{i+2}, \ldots, x_s, x_0, \ldots, x_{i+1})$$

are both longest paths in D with $V(P') \cap \mathcal{I} = V(P'') \cap \mathcal{I} = \varnothing$. Thus, there is $y \in \mathcal{I}$ such that either $\overrightarrow{y x_{i+1}} \in A(D)$ or $\overrightarrow{x_{i+1} y} \in A(D)$ but then, either $\overrightarrow{y x_{i+1}} \bullet P'$ or $P'' \bullet \overrightarrow{x_{i+1} y}$ is a path in D of length $k+1$ which is a contradiction. We conclude that

$$\{x_{i+2} : x_i \in N^-(x_0) \ \&\ r \leq i \leq s-2\} \cap \{x_i \in N^+(x_k) : r+2 \leq i \leq s\} = \varnothing.$$

Since both $\{x_{i+2} : x_i \in N^-(x_0) \;\&\; r \leq i \leq s-2\}$ and $\{x_i \in N^+(x_k) : r+2 \leq i \leq s\}$ are subsets of $\{x_{r+2}, \ldots, x_s\}$, it follows that

$$\big|\{x_{i+2} : x_i \in N^-(x_0) \;\&\; r \leq i \leq s-2\}\big| + \big|\{x_i \in N^+(x_k) : r+2 \leq i \leq s\}\big|$$
$$\leq s-r-1.$$

Now just observe that $|\{x_{i+2} : x_i \in N^-(x_0) \;\&\; r \leq i \leq s-2\}| = p_0[r, s-2]$ and $|\{x_i \in N^+(x_k) : r+2 \leq i \leq s\}| = p_k[r+2, s]$. □

Lemma 2.3. *If $\delta(D) \geq 1$, then $M_0^- \leq k-2$ and $m_k^+ \geq 2$.*

Proof. Since $\mathcal{I}$ is maximal, there exists $y \in \mathcal{I}$ such that $\overrightarrow{yx_k} \in A(D)$ or $\overrightarrow{x_ky} \in A(D)$. Since P is a longest path, $\overrightarrow{yx_k} \in A(D)$. First suppose that $\overrightarrow{x_kx_0} \in A(D)$. Then $(y, x_k, x_0, \ldots, x_{k-1})$ is a path in D of length $k+1$ which is a contradiction. Now suppose that $\overrightarrow{x_{k-1}x_0} \in A(D)$. Since $\delta(D) \geq 1$, $\overrightarrow{x_kx_r} \in A(D)$ for some $r \leq k-1$. Hence $(y, x_k, x_r, \ldots, x_{k-1}, x_0, \ldots, x_{r-1})$ is a path in D of length $k+1$ which is a contradiction. Thus $\{x_k, x_{k-1}\} \cap N^-(x_0) = \varnothing$ and so $M_0^- \leq k-2$. Analogously we can see that $m_k^+ \geq 2$. □

Lemma 2.4. *Suppose that $\delta(D) \geq 2$ and $k \leq \mathcal{C}(D) + 2\delta(D) - 6$. Then $\widehat{M}_0^- - \widehat{m}_k^+ \geq 2\delta(D) - 5$.*

Proof. Since $k \leq \mathcal{C}(D) + 2\delta(D) - 6$, $k < 2(\mathcal{C}(D) + \delta(D) - 3)$. Then, by Lemma 2.1, we see that $\widehat{M}_0^- > \widehat{m}_k^+$. So, we use Lemma 2.2 to get

$$\widehat{M}_0^- - \widehat{m}_k^+ \geq p_0[\widehat{m}_k^+, \widehat{M}_0^- - 2] + p_k[\widehat{m}_k^+ + 2, \widehat{M}_0^-] + 1. \tag{2.1}$$

We are going to consider the following two cases.

Case 1. Suppose that $M_k^+ \leq \widehat{M}_0^-$. By definition, $N^+(x_k) \subseteq \{x_i : m_k^+ \leq i \leq M_k^+\}$. It follows that if $M_k^+ \leq \widehat{M}_0^-$, then $p_k[\widehat{m}_k^+ + 2, M_k^+] = p_k[\widehat{m}_k^+ + 2, \widehat{M}_0^-]$. Also observe that $|N^+(x_k) - \{x_i : \widehat{m}_k^+ + 2 \leq i \leq M_k^+\}| \leq 3$ and therefore $p_k[\widehat{m}_k^+ + 2, M_k^+] = p_k[\widehat{m}_k^+ + 2, \widehat{M}_0^-] \geq d^+(x_k) - 3 \geq \delta(D) - 3$. So, by (2.1),

$$\widehat{M}_0^- - \widehat{m}_k^+ \geq p_0[\widehat{m}_k^+, \widehat{M}_0^- - 2] + \delta(D) - 2. \tag{2.2}$$

Case 1.A. Suppose that $m_0^- \geq \widehat{m}_k^+$. We can argue similarly as above and get $p_0[\widehat{m}_k^+, \widehat{M}_0^- - 2] \geq \delta(D) - 3$. From (2.2) we have $\widehat{M}_0^- - \widehat{m}_k^+ \geq 2\delta(D) - 5$ and the result follows (see Figure 2).

Case 1.B. Suppose that $m_0^- < \widehat{m}_k^+$. This never occurs for in this case $\widehat{m}_k^+ \geq \mathcal{C}(D) + p_0[m_0^-, \widehat{m}_k^+ - 1] - 1$ and hence, by (2.2) (after adding $\widehat{m}_k^+$ in both sides of the inequality),

$$\widehat{M}_0^- \geq \mathcal{C}(D) + p_0[m_0^-, \widehat{m}_k^+ - 1] - 1 + p_0[\widehat{m}_k^+, \widehat{M}_0^- - 2] + \delta(D) - 2 =$$

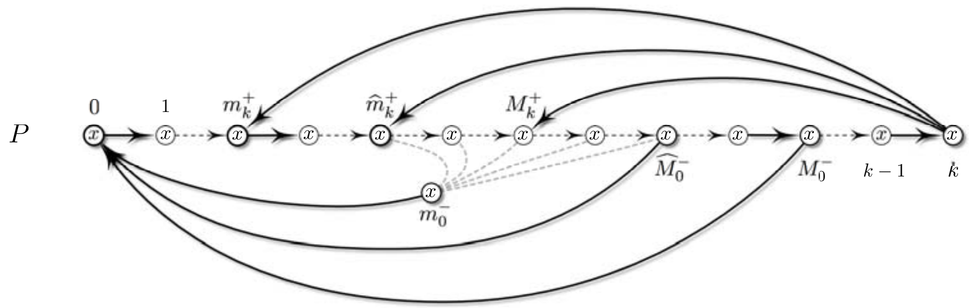

Figure 2: Location of distinguished elements of $N^+(x_k)$ and $N^-(x_0)$ under Case 1.A.

$$\mathcal{C}(D) + p_0[m_0^-, \widehat{M_0^-} - 2] + \delta(D) - 3.$$

As above, we can see that $p_0[m_0^-, \widehat{M_0^-} - 2] \geq \delta(D) - 3$ and therefore $\widehat{M_0^-} \geq \mathcal{C}(D) + 2\delta(D) - 6$, but we know from Lemma 2.3 that $k - 2 \geq M_0^- > \widehat{M_0^-}$ and thus $k - 2 > \mathcal{C}(D) + 2\delta(D) - 6$, a contradiction.

Case 2. Suppose that $M_k^+ > \widehat{M_0^-}$. This never occurs. Let us distinguish two possibilities.

Case 2.A. Suppose $m_0^- < \widehat{m}_k^+$. As above, this implies that $\widehat{m}_k^+ \geq \mathcal{C}(D) + p_0[m_0^-, \widehat{m}_k^+ - 1] - 1$, and since $M_k^+ > \widehat{M_0^-}$, it follows that $\widehat{M_0^-} \leq k - (\mathcal{C}(D) - 1 + p_k[\widehat{M_0^-} + 1, M_k^+])$. So, by (2.1) (after adding $\widehat{m}_k^+$ in both sides of the inequality),

$$k - (\mathcal{C}(D) - 1 + p_k[\widehat{M_0^-} + 1, M_k^+]) \geq$$

$$\mathcal{C}(D) + p_0[m_0^-, \widehat{m}_k^+ - 1] - 1 + p_0[\widehat{m}_k^+, \widehat{M_0^-} - 2] + p_k[\widehat{m}_k^+ + 2, \widehat{M_0^-}] + 1 =$$

$$\mathcal{C}(D) + p_0[m_0^-, \widehat{M_0^-} - 2] + p_k[\widehat{m}_k^+ + 2, \widehat{M_0^-}]$$

and therefore

$$k \geq \mathcal{C}(D) + p_0[m_0^-, \widehat{M_0^-} - 2] + p_k[\widehat{m}_k^+ + 2, \widehat{M_0^-}] + \mathcal{C}(D) - 1 + p_k[\widehat{M_0^-} + 1, M_k^+] =$$

$$2\mathcal{C}(D) + p_0[m_0^-, \widehat{M_0^-} - 2] + p_k[\widehat{m}_k^+ + 2, M_k^+] - 1,$$

but $p_0[m_0^-, \widehat{M_0^-} - 2] \geq \delta(D) - 3$ and $p_k[\widehat{m}_k^+ + 2, M_k^+] \geq \delta(D) - 3$, thus $k \geq 2\mathcal{C}(D) + 2\delta(D) - 7$ which is a contradiction.

Case 2.B. If $m_0^- \geq \widehat{m}_k^+$ we reach a contradiction in an analogous way as in the Case 1.B when $M_k^+ \leq \widehat{M_0^-}$ and $m_0^- < \widehat{m}_k^+$. The analogy allows us to let the reader verify this in detail. □

Observe from the proof that the statement of the lemma can be extended by noting that the conditions of Case 1.A are the ones actually holding.

3. Proofs of Main Results

Proof of Theorem 1.1. First, it is easy to see that condition (1.1) implies that $k < 2\mathcal{C}(D) + 2\delta(D) - 6$. Indeed, if $\frac{3\mathcal{C}(D)}{2} + \delta(D) - 4 \leq \mathcal{C}(D) + 2\delta(D) - 6$, then $k \leq \mathcal{C}(D) + 2\delta(D) - 6 < 2\mathcal{C}(D) + 2\delta(D) - 6$ otherwise, $k \leq \frac{3\mathcal{C}(D)}{2} + \delta(D) - 4 < 2\mathcal{C}(D) + 2\delta(D) - 6$, the last inequality holding because $\mathcal{C}(C) + 2\delta(D) > 4$. It follows from lemma 2.1 that for every longest path $P = (x_0, \dots, x_k)$ in D, we have $\widehat{M}^-_{0,P} > \widehat{m}^+_{k,P}$. Let $\mathcal{I} \subseteq V(D)$ be a maximal independent set and let $\mathcal{F}$ be the set of all longest paths P in D such that $V(P) \cap \mathcal{I} = \varnothing$. We are to show that assuming $\mathcal{F}$ nonempty yields a contradiction.

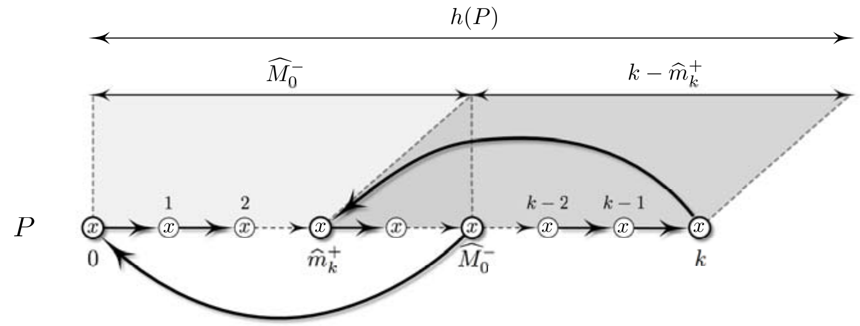

Figure 3: Choose $P \in \mathcal{F}$ so that the value $h(P)$ is minimal.

For each $P \in \mathcal{F}$, let $h(P) = \widehat{M}^-_{0,P} + k - \widehat{m}^+_{k,P}$. Choose P such that $h(P) = \min\{h(Q) : Q \in \mathcal{F}\}$ (see Figure 3).

Let $s = \widehat{M}^-_{0,P}$, $r = \widehat{m}^+_{k,P}$, $M^-_{0,P} = s + n$ with $n \geq 1$ and $m^+_{k,P} = r - m$ with $m \geq 1$. Since $s > r$ we see that

$$\begin{aligned} P' &= (x_{s+1}, \dots, x_{s+n}, \dots, x_k, x_r, \dots, x_s, x_0, \dots, x_{r-m}, \dots, x_{r-1}) \\ &= (y_0, \dots, y_k) \end{aligned}$$

is a longest path in D and is such that $V(P') \cap \mathcal{I} = \varnothing$. Therefore $P' \in \mathcal{F}$ and $h(P') \geq h(P)$. Observe that $y_0 = x_{s+1}$, $y_{k-s-1} = x_k$, $y_{k-s} = x_r$, $y_{k-r} = x_s$, $y_{k-r+1} = x_0$ and $y_k = x_{r-1}$. Let $p = k - m + 1$ and $q = n - 1$ so that $y_p = x_{r-m}$ and $y_q = x_{s+n}$ (see Figure 4, next page).

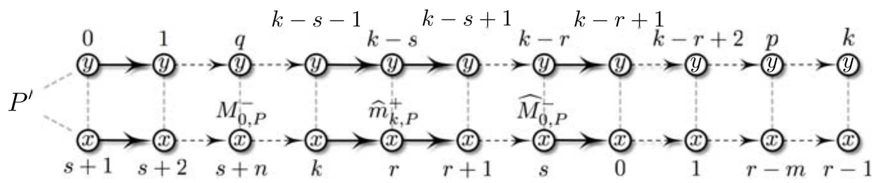

Figure 4: Correspondence of indices of P on P'.

It is not hard to see that $M^-_{0,P'} + k - m^+_{k,P'} \geq h(P') + 2$, and since $h(P') \geq h(P)$,

$$M^-_{0,P'} + k - m^+_{k,P'} \geq \widehat{M}^-_{0,P} + k - \widehat{m}^+_{k,P} + 2$$

and then, either $M^-_{0,P'} \geq k - \widehat{m}^+_{k,P} + 1$ or $k - m^+_{k,P'} \geq \widehat{M}^-_{0,P} + 1$.

Case 1. $M^-_{0,P'} \geq k - \widehat{m}^+_{k,P} + 1 = k - r + 1$.

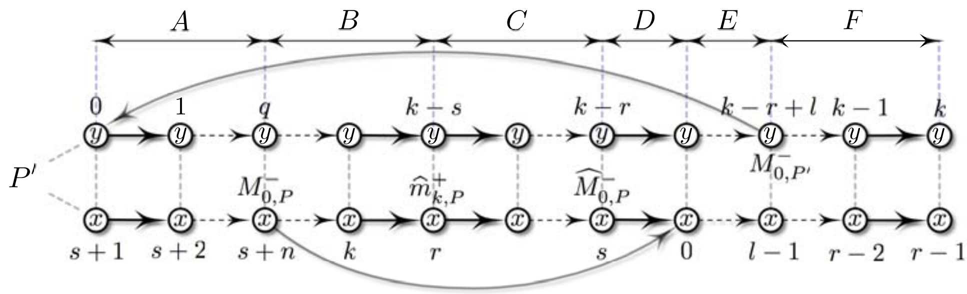

Figure 5: Partition of P' in Case 1.

Let $M^-_{0,P'} = k-r+l$ with $l \geq 1$ and consider the following set of edge-disjoint P'-subpaths (see Figure 5):

$$\begin{array}{lll} A = (y_0, \dots, y_q) & B = (y_q, \dots, y_{k-s}) & C = (y_{k-s}, \dots, y_{k-r}) \\ D = (y_{k-r}, y_{k-r+1}) & E = (y_{k-r+1}, \dots, y_{k-r+l}) & F = (y_{k-r+l}, \dots, y_k) \end{array}$$

First observe that since $A \bullet \overrightarrow{y_q y_{k-r+1}} \bullet E \bullet \overrightarrow{y_{k-r+l} y_0}$ is a directed cycle,

$$\ell(A) + \ell(E) + 2 \geq C(D). \tag{3.1}$$

Also, since $B = (y_q, \dots, y_{k-s}) = (x_{s+n}, \dots, x_k, x_r)$, it follows that $\ell(B) = k + 1 - M^-_{0,P}$ which, by Lemma 2.3, implies $\ell(B) \geq 3$. On the other hand, $\ell(F) = k - (k - r + l) = k - M^-_{0,P'}$ and then, again by Lemma 2.3, $\ell(F) \geq 2$. So

$$\ell(B) + \ell(D) + \ell(F) \geq 6 \tag{3.2}$$

because $\ell(D) = 1$. On the other hand,

$$\ell(A) + \ell(B) + \ell(C) = k - r \text{ and } \ell(C) + \ell(D) + \ell(E) + \ell(F) = s. \tag{3.3}$$

From (3.1) and (3.3) we see that

$$2\ell(A) + \ell(B) + 2\ell(C) + \ell(D) + 2\ell(E) + \ell(F) + 2 \geq C(D) + k - r + s$$

and since $\ell(A) + \ell(B) + \ell(C) + \ell(D) + \ell(E) + \ell(F) = k$,

$$2k - \ell(B) - \ell(D) - \ell(F) + 2 \geq C(D) + k - r + s.$$

Then

$$k \geq \mathcal{C}(D) + \ell(B) + \ell(D) + \ell(F) + s - r - 2$$

which by (3.2) implies that

$$k \geq \mathcal{C}(D) + 4 + s - r = \mathcal{C}(D) + 4 + \widehat{M}^-_{0,P} - \widehat{m}^+_{k,P}.$$

Therefore, by Lemma 2.1, we have

$$k \geq \mathcal{C}(D) + 4 + 2(\mathcal{C}(D) + \delta(D) - 3) - k$$

and thus

$$2k \geq 3\mathcal{C}(D) + 2\delta(D) - 2.$$

Since $k \leq \max\left\{\frac{3\mathcal{C}(D)}{2} + \delta(D) - 4, \mathcal{C}(D) + 2\delta(D) - 6\right\}$, it follows that $k \leq \mathcal{C}(D) + 2\delta(D) - 6$. Then Lemma 2.4 implies

$$k \geq \mathcal{C}(D) + 4 + 2\delta(D) - 6 \geq \mathcal{C}(D) + 2\delta(D) - 1,$$

a contradiction.

Case 2. $k - m^+_{k,P'} \geq \widehat{M}^-_{0,P} + 1 = s + 1$.

We reach a contradiction for this case in an analogous way as in Case 1. Just let $k-s-l = m^+_{k,P'}$ with $l \geq 1$ and consider the partition

$$\begin{array}{lll} A = (y_0, \ldots, y_{k-s-l}) & B = (y_{k-s-l}, \ldots, y_{k-s-1}) & C = (y_{k-s-1}, y_{k-s}) \\ D = (y_{k-s}, y_{k-r}) & E = (y_{k-r}, \ldots, y_p) & F = (y_p, \ldots, y_k) \end{array}$$

Here $\ell(D)+\ell(E)+\ell(F) = s$, $\ell(A)+\ell(B)+\ell(C)+\ell(D) = k-r$ and $F \bullet \overrightarrow{y_k y_{k-s+l}} \bullet B \bullet \overrightarrow{y_{k-s-1} y_p}$ is a directed cycle (see Figure 6).

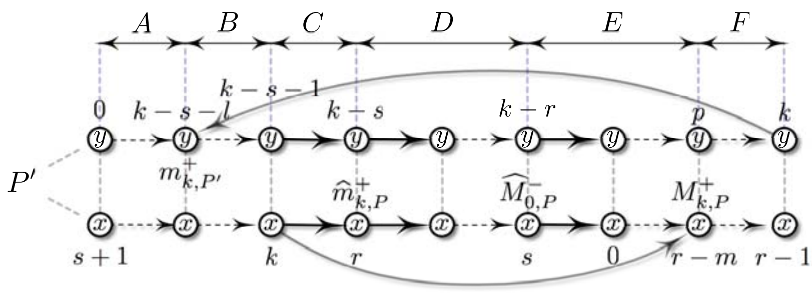

Figure 6: Partition of P' in Case 2.

□

Remark 3.1. *The assumption $\delta(D) \geq 2$ in the Theorem 1.1 can be weakened by only asking $d^-(x) \geq 2$ for all $x \in V(D)$ for which there exists a longest path starting at x and $d^+(y) \geq 2$ for all $y \in V(D)$ for which there exists a longest path ending at y.*

Proof of Theorem 1.2. Let $P = (x_0, \ldots, x_k)$ be a longest path in D and suppose that there is a maximal independent set $\mathcal{I}$ which does not intersect P. We distinguish two cases.

Case 1. There is no $j \leq k$ such that $x_j \in N^-(x_0)$ and $x_{j+1} \in N^+(x_k)$.

Since, by Lemma 2.3, $M_0^- \leq k-2$ and $m^+(x_k) \geq 2$ it follows that

$$N^-(x_0) \subseteq \{x_2, \ldots, x_{k-2}\}$$

and

$$N^+(x_k) \subseteq \{x_2, \ldots, x_{k-2}\}.$$

Let $A = \{x_{i+1} : x_i \in N^-(x_0)\} \subseteq \{x_3, \ldots, x_{k-1}\}$. Since there is no $j \leq k$ such that $x_j \in N^-(x_0)$ and $x_{j+1} \in N^+(x_k)$, then $A \cap N^+(x_k) = \varnothing$ and thus,

$$|A - \{x_{k-1}\}| + |N^+(x_k) - \{x_2\}| \leq k-4.$$

Since $|N^+(x_k) - \{x_2\}| \geq \delta(D) - |N^+(x_k) \cap \{x_2\}|$ and $|A - \{x_{k-1}\}| = d^-(x_0) - |N^-(x_0) \cap \{x_{k-2}\}| \geq \delta(D) - |N^-(x_0) \cap \{x_{k-2}\}|$ it follows that

$$k - 4 \geq 2\delta(D) - |N^-(x_0) \cap \{x_{k-2}\}| - |N^+(x_k) \cap \{x_2\}| \geq 2\delta(D) - 2$$

which is not possible since $k \leq 2\delta(D) + 1$.

Case 2. There is $j \leq k$ such that $x_j \in N^-(x_0)$ and $x_{j+1} \in N^+(x_k)$.

There is some $z \in \mathcal{I}$ such that either $\overrightarrow{zx_0} \in A(D)$ or $\overrightarrow{x_0z} \in A(D)$ because $\mathcal{I}$ is maximal. Since P is a longest path, $\overrightarrow{x_0z} \in A(D)$. In the same way we see that there is some $z' \in \mathcal{I}$ such that $\overrightarrow{z'x_k} \in A(D)$.

Let $P_0 = (y_0, \ldots, y_t)$ be a path in D such that $V(P) \cap V(P_0) = y_0 = x_0$ and t maximum. In the same way, let $P_k = (w_0, w_1, \ldots, w_q)$ be a path in D such that $V(P) \cap V(P_k) = x_k = w_q$ and q maximum.

Claim 1. $N^+(y_t) \cap \{x_0, \ldots, x_t\} = \varnothing$ and $N^-(w_0) \cap \{x_{k_q}, \ldots, x_k\} = \varnothing$.

From the proof of Claim 1 we observe that if $x_j \in N^+(y_t) \cap \{x_0, \ldots, x_t\}$, with $j \leq t-1$, then the length of the path $P_0 \bullet (x_j, x_{j+1}, \ldots, x_k)$ is greater than the length of P, a contradiction. To prove $N^-(w_0) \cap \{x_{k_q}, \ldots, x_k\} = \varnothing$ is analogous.

Now, let $r = \min\{i : x_i \in N^-(x_0) \mathbin{\&} x_{i+1} \in N^+(x_k)\}$ and $s = \max\{i : x_i \in N^-(x_0) \mathbin{\&} x_{i+1} \in N^+(x_k)\}$ (r and s are not necessarily different).

Claim 2. $N^+(y_t) \cap \{x_{r+1}, \ldots, x_k\} = \varnothing$ and $N^-(w_0) \cap \{x_0, \ldots, x_s\} = \varnothing$.

Suppose that $x_j \in N^+(y_t) \cap \{x_{r+1}, \ldots, x_k\}$. Then the length of the path

$$(x_1, \ldots, x_r, x_0 = y_0, y_1, \ldots, y_t, x_j, \ldots, x_k, x_{r+1}, \ldots, x_{j-1})$$

is greater than the length of P, a contradiction. In an analogous way we can show that $N^-(w_0) \cap \{x_0, \ldots, x_s\} = \varnothing$.

Claim 3. $\max\{t, q\} \leq \delta(D)$.

Let $P' = (x_1, \ldots, x_{M^-_{0,P}}, x_0 = y_0, \ldots, y_t)$. Since $M^-_{0,P} \geq \mathcal{C}(D) + \delta(D) - 2 \geq \delta(D) + 1$, it follows that $t + 1 + \delta(D) + 1 \leq k + 1 \leq 2\delta(D) + 2$. In an analogous way we see that $q \leq \delta(D)$.

Claim 4. $|V(P) \cap N^+(y_t)| \geq \min\{\delta(D), \delta(D) - t + 2\} \geq 2$ and $|V(P) \cap N^-(w_0)| \geq \min\{\delta(D), \delta(D) - q + 2\} \geq 2$.

Since t is maximum, it follows that $N^+(y_t) \subseteq V(P_0) \cup V(P)$, and since $y_0 = x_0, y_t, y_{t-1} \notin N^+(y_t)$, then $|N^+(y_t) \cap V(P_0)| \leq t - 2$. Then, from Claim 3, we see that $|V(P) \cap N^+(y_t)| \geq \min\{\delta(D), \delta(D) - t + 2\} \geq 2$. To prove $|V(P) \cap N^-(w_0)| \geq \min\{\delta(D), \delta(D) - q + 2\} \geq 2$ is analogous.

From Claims 1 and 2 we see that $|V(P) \cap N^+(y_t)| \leq r + 1 - (t + 1) = r - t$ and $|V(P) \cap N^-(w_0)| \leq k - q - (s + 1)$, which, from Claim 4, implies that $r - t \geq \min\{\delta(D), \delta(D) - t + 2\}$ and $k - q - (s + 1) \geq \min\{\delta(D), \delta(D) - q + 2\}$. So $r \geq \min\{\delta(D), \delta(D) - t + 2\} + t$ and $k \geq \min\{\delta(D), \delta(D) - q + 2\} + q + s + 1$, and since $s \geq r$ it follows that

$$k \geq \min\{\delta(D), \delta(D) - q + 2\} + q + 1 + \min\{\delta(D), \delta(D) - t + 2\} + t$$

which implies that $k \geq 2\delta(D) + 3$, a contradiction. □

Acknowledgements

We sincerely thank the anonymous referee for reading the paper and making suggestions to improve this paper.

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