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Proper interval graph extention problems of the complements of trees

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Abstract

The extended profile problem is to find a proper interval supergraph with the smallest possible number of edges; the bandwidth problem is to find a proper interval supergraph with the smallest possible cliquesize. These two problems have important applications in numerical algebra, $VLSI$ designs and molecular biology. In this paper the extended profile and bandwidth values of the complement graph $\overline{T}$ of any tree T are given.

Keywords: proper interval graph, extended profile, bandwidth, tree, complement of tree.

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1. Introduction

An undirected graph G is said to be a *proper interval graph* if its vertices can be put into one-to-one correspondence with a set of intervals $\mathcal{C}$ of a linearly ordered set (like the real line) such that two vertices are adjacent if and only if their corresponding intervals have nonempty intersection, and all these intervals have equal length. The *extended profile problem* of a graph G is to find an added edge set F such that supergraph $G+F$ is a proper interval graph, and the number of added edges, denoted by $|F|$, is minimized, where the edge number $|E(G+F)|$ is called the *extended profile* of G, denoted as $\tilde{P}(G)$; and $|F|$ is called the *proper interval completion number* of G, denoted as $pic(G)$. The problem stems from the storage and elimination techniques of a sparse symmetric matrix A in 1950's. For instance, in the finite element method [17], we want to solve a system of linear equations $Ax = b$ where $A = (a_{ij})$ is a sparse symmetric $n \times n$ matrix with $a_{ii} \neq 0$ for all $1 \leq i \leq n$. For each row i, the set $\{a_{ij} : h \leq j < i \leq k, a_{kh} \neq 0\}$ is called the extended envelope of A. By viewing the extended envelope from each row, we call $w_i = i - min\{h : h \leq i \leq k, a_{kh\neq 0}\}$ the extended width of row i of A, and call

$\tilde{P}(A) = \sum_{i=1}^{n} w_i$ the extended profile of matrix A. To store A, we need only store $w_i + 1$ elements in each row i, which are from position $min\{h : h \le i \le k, a_{kh \neq 0}\}$ to position i. The total amount of storage for this scheme is then $P(A)+n$. In order to reduce the amount of storage, we need only permute the rows and columns of A simultaneously such that the resulting matrix has minimum extended profile, i.e., we need to find permutation matrix Q such that the profile $\tilde{P}(QAQ')$ is minimized. The problem of finding a permutation matrix Q for a matrix A such that fewest elements are introduced is equivalent to the minimum extended profile problem on a graph G [13], whose vertices correspond to the rows of A and in which (i, j) is an edge if and only if $a_{ij} \neq 0$. Based on [13], the extended profile problem on a graph has important applications to numerical algebra and even mathematical models in molecular biology.

A related problem is the bandwidth problem on graphs, which has also arisen from the storage and elimination techniques of a sparse symmetric matrix in 1950's [3, 5], and computing the bandwidth of a graph G is equivalent to finding a proper interval supergraph H of G with the minimum clique size [13]. It plays an important role in circuit designs, data structures, Gauss elimination and the dilation in an embedding of interconnection networks etc [4, 5, 10].

As an important graph-theoretic parameter, the extended profile problem for matrices and graphs has been less considered so far in graph theory. So, the computational complexity of this problem for general graphs has not been known. Since this problem is also equivalent to the proper interval graph complete number problem, by [16], this problem is solvable polynomially for some special graphs such as claw-free d-trapezoid graphs and claw-free permutation graphs, but is NP-complete for special co-bipartite graphs. Others still remain to be further investigated. The bandwidth problem for graphs has been extensively investigated in different aspects since the bandwidth value of $n-$cube Q_n, which equals $\sum_{i=0}^{n-1} \binom{i}{\lfloor \frac{i}{2} \rfloor}$, was found by Harper in 1966 [8], including its computational complexity, heuristic algorithms, lots of polynomially solvable special cases, lower and upper bounds and others (see surveys [3, 4, 10, 12]). For example, the bandwidth problem is NP-complete for general graphs [14]. Even for the trees with the degree of any vertex at most three [6], the bandwidth problem is also $NP-$complete, even though it is solvable in polynomial time for the caterpillars [12]. However, the bandwidth of the complement of a graph G was not found to our knowledge. Since there exists no polynomial time algorithms for $NP-$complete problems (unless $P = NP$), finding some special cases which can be solved polynomially is one of the important tasks for us to investigate. In this paper, the extended profile and bandwidth values of the complement $\overline{T}$ of a tree T (a connected acyclic simple graph) are given.

The rest of this paper is organized as follows. Section 2 presents some preliminaries. Section 3 is devoted to finding the extended profile value $\tilde{P}(\overline{T})$ of $\overline{T}$. Section 4 gives the bandwidth $B(\overline{T})$ of $\overline{T}$.

Terminologies and notations not defined in this paper can be found in [2, 15].

2. Preliminaries

Let $G = (V(G), E(G))$ be a finite, simple, connected graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$. We denote by $\overline{G} = (V(\overline{G}), E(\overline{G}))$ the *complement* of G, where $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{uv \notin E(G) : u, v \in V(G)\}$. If $S \subseteq V(G)$, then the *induced subgraph* $G[S]$ is defined as the subgraph of G with vertex set S and edges of $E(G)$ with both endvertices in S. The *neighbor sets* of a vertex v and of a vertex set $S \subseteq V(G)$ are $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G(S) = \{v \in V(G) \setminus S : u \in S, uv \in E(G)\}$, respectively. We refer to the cardinality of a maximum clique of G, denoted by $\omega(G)$, as the *clique size* of G. The *deficiency* of vertex v is defined as $D_G(v) = \{xy \in E(\overline{G}) : vx, vy \in E(G)\}$ and the *v-elimination graph* is defined as $G_v = G - v + D_G(v)$, which represents such an elimination operation: (i) deleting vertex v and its incident edges from G; (ii) adding edges in $N_G(v)$ so that $N_G(v)$ becomes a clique. For graph G, an *ordering* π of G is a bijection $\pi : \{1, 2, ..., n\} \to V$; its inverse bijection $\pi^{-1} : V \to \{1, 2, ..., n\}$ is called a *labelling* of G, where for each vertex v, $\pi^{-1}(v) \in \{1, 2, ..., n\}$ is called the *label* of v.

For an ordering π of G, $\pi(i) = v_i \in V(G)(i = 1, 2, ..., n)$, then the *elimination process* of the ordering $\pi = (v_1, v_2, ..., v_n)$ is

$$G^0 = G; \quad G^i = (G^{i-1})_{v_i} \quad (1 \leq i \leq n).$$

In this process, the set of added edges $F(G, \pi) = \bigcup\limits_{i=1}^{n-1} D_{G_i}(v_i)$ is called the *fill-in*; that is to say, $F(G, \pi)$ is a set formed by the edges in all deficiencies. The *minimum fill-in problem* is to find an elimination ordering π so that the number $|F(G, \pi)|$ is minimized. The *fill-in number* of G is defined as $f(G) = \min\limits_{\pi} |F(G, \pi)|$, where the minimum is taken over all orderings. An ordering π attaining the above minimum value is called an *optimal ordering* of G. An edge set $F \subseteq E(\overline{G})$ is called a *minimal fill-in* of G, if F is a fill-in of G, but no proper subset of F is a fill-in of G.

A graph G is a *chordal graph* if any cycle of G with length at least four has a chord (an edge between two nonconsecutive vertices on the cycle). A graph G is called a *proper interval graph* if its vertices can be put into one-to-one correspondence with a set of intervals on a line such that two vertices are adjacent if and only if their corresponding intervals intersect, and all these intervals have equal length. There have been many theoretic results on the characterization of chordal graphs, proper interval graphs, which can be seen in [1, 7].

Regarding the fill-in problem, two well-known properties are as follows.

Lemma 2.1. [1] *G is a chordal graph if and only if its fill-in number $f(G) = 0$.*

Lemma 2.2. [11] *If $S \subseteq V(G)$ is a clique of a graph G, then there exists an optimal ordering π in which the vertices of S are ordered last (or simply say, S is ordered last).*

The *treewidth* of a graph G, denoted as $TW(G)$, is one less than the least clique size of any chordal supergraph H of G, i.e.,

$$TW(G) = \min\{\omega(H) - 1 : G \subseteq H, H \text{ is a chordal supergraph of } G\}.$$

For graph G and a labelling $\pi^{-1} : V \to \{1, 2, ..., n\}$, the *extended profile of π^{-1} of G* is defined as

$$\tilde{P}(G, \pi^{-1}) = \sum_{v \in V(G)} [\pi^{-1}(v) - \min\{\pi^{-1}(x) : \pi^{-1}(x) \leq \pi^{-1}(v) \leq \pi^{-1}(y) \text{ and } xy \in E(G)\}].$$

The *extended profile* of G is the minimum value

$$\tilde{P}(G) = \min_{\pi^{-1}} \tilde{P}(G, \pi^{-1}),$$

where π^{-1} runs through all labellings of G. A *labelling* π^{-1} that attains the minimum is called an *optimal labelling* of G.

For graph G and a labelling $\pi^{-1} : V \to \{1, 2, ..., n\}$, the *bandwidth of π^{-1} of G* is defined as

$$B(G, \pi^{-1}) = \max_{uv \in E(G)} |\pi^{-1}(u) - \pi^{-1}(v)|.$$

The *bandwidth* of G is defined as

$$B(G) = \max_{uv \in E(G)} B(G, \pi^{-1}),$$

where π^{-1} is taken over all possible labelling.

In the version of graph extension, we have the following equivalent definitions [13].

The extended profile of graph G is the least size of any proper interval supergraph to which G can be extended, i.e.,

$$\tilde{P}(G) = \min\{|E(G + F)| : F \subseteq E(\overline{G}),\ G + F \text{ is a proper interval supergraph}\},$$

where the minimum value $|F|$ is called the *proper interval completion number*, denoted as $pic(G)$.

From the definition, we can see that $\tilde{P}(G) = |E(G)| + pic(G)$.

The bandwidth of G is one less than clique size of a proper interval supergraph to which G can be extended, i.e.,

$$B(G) = \min\{\omega(H) - 1 : G \subseteq H, H \text{ is a proper interval supergraph of } G\}.$$

A graph G is called *claw-free* if G does not contain the star $K_{1,3}$ (the claw) as an induced subgraph. Three mutually independent vertices x, y, z of a graph G are called an *asteroidal triple* (AT for short) if, between any two of them, there exists a path that avoids the neighborhood of the third.

A graph G without asteroidal triples is called *asteroidal triple-free* (AT-free for short). Proper interval graphs are AT-free and claw-free. Other examples of AT-free and claw-free graphs are co-bipartite graphs and AT-free line graphs.

Lemma 2.3. [7] *A graph G is a proper interval graph if and only if G is an AT-free and claw-free chordal graph.*

Lemma 2.4. [16] *If a graph G is AT-free and claw-free, then*

$$pic(G) = f(G), \quad B(G) = TW(G).$$

3. Extended Profile of the Complements of Trees

In this section, we will obtain the extended profile value of the complement $\overline{T}$ of tree T by means of the minimum fill-in of $\overline{T}$.

Lemma 3.1. *Let $T = (V(T), E(T))$ be a tree. Then the complement $\overline{T}$ of T is a proper interval graph if and only if the diameter $D(T) \leq 3$ in T.*

Proof. We first prove the "only if" part. Suppose that $\overline{T}$ is a proper interval graph, but the condition does not hold, i.e., $D(T) \geq 4$. It follows that there exists a path $P = xy...uv$ in T such that the distance between vertices x and v $d_T(x, v) \geq 4$. Then each of x and y is not adjacent to any of u and v in T. But $xy, uv \in E(G)$. Therefore these four vertices x, y, u, v form a cycle $xuyvx$ without chords in $\overline{T}$ by the definition of $\overline{T}$, which contradicts the chordality of $\overline{T}$. So $\overline{T}$ is not a proper interval graph by Lemma 2.3.

Next, we show the "if" part. Assume that $D(T) \leq 3$. When $D(T) \leq 2$, i.e., T is a star $K_{1,n}$ $(n \geq 1)$, $\overline{T}$ is a union of a vertex and a complete graph by the definition of $\overline{T}$. Clearly, $\overline{T}$ is a proper interval graph in this case. Now let $D(T) = 3$, then the structure of T is as follows: $V(T) = \{x_0, x_1, ..., x_s, y_0, y_1, ..., y_t\}$, $E(T) = \{x_0y_0\} \cup \{x_0x_i : 1 \leq i \leq s\} \cup \{y_0y_j : 1 \leq j \leq t\}$, where $s, t \geq 1$. By the definition of $\overline{T}$ again, we know that the neighbor sets of x_0 and y_0 are $\{y_j : 1 \leq j \leq t\}$ and $\{x_i : 1 \leq i \leq s\}$ in $\overline{T}$ respectively, $\{x_i : 1 \leq i \leq s\} \cup \{y_j : 1 \leq j \leq t\}$ is an independent set in T and is a clique in $\overline{T}$; And $\{x_0, y_0\}$ is a clique in T and is an independent set in $\overline{T}$ respectively. Consequently, if we perform the elimination operations of vertices in $\overline{T}$ according to the ordering

$$x_0, y_0, x_1, x_2, ..., x_s, y_1, y_2, ..., y_t,$$

then no fill-in edges occur. Thus, the fill-in number of $\overline{G}$ is zero. By Lemma 2.1, $\overline{T}$ is chordal. On the other hand, based on the structure of $\overline{T}$, we can verify that $\overline{T}$ is AT-free. In fact, if x, y, z is an asteroidal triple in $\overline{T}$, then $xy, yz, xz \in E(T)$, i.e., $xyzx$ is a cycle with length three in T, which contradicts that T is a tree. In addition, since $\{x_0\} \cup \{y_j : 1 \leq j \leq t\}$, $\{y_0\} \cup \{x_i : 1 \leq i \leq s\}$ and $\{x_i : 1 \leq i \leq s\} \cup \{y_j : 1 \leq j \leq t\}$ are maximal cliques in $\overline{T}$, it is easy to show that $\overline{T}$ is also claw-free. Hence $\overline{T}$ is a proper interval graph by Lemma 2.3. This establishes the proof. □

If a tree T satisfies the condition of Lemma 3.1, that is, T has diameter at most three, then $\overline{T}$ is a proper interval graph, and we need not consider this case below.

For any nonpendent edge uv of a tree T, let E_{uv} be the set of edges incident to the vertices of $\{u, v\}$, and let $T_{uv} = (V(T_{uv}), E(T_{uv})) = (V_{uv}, E_{uv}) = G[E_{uv}]$ be the subtree of T induced by E_{uv}. Obviously, the T_{uv} subtrees correspond one to one to the nonpendent edges; and T_{uv} has two properties as follows: (i) $V_{uv} = \{u, v\} \cup \{x_i : ux_i \in E(T)\} \cup \{y_j : vy_j \in E(T)\}$ $(i, j \geq 1)$; (ii) Its diameter $D(T_{uv}) \leq 3$.

By Lemma 3.1, the complement $\overline{T}_{uv}$ of T_{uv} is a proper interval subgraph in $\overline{T}$. And $\{x_i : ux_i \in E(T), i \geq 1\} \cup \{y_j : vy_j \in E(T), j \geq 1\}$, $\{u\} \cup \{y_j : vy_j \in E(T), j \geq 1\}$ and $\{v\} \cup \{x_i : ux_i \in E(T), i \geq 1\}$ are maximal cliques in the subgraph $\overline{T}_{uv}$ respectively. Moreover, we have

Lemma 3.2. *For each nonpendent edge $uv \in E(T)$, T_{uv} is a maximal subtree of T such that its complement $\overline{T}_{uv}$ is a proper interval graph.*

Proof. As we have just mentioned above, $\overline{T}_{uv}$ is a proper interval graph. Next, we show that T_{uv} is a maximal subtree with respect to this property. Assume this is not the case. Then there would be another subtree H of T such that T_{uv} is a proper subtree of H and $\overline{H}$ is a proper interval graph. Since $T_{uv} \subset H$, H must contain an edge xy such that $xy \notin E_{uv}$, and $\{u, v\} \cap \{x, y\} = \emptyset$. So there is at most a common vertex between $\{x, y\}$ and V_{uv}. Thus, there are two cases to consider:

Case 1. $|\{x, y\} \cap V_{uv}| = 1$.

Let $y_{j_0} \in N_T(v) = \{y_j : vy_j \in E(T), j \geq 1\}$ such that $y_{j_0} = y$. Since uv is not a pendent edge of tree T, for each vertex $x_i \in N_T(u) = \{x_i : ux_i \in E(T), i \geq 1\}$, the distance $d_T(x_i, x)$ between x_i and x is four. In addition, x_iuvyx is unique path between x_i and x in H. So, each of x_i and u is not adjacent to any of x and y in H, that is to say, x_i, y, u, x form a cycle x_i, y, u, x, x_i of length four without chords in $\overline{H}$. Thus $\overline{H}$ is not chordal, and further is not a proper interval graph by Lemma 2.3, a contradiction.

Case 2. $|\{x, y\} \cap V_{uv}| = 0$, i.e., each of u and v is not adjacent to any of x and y.

Thus, x, u, y, v form a cycle x, u, y, v, x of length four without chords in $\overline{H}$, which is also a contradiction by Lemma 2.3. The result follows. □

Corollary 3.3. *For any nonpendent edge $uv \in E(T)$, $F_{uv} = E(T) \setminus E_{uv}$ is a minimal edge set added such that $\overline{T} + F_{uv}$ is a proper interval graph.*

Proof. By Lemma 3.2, F_{uv} is a minimal fill-in such that $\overline{T} + F_{uv}$ is chordal. In addition, since $V(T) \setminus \{u, v\}$ is a maximal clique, and $\{u, v\}$ is an independent set in $\overline{T} + F_{uv}$, as the proof of Lemma 3.1, it is easy to verify that there exists no any asteroidal triple and $K_{1,3}$ as an induced subgraph in $\overline{T}+F_{uv}$. That is to say, $\overline{T}+F_{uv}$ is an AT-free and claw-free chordal graph. Thus, $\overline{T}+F_{uv}$ is a proper interval graph by Lemma 2.3, completing the proof. □

Furthermore, we will assert that a minimum edge set added in $\overline{T}$ can be found from among these $F_{uv}^{'}s$.

Lemma 3.4. *Let $F_{u_0v_0}$ be such that $|F_{u_0v_0}| = \min\{|F_{uv}| : uv$ is a nonpendent edge of $T\}$. Then $F_{u_0v_0}$ is a minimum added edge set of $\overline{T}$ such that $\overline{T}+F_{u_0v_0}$ is a proper interval graph.*

Proof. We show that $F_{u_0v_0}$ is a minimum fill-in of $\overline{T}$ first. Let F^* be a minimum fill-in of $\overline{T}$ produced by an optimal elimination ordering π^*. By Lemma 2.2, we know that, for a clique X of G, there exists an optimal elimination ordering π such that the vertices of X are ordered last. Now, let $v_{i_1}, v_{i_2}, ..., v_{i_q}$ be all pendent vertices of T. Then $X = \{v_{i_1}, v_{i_2}, ..., v_{i_q}\}$ is a clique in $\overline{T}$. So, we may assume that X is ordered last in ordering π^*. Thus the first vertex, say v_1, of π^* is chosen from a nonpendent edge u_1v_1. Note that v_1 is adjacent to all vertices in $V(T)\setminus(\{v_1\}\cup N_T(v_1))$ in $\overline{T}$. Hence, when v_1 is eliminated, $V(T)\setminus(\{v_1\}\cup N_T(v_1))$ will become a clique. That is to say, all edges belonging to $E(\overline{T})$ between the vertices of this clique are contained in F^*. Therefore, we can ignore the vertices of $V(T)\setminus(\{v_1\}\cup N_T(v_1))$ and only consider the fill-in of F^* between $N_T(v_1)\setminus\{u_1\}$ and $V(T)\setminus N_T(v_1)$. We have the following

Claim. All edges of T between the vertices of $N_T(v_1)\setminus\{u_1\}$ and $V(T)\setminus N_T(v_1)$ are in F^*.

Assume to the contrary that there exists an edge $xy \in E(T)\setminus F^*$. Without loss of generality, let $x \in N_T(v_1)\setminus\{u_1\}$ and $y \in N_T(N_T(v_1)\setminus\{u_1\})$, i.e., $v_1x \in E(T), v_1y \notin E(T)$. Since u_1v_1 is a nonpendent edge, there exists at least a vertex $w \in N_T(u_1)$ such that the distance between w and y is four in T. There are two cases to consider:

Case 1. If $wu_1 \notin F^*$, then we can find a cycle w, x, u_1, y, w of length four without chords in $\overline{T}+F^*$, which contradicts that F^* is a fill-in of $\overline{T}$ (i.e., $\overline{T}+F^*$ is chordal).

Case 2. If $wu_1 \in F^*$, then we may set $F' = (F^*\setminus\{wu_1\})\cup\{xy\}$. First, we show that F' is also a fill-in of $\overline{T}$. For edge u_1v_1, let vertex sets $X = N_T(v_1)\setminus\{u_1\} = \{x_i : 1 \le i \le i_0\}$, $W = N_T(u_1)\setminus\{v_1\} = \{w_j : 1 \le j \le j_0\}$, $Y = N_T(N_T(v_1)\setminus\{u_1\}) = \{y_l : 1 \le l \le l_0\}$, $S = V(T)\setminus(N_T(\{u_1,v_1\})\cup\{u_1,v_1\}\cup Y)$, $\overline{T}^0 = \overline{T}$. Clearly, $x = x_{i_1} \in X$, $y = y_{l_1} \in Y$, $w = w_{j_1} \in W$. According to π^*, the structure of the elimination graph $\overline{T}^1 = (\overline{T}^0)_{v_1}$ is as follows: $W\cup Y\cup S = V(T)\setminus(N_T(v_1)\cup\{v_1\})$, $X\cup W\cup S$, $Y\cup S\cup\{u_1\}$ and $X\cup S\cup\{u_1\}$ are maximal cliques respectively, and for any vertex $x_i \in X$, $w_j \in W$, $y_l \in Y$, w_j, x_i, u_1, y_l, w_j forms a cycle with length four in $\overline{T}^1$. Since the clique size of $W\cup Y\cup S$ is maximum among these cliques clearly, the vertices of $X\cup\{u_1\}$ can be ordered prior to those of $W\cup Y\cup S$ in π^* by Lemma 2.2. On the other hand, since $wu_1 \in F^*$, there exists at least a vertex, say $x \in X$, such that x is ordered prior to u_1 in π^*, which leads to $xy \notin F^*$ and $wu_1 \in F^*$. Now let π' be an elimination ordering obtained by exchanging the label of x with that of u_1 in π^*, and $F' = (F^* \setminus \{wu_1\}) \cup \{xy\}$. Then for any $x_i \in X$, $y_l \in Y$, $w_j \in W$, $x_i y_l \in F'$, $w_j u_1 \notin F'$, so $F(\overline{T}^1, \pi') \cup D_{\overline{T}^0}(v_1) = F'$. That is to say, F' is a fill-in of $\overline{T}$. Second, because $|F'| = |F^*|$, F' is still a minimum fill-in of $\overline{T}$. We may still denote this F' by F^*. If this case occurs for another edge $xy \notin F^*$, then we can carry out the same transformation continually. In this way, we will eventually obtain a minimum fill-in F^* satisfying the claim.

By the claim, it follows that $F_{u_1v_1} = E(T) \setminus E_{u_1v_1} \subseteq F^*$. Further, by the minimality of F^*, we have $F_{u_1v_1} = F^*$. That is to say, $F_{u_1v_1}$ is a minimum fill-in of $\overline{T}$. Thus, any $F_{u_1v_1}$ with $|F_{u_1v_1}| = |F_{u_0v_0}|$ is minimum fill-in of $\overline{T}$.

Next, we show that $\overline{T} + F_{u_0v_0}$ is also a proper interval graph. Since $\{u_0, v_0\}$ and $V(\overline{T} + F_{u_0v_0}) \setminus \{u_0, v_0\}$ are an independent set and a clique in $\overline{T} + F_{u_0v_0}$ respectively, we can verify that $\overline{T} + F_{u_0v_0}$ is AT-free. On the other hand, $\{v_0\} \cup (V(T) \setminus N_T(u_0)))$ and $\{u_0\} \cup (V(T) \setminus N_T(v_0)))$ are also maximal cliques in $\overline{T} + F_{u_0v_0}$, so $\overline{T} + F_{u_0v_0}$ is claw-free. Thus, $\overline{T} + F_{u_0v_0}$ is a proper interval graph by Lemma 2.3. This completes the proof. □

Theorem 3.5. *Let T be a tree. Then the proper interval completion number of the complement $\overline{T}$ of T is*

$$pic(\overline{T}) = |E(T)| - \max\{d_T(u) + d_T(v) : uv \in E(T)\} + 1,$$

where $d_T(v)$ denotes the degree of vertex v in T.

Proof. Let $\mathcal{S}$ be the set of all nonpendent edges in T. By Lemma 2.4 and Lemma 3.4, we have

$$pic(\overline{T}) = f(\overline{T}) = \min_{uv \in \mathcal{S}} |F_{uv}|.$$

Let $|F_{u_0v_0}| = \min_{uv \in \mathcal{S}} |F_{uv}|$. Then $pic(\overline{T}) = |F_{u_0v_0}| = |E(T)| - |E_{u_0v_0}|$. And it is clear that

$$|E_{u_0v_0}| = d_T(u_0) + d_T(v_0) - 1 = \max_{uv \in \mathcal{S}} (d_T(u) + d_T(v)) - 1.$$

Hence

$$\begin{aligned} pic(\overline{T}) &= |E(T)| - |E_{u_0v_0}| \\ &= |E(T)| - \max_{uv \in \mathcal{S}} (d_T(u) + d_T(v)) + 1. \end{aligned}$$

Thus, the proof is complete. □

Corollary 3.6. *Let $T = (V(T), E(T))$ be a tree with $|V(T)| = n$. Then the extended profile of the complement $\overline{T}$ of T is*

$$\tilde{P}(\overline{T}) = \frac{1}{2}n(n-1) - \max\{d_T(u) + d_T(v) : uv \in E(T)\} + 1,$$

where $d_T(v)$ denotes the degree of vertex v in T.

4. Bandwidth of the Complements of Trees

Note that the complement $\overline{T}$ of tree T with n vertices may be a complete graph K_{n-1} (the nontrivial component) together with one isolated vertex (trivial components). For example, if the diameter of T is two, i.e., T is a star $K_{1,n-1}$, then $\overline{T}$ is a union of one isolated vertex and a complete graph K_{n-1}. In this case, when no confusion can arise, we may say that $\overline{T}$ is a K_{n-1} for short. In general, we have the following result.

Proposition 4.1. *Let T be a tree with n vertices. Then the complement $\overline{T}$ is a complete graph K_{n-1} if and only if T is a star $K_{1,n-1}$.*

Lemma 4.2. [9] *Let T be a tree with n vertices. Then*

$$TW(\overline{T}) = \begin{cases} n-2 & \text{if the diameter of } T \text{ is at most two,} \\ n-3 & \text{otherwise.} \end{cases}$$

From Lemma 2.3 and Lemma 4.2, when the diameter $D(T)$ of tree T is at most two, $B(\overline{T}) = TW(\overline{T}) = n-2$ by the structure of $\overline{T}$; Next, when the diameter $D(T)$ of T is at least three, the maximum clique of any chordal supergraph H of $\overline{T}$, which achieves its treewidth value, has $n-2$ vertices. By the proof of Corollary 3.3, for any minimal added edge set $F_{uv} = E(T) \setminus E_{uv}$ corresponding to nonpendent edge $uv \in E(T)$, $\overline{T} + F_{uv}$ is not only a chordal supergraph but also a proper interval graph, and the maximum clique size of it equals $n-2$; Consequently, F_{uv} is one of added edge sets of $\overline{T}$ which achieves treewidth $TW(\overline{T})$. So by Lemma 2.4, we can obtain

Theorem 4.3. *Let T be a tree with n vertices. Then the bandwidth of the complement $\overline{T}$ of T is*

$$B(\overline{T}) = \begin{cases} n-2 & \text{if the diameter of } T \text{ is at most two,} \\ n-3 & \text{otherwise.} \end{cases}$$

This paper gives the extended profile and the bandwidth of the complement $\overline{T}$ of a tree T. Its other parameters, such as the cutwidth, topological bandwidth and the like, are worthy of further research.

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