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On Determination and Construction of Critically 2-Connected Graphs*

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Abstract

Kriesell proved that every almost critical graph of connectivity 2 nonisomorphic to a cycle has at least 2 removable ears of length greater than 2. We improve this lower bound on the number of removable ears. A necessary condition for critically 2-connected graphs in terms of a forbidden minor is obtained. Further, we investigate properties of a special class of critically 2-connected series-parallel graphs.

Keywords: almost critical, critically 2-connected, series-parallel graphs, removable ear, $\mathcal{G}$-series.

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1. Introduction

All graphs considered here are simple. Let G be a graph and B be a block of G. An *internal* vertex of B is a vertex of B which is not a cut-vertex of G. We shall say that B is an *end block* of G if B contains exactly one cut-vertex of G. Given $U \subseteq V(G)$, we use $N(U)$ to denote the set of vertices of $V(G) \setminus U$ adjacent with a vertex of U, and $G[U]$ to denote the subgraph of G induced by U. For a path P in G, put $|P| = |E(P)|$. We refer [17] for terms not defined here.

An *ear* of a graph G is a maximal nonempty path whose internal vertices have degree 2 in G. Let H be a subgraph of G. The graph induced by $E(G) \setminus E(H)$ is denoted by $G - H$. (i.e. $E(G-H) = E(G) \setminus E(H)$ and $V(G-H) = \{v \mid v$ is incident with an edge in $E(G) \setminus E(H)\}$). Let G be a 2-connected graph. A vertex u is *removable* if $G - u$ is 2-connected, and a nonempty proper subset U of $V(G)$ is *removable* if $G - U$ is 2-connected. An ear P of G is *removable* if $G - P$ is 2-connected.

A 2-connected graph G is *critically (minimally)* 2-connected if $G - v$ is not 2-connected for each vertex (edge) v of G. An *almost critical* graph of connectivity 2 is a

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noncomplete graph G such that given any 2-vertex cut Q of G, each component of $G - Q$ contains a vertex which is not removable in G. Obviously, every critically 2-connected graph nonismorphic to a triangle is almost critical.

Dirac [3] and Plummer [12], independently, proved that a minimally 2-connected graph has a vertex of degree two. Chartrand, Kaugars and Lick [2] extended this result to n-connected graphs. Results of [2] are generalized by Nebeský [11] to the class of almost critical graphs of connectivity 2. Kriesell [7], [8] improved a result of Nebeský as follows.

Theorem 1.1. *Every almost critical graph of connectivity* 2 *nonisomorphic to a cycle has at least two removable ears of length at least* 3.

Kriesell used the above result to provide a partial solution to the following conjecture of McCuaig and Ota [10].

Conjecture 1.2. *For any given natural number k there exists an $f(k)$ such that any* 3-*connected graph G on at least $f(k)$ vertices contains a connected subgraph H on k vertices such that $G - V(H)$ is* 2-*connected.*

We obtain the following strengthening of Theorem 1.1.

Theorem 1.3. *Let G be an almost critical graph of connectivity* 2 *nonisomorphic to a cycle and let v be a vertex of G. Then the number of removable ears of G of length at least* 3 *is greater than or equal to the number of end blocks of $G - v$.*

It follows immediately from the above theorem that the number of removable ears of length at least 3 of a critically 2-connected graph G nonisomorphic to a cycle is greater than or equal to the number of end blocks of $G - v$ for each vertex v of G.

The following theorem characterizes minimal removable sets of vertices of a 2-connected graph.

Theorem 1.4. *Let G be a* 2-*connected graph and let $U \subset V(G)$ with $|U| \geq 2$. Then U is a minimal removable set of G if and only if U is the set of internal vertices of a removable ear of G.*

It follows from Theorem 1.4 that every minimal removable set of vertices of any critically 2-connected graph is the set of internal vertices of a removable ear. Using this fact, a special case of Theorem 1.1 for critically 2-connected graphs can easily be derived.

If every vertex of a 2-connected graph G has a neighbour of degree 2, then, obviously, G is critically 2-connected. Let $\mathcal{G}'$ be the class of critically 2-connected graphs each of which has a vertex v such that $d(v) > 2$ and $d(x) > 2$ for all $x \in N(v)$. We find a member of $\mathcal{G}'$ which is a minor of all members of $\mathcal{G}'$.

A graph is *series-parallel* if it can be created from K_2 by repeatedly duplicating and subdividing edges. Series-parallel graphs are very well studied in connection with algorithmic graph theory (*see* [4],[5],[6],[9],[13],[14],[16]). We investigate properties of a special class of critically 2-connected series-parallel graphs.

2. Removable Ears

Lemma 2.1. *Let G be an almost critical graph of connectivity* 2 *and v be a vertex of G. Let B be a* 2*-connected block of $G-v$ containing exactly one cut-vertex of $G-v$. Then B contains a removable ear of G of length at least* 3.

Proof. Let $x \in V(B)$ be the cut-vertex of $G-v$. Since G is 2-connected, an internal vertex y of B is a neighbour of v. Let G_1 be the graph obtained from the graph $G[V(B) \cup \{v\}]$ by adding two new vertices x_1, x_2 and three new edges vx_1, x_1x_2 and x_2x. Certainly, G_1 is 2-connected. Let L be the ear of G_1 containing x, x_1, x_2 and v. Since B is 2-connected, $d_{G_1}(x) \geq 3$. Hence x is an end vertex of L. The other end vertex of L is v if $d_{G_1}(v) \geq 3$, otherwise it is y.

We claim that G_1 is almost critical. Obviously, the connectivity of G_1 is 2. Let Q be a 2-vertex cut of G_1 and let D be a component of $G_1 - Q$. It suffices to prove that D contains a vertex which is not removable in G_1. If D contains a vertex of L, then it is not removable G_1 because it has a neighbour of degree two in L. Suppose that D is vertex disjoint from L. Assume that Q contains an internal vertex of L. Then $Q \subset V(L)$. It follows that $G_1 - Q$ has exactly two components each containing a vertex of L. Therefore D contains a vertex of L, which is a contradiction. Thus $Q \subset V(B) \cup \{v\}$. Suppose that $v \in Q$. Then $d_{G_1}(v) \geq 3$ and $Q = \{v, x\}$. Therefore $D = B - x$. If $v \notin Q$, then $Q \subset V(B)$ and hence D is a component of $B - Q$. In any case, D is a component of $G - Q$. Since G almost critical, D contains a vertex u which is not removable in G. It follows that u is not removable in G_1. Thus the claim is satisfied. Since G_1 is not isomorphic to a cycle, by Theorem 1.1, G_1 contains a removable ear $L' \neq L$ with $|L'| \geq 3$. It follows that L' is contained in B and it is a removable ear of G. □

Proof of Theorem 1.3. If $G - v$ has at most two end blocks, then the result follows from Theorem 1.1. Suppose that $G - v$ has at least three end blocks. Hence $d_G(v) \geq 3$. Every end block of $G - v$ contains a neighbour of v as an internal vertex. We prove that there is a one-one correspondence from the set of end blocks of $G - v$ to the set of removable ears of G of length at least 3. Let B be an end block of $G - v$. If B has at least three vertices, then it is 2-connected and hence, by Lemma 2.1, it contains a removable ear of G of length at least 3. Assign this ear to B. Suppose that B is isomorphic to K_2. Let x be the internal vertex of B. Then x is adjacent with v and $d_G(x) = 2$. Let L_1 be the ear of G containing x. Since $d_G(v) \geq 3$, v is an end vertex of L_1. Let z be the other end vertex of L_1. If L_1 is removable, then $|L_1| \geq 3$ because G is almost critical. In this case assign L_1 to B.

Assume that $G - L_1$ is not 2-connected. Then it has exactly two end blocks and each end block of it contains either z or v as an internal vertex. Let B_1 be the end block of $G - L_1$ containing z. Let z_1 be the cut-vertex of $G - L_1$ that is contained in B_1. Obviously, v has no neighbour in $V(B_1) \setminus \{z_1\}$.

We claim that B_1 contains a removable ear L of G with $|L| \geq 3$. Let G' be the graph obtained from G be contracting all but one internal vertices of L_1. Let L_1' be the ear of G' corresponding to L_1. Then $|L_1'| = 2$. Let v' be the only internal vertex of L_1'. Clearly, G' is has connectivity 2. Since G is almost critical, G' is almost critical. Further, B_1 is a 2-connected end block of $G' - v'$. By Lemma 2.1, B_1 contains an ear L with $|L| \geq 3$ which is removable in G'. Obviously, L is removable in G also.

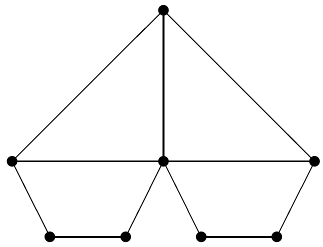
Assign L to the end block B of $G - v$. It suffices to prove that L does not correspond to any end block of $G - v$ other than B.

Assume that $B' \neq B$ is an end block of $G - v$ that corresponds to L. Then, by Lemma 2.1, B' is isomorphic to K_2. Let y be the internal vertex of B'. Then $d_G(y) = 2$ and y is adjacent with v. Let L_2 be the ear of G containing y. Certainly, L_2 is edge-disjoint from L_1 as well as L. Obviously, v is end vertex of L_2. Replacing the roles of B and L_1 by B' and L_2 respectively in the above arguments, we see that L_2 is not removable and B_1 is an end block of it. As L_2 is contained in $G - L_1$ and z_1 is a cut-vertex that is contained in the end block B_1 of $G - L_1$, z_1 is an end vertex of L_2. Further, each end block of $G - L_2$ contains z_1 or v as an internal vertex. It follows that z is a cut-vertex of $G - L_2$. This shows that v is not adjacent with z_1 and hence v has no neighbour in B_1. Since the end block of $G - L_1$ containing v is 2-connected, we c an find a v, z_1-path P in $G - L_1$ that avoids y. Hence $V(P) \cap V(L_2) = \{v, z_1\}$. Therefore P is a path in $G - L_2$. As v and z_1 are internal vertices of end blocks of $G - L_2$, every v, z_1-path must contain all cut-vertices of $G - L_2$. Hence $z \in V(P)$. Since z_1 a cut-vertex of $G - L_1$ and $v \notin V(B_1)$, $V(P) \cap V(B_1) = \{z_1\}$. This is a contradiction. □

Proof of Theorem 1.4. Suppose that U is a minimal removable set of G. Let $U = \{u_1, u_2, \cdots, u_n\}$ and let $H = G - U$. Then H is 2-connected. If u_i, for some i, has more than one neighbours in H, then $G[V(H) \cup \{u_i\}]$ is 2-connected, which contradicts the minimality of U. Hence every vertex of U has at most one neighbour in H. Let $z \in V(H)$ and $u_i, u_j \in U$ with $i \neq j$. Since G is 2-connected, there are internally vertex-disjoint paths from z to u_i and z to u_j in G. On each of these paths, there is an edge having one end vertex in H and the other in U. Without loss of generality, we assume that these edges are u_1x and u_ny, where $x, y \in V(H)$. As G is 2-connected, there is a u_1, u_n-path, say P, not containing y. Suppose that P contains a vertex v of H. Then $G[V(H) \cup V(v, u_n\text{-subpath of } P)]$ is 2-connected, which is a contradiction to the minimality of U. Hence $V(P) \subseteq U$. Further, $G[V(H) \cup V(P)]$ is 2-connected. By minimality, $V(P) = U$. Therefore P is contained in $G[U]$. If $G[U] \neq P$, then we can find a u_1, u_n-path P' in $G[U]$ such that $|P'| < |P|$. Since G does not have a cut-vertex, $G[V(H) \cup V(P')]$ is 2-connected. This is a contradiction to the minimality of U. Thus $G[U] = P$. If an internal vertex of P, say u_i, has a neighbour in H, then $G[V(H) \cup V(u_1, u_i\text{-subpath of } P)]$ or $G[V(H) \cup V(u_i, u_n\text{-subpath of } P)]$ is 2-connected, which is a contradiction. Thus every internal vertex of P has degree 2. It is easy to see that $d_G(u_1) = d_G(u_n) = 2$ and $d_G(x), d_G(y) \geq 3$. Let $L = xu_1 + P + u_ny$. Then L is an ear of G such that U is the set of internal vertices of L. As $G - U$ is 2-connected, L is removable.

Converse is obvious. □

Theorem 2.2. *Let $\mathcal{G}'$ be the class of critically 2-connected graphs each of which has a vertex v such that $d(v) > 2$ and $d(x) > 2$ for all $x \in N(v)$. Let $G \in \mathcal{G}'$. Then the graph H of Figure 1 is a minor of G. Further, H is a member of $\mathcal{G}'$.*



H
Figure 1

Proof. Obviously, H is a member of $\mathcal{G}'$. We prove that H is a minor of G. Let v be a vertex of G such that $d(v) > 2$ and $d(x) > 2$ for all $x \in N(v)$. Let B_1, B_2 be two end blocks of $G - v$. Let $i \in \{1, 2\}$. An internal vertex v_i of B_i is adjacent with v. Certainly, B_i is 2-connected. Hence, by Lemma 2.1, B_i contains a removable ear L_i of G with $|L_i| \geq 3$. Let x_i be the cut-vertex of $G - v$ contained in B_i. Let $G' = G[V(B_1) \cup V(B_2) \cup \{v\}]$. Now, obtain the graph G'' from G' as follows: (i) if $x_1 \neq x_2$, and x_1 and x_2 are not adjacent in G', then add the edge x_1x_2 to G'; (ii) if v is neither adjacent with x_1 nor adjacent with x_2 and, if it is adjacent with a vertex of $V(G) \setminus V(G')$, then add the edge vx_1 to G'. Thus $V(G') = V(G'')$, $E(G') \subseteq E(G'')$. It is easy to see that G'' is a minor of G. Now obtain the minor G_1 of G'' as follows: (i) If x_1x_2 is an edge in G'', then contract it and denote by z the corresponding new vertex; (ii) If $x_1 = x_2$, then put $z = x_1$. Certainly, G_1 is a minor of G. It follows that G_1 is 2-connected. Further, B_1 and B_2 are the only blocks and z is the only cut-vertex of $G_1 - v$. Note that L_i is a removable ear of G_1.

Case (i) z and v are adjacent in G_1.

Delete all edges of G_1 incident at v except the edges vv_1, vv_2, vz. Let x be an internal vertex of the ear L_1. Since B_1 is 2-connected, there is a z, v_1-path, say P, in B_1 not containing x and hence not containing any internal vertex of L_1. Further, there are two internally vertex-disjoint paths from x to z and x to v_1, say P_1 and P_2, respectively. Traverse P_i beginning at x let z_i be the first vertex of P_i common to P for $i = 1, 2$. Then (x, z_1-subpath of P_1) $\cup$ (z_1, z_2-subpath of P) $\cup$ (z_2, x-subpath of P) is a cycle containing all vertices of L_1. Assume that z_1 is nearer to z than z_2 on P. By contraction, identify the z, z_1-subpath of P to z, and the z_2, v_1-subpath of P to v_1 so that there is a cycle containing z, v_1 and all vertices of L_1. Now reduce this cycle to a 4-cycle C_1 containing z, v_1 and exactly two internal vertices of L_1. By contraction and deletion, eliminate all the remaining edges and vertices so that B_1 get reduced to C_1 only. Similarly, B_2 can be reduced to a 4-cycle C_2 containing z, v_2 and only two internal vertices of L_2. Thus the graph H of Figure 1 is a minor of G_1 and hence it is a minor of G.

Case (ii) z and v are not adjacent in G_1.

Then from the construction of G_1, it is easy to see that all neighbours of v are internal vertices of B_1 and B_2 in G_1 as well as in G. Hence $d_{G_1}(v) = d_G(v) \geq 3$. By deleting all but three edges incident on v and by relabelling, if necessary, we may assume that exactly two internal vertices v_1, v_1' of B_1 are adjacent with v and exactly one internal vertex v_2 of B_2 is adjacent with v.

As in Case (i), reduce B_2 to a 4-cycle containing z, v_2 and two internal vertices of L_2. We will identify either v_1 or v_1' to z, and reduce B_1 to a 4-cycle containing z, two internal vertices of L_1 and exactly one of v_1, v_1' which is not identified with z.

Since B_1 is 2-connected, there are internally vertex-disjoint paths from z to v_1 and z to v_1'. Combining these paths we get a v_1, v_1'-path, say P.

Subcase (i) P contains an internal vertex of L_1.

Since all internal vertices of L_1 are of degree two, L_1 is a subpath of P contained in either the v_1, z-subpath of P or the v_1', z-subpath of P. Assume that L is a subpath of the v_1, z-subpath of P. Since B_1 is 2-connected, there is a cycle containing z and an internal vertex of L and hence all vertices L. Let z_1 be the first vertex of P from v_1 which is on this cycle. Then z_1 is on the v_1, z-subpath of P. Contracting the v_1, z_1-subpath and the v_1', z-subpath of P, identify z_1 to v_1, and v_1' to z. Therefore we get a cycle containing v_1, z and all internal vertices of L_1. Since v is adjacent with v_1', v is adjacent with z after identifying v_1' to z. Thus, as in Case (i), we can reduce B_1 to a 4-cycle containing z, v_1 and two internal vertices of L_1. This shows that the graph H of Figure 1 is a minor of G.

Subcase (ii) P contains no internal vertex of L_1.

Let x be an internal vertex of L_1. There are internally vertex-disjoint paths from x to v_1, and x to v_1', say P_1 and P_2, respectively. Let z_i be the first vertex of P_i common to P. There is a cycle containing the z_1, z_2-subpath of P, the z_2, x-subpath of P_2 and the x, z_1-subpath of P_1. We may assume that z_1 is nearer to v_1 than z_2 on P. Identifying z_1 to v_1, and z_2 to v_1' by contracting edges of P, we get a cycle containing v_1, v_1' and all internal vertices of L_1. If z is not identified with v_1 or v_1', then identify it by contraction with one of them so that v is adjacent with z. Thus, in any case, there is a cycle containing z, all internal vertices of L_1, a neighbour of v different from z. It follows that we can reduce B_1 to a 4-cycle containing z, a neighbour of v, two internal vertices of L_1. Hence the graph H of Figure 1 is a minor of G. □

3. Critically 2-Connected Series-Parallel Graphs

Whitney [18] proved that every 2-connected graph has an ear decomposition $E_0, E_1, \cdots, E_n$, where E_0 is a cycle and E_i is an ear of $E_0 \cup E_1 \cup \cdots \cup E_i$. We say that an ear E_i is *nested* in E_j if $j < i$ and both end vertices of E_i are in E_j. A subpath of E_j between the end vertices of E_i is *nested interval* of E_i in E_j. Clearly, E_i has unique nested interval in E_j if E_j is an ear, and has two nested intervals if E_j is a cycle. The following lemma follows from Theorem 1 of [4].

Lemma 3.1. *Every 2-connected series-parallel graph G has an ear decomposition $E_0, E_1, \cdots, E_n$ satisfying*

(i) for $i > 1$, there is some $j < i$ such that E_i is nested in E_j;

(ii) if two ears E_i and Ei' are both nested in the same E_j, then either the nest interval of E_i contains that of E'_i, or vice versa, or the two nest intervals are disjoint; i.e., no two nest intervals in each ear E_j cross each other.

An ear decomposition of a graph G is *nested* if it satisfies the conditions given in Lemma 3.1.

Lemma 3.2. *Let G be a 2-connected series-parallel graph. If no vertex of G of degree two is removable, then G is critically 2-connected.*

Proof. Let v be a vertex of G of degree at least 3. It suffices to prove that $G - v$ is not 2-connected. By Lemma 3.1, G has a nested ear decomposition $E_0, E_1, \cdots, E_n$. It is easy to see that v is an end vertex of an ear. Let i be the smallest positive integer such that v is an end vertex of E_i. Let v' be the other end vertex of E_i. Since G is simple, E_i or each nest interval of E_i in E_j has an internal vertex. It follows that $\{v, v'\}$ is a 2-vertex cut of G. Hence $G - v$ is not 2-connected. □

Corollary 3.3. *Every 2-connected series-parallel graph in which every vertex of degree two has a neighbour of degree two is critically 2-connected.*

The graph of Figure 2 is critically 2-connected series-parallel graph which has a vertex of degree 2 without a neighbour of degree 2. This graph belongs to the class $\mathcal{C}$ of graphs

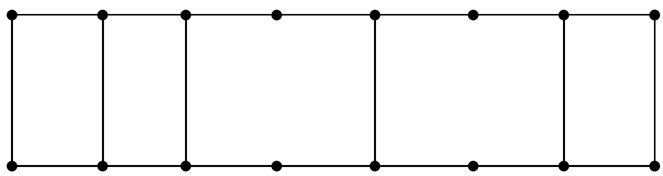


Figure 2

that is defined as follows.

Definition 3.4. *Let $\mathcal{C}$ be the class of series-parallel graphs each of which has a nested ear decomposition $E_0, E_1, \cdots, E_n$ satisfying*

(i) $|E_0| \geq 4$;

(ii) $|E_i| \neq 2$ for each i;

(iii) the length of each nested interval is different from 2.

We prove some interesting properties of this class.

Theorem 3.5. *If $G \in \mathcal{C}$, then G is critically 2-connected.*

Proof. By Theorem 3.2, it is enough to prove that no vertex of degree 2 in G is removable. Let v be a vertex of degree 2 in G. If v has a neighbour of degree 2, then v is not removable. Suppose that v_1 and v_2 are neighbours of v each having degree at least 3. Let $E_0, E_1, \cdots, E_n$ be an ear decomposition of G satisfying the conditions of Definition 3.4. Then the edges vv_1 and vv_2 belong to exactly one E_i. As degrees of v_1 and v_2 are at least 3, each is an end vertex of an ear. Further, they are not end vertices of the same ear because $|E_j| \neq 2$ for all j. If $E_i = E_0$, then a vertex of the cycle E_0 other than v_1 and v_2 is a cut-vertex of $G - v$. Suppose $E_i \neq E_0$. It follows that an end vertex of the ear E_i is a cut-vertex of $G - v$. Hence v is not removable. □

The following theorem gives a nested ear decomposition for any graph $G \in \mathcal{C}$ which starts from any given cycle of G. The proof of the theorem is routine and hence is omitted. The details of the proof can be found in [1].

Theorem 3.6. *Let $G \in \mathcal{C}$. Let C_0 be any cycle of G of length at least 4. Then G has a nested ear decomposition $E_0, E_1, \cdots, E_n$ satisfying the conditions of Definition 3.4 with $E_0 = C_0$.*

Let $\mathcal{G}$ denote the class of all simple 2-connected graphs.

Let $G \in \mathcal{G}$. Suppose that $E_0, E_1, \cdots, E_n$ is an ear decomposition of G. Let $X_i = E_0 \cup E_1 \cup \cdots \cup E_i$. Then $X_0, X_1, \cdots, X_n$ is a sequence of 2-connected edge induced subgraphs of G such that X_0 is a cycle and $X_n = G$, and $|E(X_i)| < |E(X_{i+1})|$ for $i = 1, 2, \cdots, n$. Now, we consider the such sequence of *vertex induced* subgraphs.

Definition 3.7. *Let $G \in \mathcal{G}$. A $\mathcal{G}$-series of G is a sequence $X_0, X_1, \cdots, X_s$ of vertex induced subgraphs of G such that $|V(X_i)| < |V(X_{i+1})|$ and $X_i \in \mathcal{G}$ with X_0 is a cycle and $X_s = G$. A $\mathcal{G}$-series $\{X_i\}$ of G is maximal if $V(X_{i+1}) \setminus V(X_i)$ is a minimal removable set of X_{i+1} for each i.*

Lemma 3.8. *Let G be a 2-connected graph with p vertices and q edges, and let $X_0, X_1, \cdots, X_s$ be a maximal $\mathcal{G}$-series of G with $|V(X_{i+1}) \setminus V(X_i)| \geq 2$ for each $i = 0, 1, \cdots, s-1$. Then*

(i) $X_{i+1} - X_i$ *is a removable ear of* X_{i+1};

(ii) $q = p + s$;

(iii) $q \leq \frac{3(p-1)}{2}$;

(iv) *if* $Y_0, Y_1, \cdots, Y_t$ *is a maximal* $\mathcal{G}$*-series of* G, *then* $s = t$.

Proof. (i) follows from Theorem 1.4. By (i), $|E(X_{i+1} - X_i)| = |V(X_{i+1})| - |V(X_i)| + 1$. Further, $|V(X_0)| = |E(X_0)|$. This proves (ii). It is easy to see that (iii) and (iv) follows from (ii). □

A structure theorem for dismantlable lattices in terms of adjunct of chains is proved by Thakare et al. [15]. The above lemma is analogous to some consequences of this structure theorem (*see* Lemma 2.3 and Theorem 2.5 of [15]).

Definition 3.9. *Let $G \in \mathcal{G}$ and let k be a positive integer. A $\mathcal{G}$-series $\{X_i\}$ of G is k-regular if $|V(X_{i+1})| - |V(X_i)| = k$ for all k, and $|V(X_0)| = k + 2$. The graph G is k-reducible if every maximal $\mathcal{G}$-series of G is k-regular.*

In the following result, we obtain a structure theorem for k-reducible graphs with $k \geq 2$. It follows from this result that the class of k-reducible graphs with $k \geq 2$ is a subclass of the class $\mathcal{C}$.

Theorem 3.10. *Let G be a 2-connected graph and let $k \geq 2$. Then G is k-reducible if and only if G has a k-regular $\mathcal{G}$-series $Y_0, Y_1, Y_2, \cdots, Y_s$ such that $Y_{i+1} - Y_i$ is an ear of Y_{i+1} whose end vertices are adjacent in Y_i for $i = 0, 1, 2, 3, \cdots, s - 1$.*

Proof. If part. Suppose that G is k-reducible. Then G has at least one maximal k-regular $\mathcal{G}$-series $Z_0, Z_1, \cdots, Z_s$. By Lemma 3.8(ii), $s = |E(G)| - |V(G)|$. We prove the result by induction on s. The result holds obviously for $s = 0, 1$. Suppose that $s \geq 2$. Assume that the result holds for $s-1$. We prove the result for s. By Lemma 3.8(i), $Z_{i+1} - Z_i$ is an ear of Z_{i+1} for each i. It is easy to see that Z_{s-1} is a k-reducible graph. By induction hypothesis, Z_{s-1} has a k-regular $\mathcal{G}$-series $Y_0, Y_1, Y_2, \cdots, Y_{s-1}$ such that $L_i = Y_i - Y_{i-1}$ is an ear of Y_i whose end vertices are adjacent in Y_{i-1} for $i = 1, 2, 3, \cdots, s-1$. Let $L_s = Z_s - Z_{s-1}$. Then $Y_{s-1} = Z_{s-1} = G - L_s$ and L_s is a removable ear of Z_s. Let x, y be the end vertices of L_s.

We claim that x and y are adjacent. Suppose that the claim is false. Let x_0, y_0 be the end vertices of L_1. Then x_0 and y_0 are adjacent. Let $L_0 = Y_0 - x_0y_0$. Then L_0 is an ear of Y_1 of length $k + 1$. Let m and n be smallest integers such that $x \in V(L_m)$ and $y \in V(L_n)$. We may assume that $0 \leq m \leq n \leq s - 1$. Let $X_r = Y_r$ for $r = 0, 1, \cdots, n$. Let $X_{n+1} = Y_n + L_s$ and $X_r = X_{r-1} + L_{r-1}$, for $r = n+2, \cdots, s$. It follows that $X_0, X_1, \cdots, X_s$ is a maximal k-regular $\mathcal{G}$-series of G. Note that $d_{X_n}(y) = 2$. Suppose that $x \notin V(L_n)$. Let x' and y' be end vertices of L_n. Let P_1 and P_2 be the x', y-subpath and the y, y'-subpath of L_n respectively. Since L_n has exactly $k + 2 \geq 4$ vertices, $|P_1| > 1$ or $|P_2| > 1$. Assume that $|P_1| > 1$. Since $x \notin \{x', y'\}$ and X_{n-1} is 2-connected, $X_{n-1} + P_2 + L_s = X_{n-1} + P = X_{n+1} - P_1$ is 2-connected. Thus P_1 is a removable ear of X_{n+1} with $2 < |P_1| < k + 1$. Thus $X_0, X_1, \cdots, X_{n-1}, X_{n+1} - P_1, X_{n+1}, \cdots, X_s$ is a maximal $\mathcal{G}$-series of G which is not k-regular. This is a contradiction to the fact that G is k-reducible. Hence $x \in L_n$. Let P_3 be the x, y-subpath of L_n. Since x and y are not adjacent, $2 \leq |P_3| < k + 1$. It is easy to see that $X_{n+1} - P_3$ is 2-connected. This fact leads to a contradiction. Thus the end vertices of L_s are adjacent.

Let $Y_s = Z_s = G$. Then $L_s = Y_s - Y_{s-1}$. Thus $\{Y_i\}$ is a maximal k-regular $\mathcal{G}$-series of G such that $Y_{i+1} - Y_i$ is an ear of Y_{i+1} whose end vertices are adjacent in Y_i for $i = 0, 1, 2, 3, \cdots, s-1$.

Only if part. Suppose that the later part in the hypothesis is true. It is easy to see that $\{Y_i\}$ is a maximal $\mathcal{G}$-series of G. Let $X_0, X_1, \cdots, X_t$ be any maximal $\mathcal{G}$-series of G. Then, by Lemma 3.8 (iv), $t = s$. It suffices to prove that $\{X_i\}$ is a k-regular $\mathcal{G}$-series. We prove this by induction of s.

If $s = 0$ or 1, then the result is obvious. Suppose that $s \geq 2$. Let $L_i = Y_i - Y_{i-1}$ for $i = 1, 2, \cdots, s$. Suppose that x_0 and y_0 are the end vertices of L_1. Then x_0 and y_0 are adjacent in Y_0. Let $L_0 = Y_0 - x_0y_0$. By hypothesis, L_i is a removable ear of length $k+1$ of Y_i for $i = 1, 2, \cdots, s$ and L_0 is a removable ear of length $k+1$ of X_1. Let $P_i = X_i - X_{i-1}$ for $i = 1, 2, \cdots, s$. Then, by Lemma 3.8(i), P_i is a removable ear of X_i. By Theorem 3.5, G is critically 2-connected. Therefore $3 \leq |P_s|$. Since P_s is an ear in $G = Y_s$, from the construction of Y_s, it is clear that P_s is a subpath of L_j for some $j \leq s$. Let x and y be end vertices of P_s. Suppose that $P_s \neq L_j$. Then we get at least one vertex z of L_j with $z \notin \{x, y\}$. We may assume that y is between x and z on L_j. As y is an end vertex of an ear, $d_G(y) \geq 3$. Hence at least one ear is attached to y and one of its neighbour on the y, z-subpath of L_j. Since ears are attached to adjacent vertices, it follows that z is a cut-vertex of $G - P_s$, which is a contradiction. Thus $P_s = L_j$.

Clearly, every internal vertex of L_j has degree 2 in G. This implies that no internal vertex of L_j is an end vertex of L_i for $i \neq j$. Hence L_{j+1} is attached to an edge of Y_{j-1}. Let $Z_i = Y_i$ for $r = 0, 1, \cdots, j-1$, $Z_j = Y_{j-1} + L_{j+1}$ and $Z_r = Z_{r-1} + L_{r+1}$, for $r = j+1, j+2, \cdots, s-1$. Further, $Z_{s-1} = G - L_j = G - P_s = X_{s-1}$. Thus $Z_0, Z_1, \cdots, Z_{s-1}$ is a k-regular $\mathcal{G}$-series of X_{s-1} such that $Z_{i+1} - Z_i$ is an ear of Z_{i+1} whose end vertices of are adjacent. By induction, the graph X_{s-1} is k-reducible. Since $X_0, X_1, \cdots, X_{s-1}$ is a maximal $\mathcal{G}$-series of X_{s-1}, it must be k-regular. As $|P_s| = |L_j| = k+1$, $X_0, X_1, \cdots, X_{s-1}, X_s$ is a k-regular $\mathcal{G}$-series of G. □

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