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Pentangulated graphs

Terry A. McKee
Department of Mathematics and Statistics
Wright State University
Dayton, Ohio 45435 USA.
e-mail: *terry.mckee@wright.edu*

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Abstract

One fundamental way to characterize chordal graphs—sometimes called triangulated graphs—is that every cycle C with $|C| \geq 3$ is the sum of $|C| - 2$ distinct triangles (3-cycles, where "sum" means the symmetric difference as sets of edges), or equivalently that every cycle C with $|C| \geq 4$ is the sum of a triangle and a $(|C| - 1)$-cycle. Define a graph to be pentangulated if every cycle C with $|C| \geq 5$ is the sum of $|C| - 4$ distinct pentagons (5-cycles). Conjecture: Being pentangulated is equivalent to every cycle C with $|C| \geq 6$ being the sum of a pentagon and a $(|C| - 1)$-cycle. The conjecture has been proved for distance-hereditary graphs and for interval graphs.

Keywords: chordal graph, triangulated graph, distance-hereditary graph.

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Suppose C is a cycle of a graph G. Let $|C|$ denote the length $|V(C)| = |E(C)|$ of C, and define a *k-cycle* to be a cycle of length k. A *chord* of C is an edge of G between nonconsecutive vertices of C. Chords vw and xy are *crossing chords* of C if their four distinct endpoints come in the order v, x, w, y around C. An *i-chord* of C is a chord vw such that the v-to-w distance is i within the graph C; thus, vw determines a cycle of length $i + 1$ that is formed by the edge vw and a length-i v-to-w path within C.

A cycle C *spans* another cycle C' if $V(C') \subseteq V(C)$. Call a *triangle* (meaning a 3-cycle) C' spanned by C an ECE-*cycle* if the edges of C' are, consecutively around C', an edge of C, then a chord of C, then an edge of C (the chord is, of course, a 2-chord of C). Call a *pentagon* (meaning a 5-cycle) C' spanned by C an ECECE-*cycle* if the edges of C' are, consecutively around C', an edge of C, then a chord of C, then an edge of C, then a chord of C, then an edge of C; furthermore, call this C' a *twisted* ECECE-*cycle* if the two edges of C' that are chords of C are crossing chords of C.

As motivation for the concepts introduced in this paper, Theorem 1 will state three simple, well-known characterizations of *chordal graphs*—graphs in which every cycle of length four or more has a chord [1, 9]. Chordal graphs have frequently been called *triangulated graphs* [3], since every induced cycle must be a triangle. The *sum* of cycles mentioned in condition (3) is the usual cycle space sum, where C is the sum of cycles C' and C''—denoted by $C = C' \oplus C''$—when $E(C) = [E(C') \cup E(C'')] - [E(C') \cap E(C'')]$.

Theorem 1. *Each of the following is equivalent to a graph G being chordal:*

(1) Every cycle C of G with $|C| \geq 4$ is the sum of a triangle and a cycle of length $|C|-1$.

(2) Every cycle C of G with $|C| \geq 4$ spans an ECE-*cycle.*

(3) Every cycle C of G with $|C| \geq 4$ is the sum of $|C|-2$ distinct triangles spanned by C.

The equivalence of being chordal with condition (1) apparently first appeared in print in [2]. Conditions (1) and (2) each essentially say that every C with $|C| \geq 4$ has a 2-chord. The equivalence of being chordal with (3) is from [5] (also see [6]).

Echoing condition (1), define a graph G to be *incrementally pentangulated* if every cycle C of G with $|C| \geq 6$ is the sum of a pentagon and a cycle of length $|C| - 1$. The graph in Fig. 1 is incrementally pentangulated. The (unique) 6-cycle $C = a, b, c, d, e, f, a$ is the sum of the pentagons a, b, c, e, d, a and c, d, a, f, e, c (each a twisted ECECE-cycle spanned by C). But if the chord bd were inserted, the resulting graph would not be incrementally pentangulated, because the newly formed 6-cycle a, b, d, c, e, f, a would not be the sum of two pentagons. It is easy to see that *every 6-cycle in a incrementally pentangulated graph has to have a 2-chord crossing a 3-chord.*

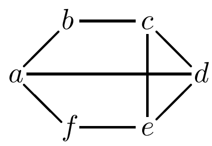

Figure 1: An incrementally pentangulated graph.

Theorem 2. *A graph is incrementally pentangulated if and only if every cycle C with $|C| \geq 6$ spans a twisted* ECECE-*cycle.*

Proof. First suppose every cycle of G of length six or more spans a twisted ECECE-cycle. Suppose C is any cycle with $|C| \geq 6$; say C spans a twisted ECECE-cycle a, b, d, c, e, a where ab, cd, and ae are edges of C and bd and ce are crossing chords of C. Let $\pi_{b,c}$ be the b-to-c path within C that does not contain a and $\pi_{d,e}$ be the d-to-e path within C that does not contain a. Then C is the sum of that ECECE-cycle and the cycle of length $|C| - 1$ that is formed by the paths $\pi_{b,c}$ and $\pi_{d,e}$ and the edges bd and ce. Therefore, G is incrementally pentangulated.

Conversely, suppose cycle C has $|C| \geq 6$ with $C = C' \oplus C''$ where C' is a 5-cycle and C'' is a $(|C| - 1)$-cycle. The pentagon C' cannot contain three or more chords of C (since $C = C' \oplus C''$ is equivalent to $C'' = C \oplus C'$, and so $C \oplus C'$ needs to be a cycle). Also, C'

cannot contain exactly one chord of C (since that would force $|C''| = |C| - 3 \neq |C| - 1$). Thus, $E(C')$ will contain exactly two chords of C, which need to be nonconsecutive edges in C' and crossing chords of C (again since $C \oplus C'$ needs to be a cycle). Therefore, C' must be a twisted ECECE-cycle spanned by C. □

It is straightforward to check that each graph in Figure 2 is incrementally pentangulated (for instance, the 7-cycle a, b, c, d, e, f, g, a in the left-most graph is the sum of the 5-cycles a, d, c, f, g, a and a, d, c, f, e, a and a, b, c, d, e, a), and indeed that Figure 2 shows all the incrementally pentangulated graphs that are spanned by a 7-cycle that has only 3-chords. Thus, *every* 7*-cycle in a incrementally pentangulated graph has to have a* 2*-chord or span a subgraph that is isomorphic to a graph in Figure* 2.

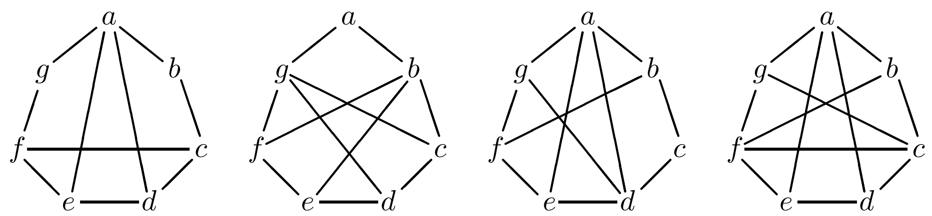

Figure 2: Four more incrementally pentangulated graphs.

Echoing condition (3), define a graph G to be *pentangulated* if every cycle C of G with $|C| \geq 6$ is the sum of $|C| - 4$ distinct pentagons spanned by C. (Requiring the pentagons to be distinct prevents the graph formed from C_8 by inserting one 4-chord from being pentangulated.) It is simple to show that every incrementally pentangulated graph is pentangulated, and that graphs of orders at most seven are pentangulated if and only if they are incrementally pentangulated.

Conjecture 3. *A graph is pentangulated if and only if it is incrementally pentangulated.*

Corollary 5 will show that Conjecture 3 holds for all *distance-hereditary graphs*—graphs G in which the distance between vertices in a connected induced subgraph of G is always equal to their distance in G, see [1]. Theorem 6 can also be combined with Theorem 6 of [8] to show that Conjecture 3 holds for all *interval graphs*—intersection graphs of intervals of the real line, see [1, 3, 9].

It is important to recognize that (incrementally) pentangulated graphs do not enjoy many of the nice properties of chordal graphs. In particular, they are not always perfect graphs [1, 3] (since C_5 is incrementally pentangulated), and they do not constitute an intersection class in the sense of [9, 10] (since the class is not closed under vertex expansion, with C_5 again a counterexample: if a vertex v of the C_5 is replaced with two new adjacent vertices that both have the same open neighborhood as v, then a 6-cycle results that has only 2-chords). Theorem 4 will give multiple characterizations of a narrower graph class that is contained in the class of perfect graphs and is an intersection class, and that also might be viewed as a potential contender for being the appropriate pentagon analogue of chordal (sometimes called "triangular") graphs.

A graph is chordal if and only if "every cycle large enough to have a chord does have a chord" or, in the present context, "every cycle large enough to span an ECE-cycle does span an ECE-cycle." But $|C| \geq 6$ is not necessary in order for C to span a twisted ECECE-cycle, since every 5-cycle spans a twisted ECECE-cycle in the graph formed from C_5 by inserting two crossing chords.

Theorem 4 will characterize the graphs for which every cycle C with $|C| \geq 5$ spans a twisted ECECE-cycle, using the following standard terminology: A graph is *nonseparable* if it is connected and remains connected whenever one vertex is deleted; a *block* of a graph is an inclusion-maximal nonseparable subgraph; a graph G is $\{H_1, \ldots, H_k\}$*-free* if no induced subgraph of G is isomorphic to any of the the subgraphs $H_1, \ldots, H_k$; P_4 is the path with four vertices; $2K_2$ consists of two vertex-disjoint edges; $K_{3,3}$ consists of a 6-cycle together with all three 3-chords; and $K_{2,2,2}$ consists of a 6-cycle together with all six 2-chords. The proof of Theorem 4 makes repeated use of Howorka's characterization [4] (also see [1]) of distance-hereditary graphs as those graphs in which every cycle C with $|C| \geq 5$ has crossing chords.

Theorem 4. *The following are equivalent for every graph* G:

(4.1) *Every cycle* C *of* G *with* $|C| \geq 5$ *spans a twisted* ECECE*-cycle.*

(4.2) G *is distance-hereditary and incrementally pentangulated.*

(4.3) G *is distance-hereditary and pentangulated.*

(4.4) G *is distance-hereditary and every* 6*-cycle of* G *has a* 2*-chord that crosses a* 3*-chord.*

(4.5) *Every block of* G *is* $\{P_4, 2K_2, K_{3,3}, K_{2,2,2}\}$*-free.*

Proof. The implication (4.1)⇒(4.2) follows from Theorem 2 and Howorka's characterization of distance-hereditary. The implications (4.2)⇒(4.3) and (4.3)⇒(4.4) were observed above.

(4.4) ⇒ (4.5): Suppose G is distance-hereditary, every 6-cycle of G has a 2-chord that crosses a 3-chord, and also that G is nonseparable (in other words, G consists of a single block). We first prove that G is P_4-free.

We begin by showing that if G contains an induced P_4, then G contains an induced P_4 that is a subpath of some cycle of G. Let $\pi = a, b, c, d$ be an induced path in G. Since G is nonseparable and $G \ncong K_2$, there is a cycle C that contains both edges ab and cd. Suppose bc is a chord of C with the vertices a, b, d, c in that order around C (otherwise π would be a subpath of a cycle, as desired). Let $E(C) = \{ab, cd\} \cup E(\pi_{a,c}) \cup E(\pi_{b,d})$ where $\pi_{a,c}$ is an induced a-to-c path and $\pi_{b,d}$ is an induced b-to-d path (each of $\pi_{a,c}$ and $\pi_{b,d}$ has length two or more, since π is an induced path, and so $|C| \geq 6$). Because $\pi_{a,c}$ and $\pi_{b,d}$ are induced paths, the only possible chords of C are of the form xy where $x \in V(\pi_{b,d})$ and $y \in V(\pi_{a,c})$. Since G is distance-hereditary, C must have crossing chords. Therefore, C must have a chord xy, crossing bc, where $x \in V(\pi_{b,d}) - \{b\}$ and $y \in V(\pi_{a,c}) - \{c\}$ and

$\{x, y\} \neq \{a, d\}$. Then π followed by the d-to-x portion of $\pi_{b,d}$ followed by the edge xy followed by the y-to-a portion of $\pi_{a,c}$ would form a cycle that contains π.

Suppose G contains an induced P_4 [arguing toward a contradiction]. Using the preceding paragraph, suppose $\pi = a, b, c, d$ is an induced subpath of some cycle C, chosen so that $|C| \geq 5$ is minimum. Let $\pi_{a,d}$ be the a-to-d subpath of C with $E(\pi_{a,d}) = E(C) - \{ab, bc, cd\}$. The minimality of $|C|$ implies that $\pi_{a,d}$ also is an induced path. Thus, the only possible chords of C are of the form bx or cy where $x, y \in V(\pi_{a,d}) - \{a, d\}$. Let a' and d' be the neighbors of a and d respectively in $\pi_{a,d}$. If $a' = d'$, then $|C| = 5$ and C could not have crossing chords [contradicting G being distance-hereditary]. So $a' \neq d'$ and $|C| \geq 6$. If $a'd' \in E(C)$, then $|C| = 6$ and C would not have a 2-chord crossing a 3-chord [contradicting condition (4.4)]. So $a'd' \notin E(C)$ and $|C| \geq 7$.

Since G is distance-hereditary, C must have crossing chords bx and cy, where x can be chosen as close as possible to d along $\pi_{a,d}$ and where y can be chosen as close as possible to a along $\pi_{a,d}$. If $a' \neq y$, then the cycle consisting of the edges ab, bc, cy, and the portion of $\pi_{a,d}$ between a and y would form a cycle of length five or more without crossing chords [contradicting G being distance-hereditary]. Therefore $a' = y$. A similar argument shows that $d' = x$. Let C' be the 6-cycle a, b, d', d, c, a', a, noting that $|C'| < |C| \geq 7$. But C' would contain the induced P_4 that consists of edges ab, bd', and $d'd$, [contradicting the choice of π and C with minimum $|C|$]. Therefore, G is P_4-free.

Next, suppose G contains an induced subgraph $H \cong 2K_2$ with $E(H) = \{pq, rs\}$ [arguing by contradiction]. Since G is nonseparable, there exists a cycle C that contains both the edges pq and rs. Further assume that H and C were chosen so that $|C| \geq 6$ is minimum. Suppose the vertices come in the order p, q, r, s around C and $E(C) = \{pq, rs\} \cup E(\pi_{s,p}) \cup E(\pi_{q,r})$ where $\pi_{s,p}$ is an induced s-to-p path and $\pi_{q,r}$ is an induced q-to-r path (each of $\pi_{s,p}$ and $\pi_{q,r}$ has length two or more since H is an induced subgraph). Every chord of C must join a vertex of $\pi_{s,p}$ to a vertex of $\pi_{q,r}$. Suppose for the moment that $|C| = 6$ with $E(C) = \{pq, qq', q'r, rs, ss', s'p\}$. Then the only possible chords that C can have are pq', sq', qs', rs', and $q's'$. But then C would not have a 2-chord crossing a 3-chord [contradicting condition (4.4)]. Therefore $|C| \geq 7$ and one of the induced paths $\pi_{s,p}$ and $\pi_{q,r}$ must have length three or more, but then that path would contain an induced P_4 [contradicting that G is P_4-free]. Therefore, G is $\{P_4, 2K_2\}$-free.

Finally, G cannot contain an induced $K_{3,3}$ or $K_{2,2,2}$ since neither of these has both 2-chords and 3-chords. Therefore (4.4) $\Rightarrow$ (4.5) holds whenever G is nonseparable.

If G satisfies (4.4) and has more than one block, then each block H satisfies (4.4) and is nonseparable, and so H is $\{P_4, 2K_2, K_{3,3}, K_{2,2,2}\}$-free. Thus, (4.4) $\Rightarrow$ (4.5) always holds.

(4.5) $\Rightarrow$ (4.1): Suppose every block of G is $\{P_4, 2K_2, K_{3,3}, K_{2,2,2}\}$-free and cycle C has $|C| \geq 5$. If $|C| = 5$, then being P_4-free implies C has two crossing 2-chords and C spans a twisted ECECE-cycle that uses those two chords. If $|C| = 6$, then being $\{P_4, 2K_2\}$-free and inducing neither $K_{3,3}$ nor $K_{2,2,2}$ implies that C has a 2-chord crossing a 3-chord and C spans two twisted ECECE-cycles that use those two chords. So suppose $|C| \geq 7$. If $|C| = 7$ and C has no 2-chord, then being P_4-free implies that C has all seven 3-chords and so spans many twisted ECECE-cycles.

In all other $|C| \geq 7$ cases, each P_4 subgraph of C must span a chord that is a 2-chord or a 3-chord of C that forms a smaller cycle C' with $E(C)$ and $|C'| \geq 6$. Repeating this with a P_4 that contains the new chord of C and two edges of C' produces additional cycles $C'', \ldots$, each of length one or two less than the previous cycle. This eventually reaches either a 6- or 7-cycle C^* that is formed by a 5- or 6-chord vw of C with $E(C)$. If vw is a 5-chord, then the 6-cycle C^* will span two twisted ECECE-cycles, one of which will not contain vw and so will be a twisted ECECE-cycle spanned by C.

So, as the remaining case, suppose C has no 5-chords but has a 6-chord v_0v_6 where $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ is a subpath of C with $v_0 \not\sim v_5$ and $v_1 \not\sim v_6$ (where the symbol $\not\sim$ denotes nonadjacency). Finally, suppose C spans no twisted ECECE-cycle [arguing by contradiction]. Note that v_1 must be adjacent to v_5 (to avoid v_1, v_0, v_6, v_5 being an induced P_4). Therefore $v_0 \not\sim v_3$ (to avoid C spanning a twisted ECECE-cycle $v_0, v_1, v_5, v_4, v_3, v_0$) and $v_3 \not\sim v_6$ (similarly). Note that $v_1 \not\sim v_3$ (to avoid v_6, v_0, v_1, v_3 being an induced P_4) and $v_3 \not\sim v_5$ (similarly). Also, v_0 must be adjacent to v_2 (to avoid v_0, v_1, v_2, v_3 being an induced P_4) and v_4 must be adjacent to v_6 (similarly). But this implies that $v_1 \not\sim v_4$ (to avoid C spanning a twisted ECECE-cycle $v_0, v_1, v_4, v_3, v_2, v_0$) and $v_2 \not\sim v_5$ (similarly). The four nonadjacencies $v_2 \not\sim v_5 \not\sim v_3 \not\sim v_1 \not\sim v_4$ would then prevent the 5-cycle $v_1, v_2, v_3, v_4, v_5, v_1$ from having crossing chords [a contradiction]. Thus condition (4.1) holds. □

Theorem 4 of [7] also shows that condition (4.5) is equivalent to G being distance-hereditary and every cycle C of G with $|C| \geq 6$ having a 2-chord that crosses a 3-chord.

Corollary 5. *Conjecture 3 is true for all distance-hereditary graphs.*

Proof. This is an immediate consequence of the equivalence of conditions (4.2) and (4.3). □

Notice that the pentangulated graphs in Figs. 1 and 2 are not distance-hereditary, and so are not in the narrower graph class characterized in Theorem 4. By contrast, condition (6.3) below will involve precisely those graphs.

Theorem 6. *If every cycle C of G with $|C| \geq 8$ has an i-chord with $i = \min\{5, |C| - 5\}$, then the following are equivalent:*

(6.1) *G is incrementally pentangulated.*

(6.2) *G is pentangulated.*

(6.3) *Every 6-cycle of G has 2-chord crossing 3-chord, and every 7-cycle of G either has a 2-chord or spans a subgraph that is isomorphic to one of the four graphs in Fig. 2.*

Proof. The implications (6.1)⇒(6.2) and (6.2)⇒(6.3) were observed above.

(6.3) ⇒ (6.1): Suppose every cycle C of G with $|C| \geq 8$ has a $\min\{5, |C|-5\}$-chord, condition (6.3) holds, and C is a cycle of G with $|C| \geq 6$ [toward showing that C has a twisted ECECE-cycle]. If $|C| \geq 8$ or if $|C| = 7$ and C has a 2-chord, let vw be a $\min\{5, |C|-5\}$-chord of C and let C' be the 6-cycle with $E(C') \subset E(C) \cup \{vw\}$; if $|C| = 6$, let $C' = C$. In these cases, condition (6.3) implies that C' has a 2-chord crossing a 3-chord, and so C' spans two edge-disjoint twisted ECECE-cycles, one of which does not contain vw and so is a twisted ECECE-cycle of C. The remaining case—where $|C| = 7$ and C has only 3-chords—follows from each of the 7-cycles in Fig. 2 spanning a twisted ECECE-cycle. □

References

[1] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[2] G. J. Chang and G. L. Nemhauser, The k-domination and k-stability problems on sun-free chordal graphs, *SIAM Journal on Algebraic and Discrete Methods*, **3** (1984), 332–345.

[3] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, San Diego, 1980.

[4] E. Howorka, A characterization of distance-hereditary graphs, *Quarterly Journal of Mathematics. Oxford (second series)*, **28** (1977), 417–420.

[5] R. E. Jamison, On the null-homotopy of bridged graphs, *European Journal of Combinatorics*, **8** (1987), 421–428.

[6] T. A. McKee, Requiring chords in cycles, *Discrete Mathematics*, **297** (2005), 182–189.

[7] T. A. McKee, Constrained chords in strongly chordal and distance-hereditary graphs, *Utilitas Mathematica*, in press.

[8] T. A. McKee, *Ensuring that cycles have i-chords for specified i*, submitted.

[9] T. A. McKee and F. R. McMorris, *Topics in Intersection Graph Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[10] E. R. Scheinerman, Characterizing intersection classes of graphs, *Discrete Mathematics*, **55** (1985), 185–193.