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Stability number and even $[2, b]$-factors in graphs

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Abstract

Let $b, b \geq 2$, be an even integer. A spanning subgraph of a graph G whose vertices have even degrees belonging to the interval $[2, b]$, is called an even $[2, b]$-factor of G. In this paper, we prove a sufficient condition for the existence of an even $[2, b]$-factor which involves the stability number, the minimum degree of G and the integer b.

Keywords: factor, connectivity, minimum degree.

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1. Introduction

We consider simple graphs without loops. For notation and graph theory terminology we follow in general [14]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(x)$ the degree of a vertex x in G, and by $\delta(G)$ the minimum degree of G. A *spanning subgraph* of G is a subgraph of G with vertex set $V(G)$. If G has an (u, v)-path, then the *distance* from u to v, written $d_G(u, v)$ is the least length of the (u, v)-paths. Let a, b be fixed integers, a spanning subgraph F of G is called an $[a, b]$-*factor* of G if $a \leq d_F(x) \leq b$ for all $x \in V(G)$. A vertex of degree b in F is called a *saturated vertex*; otherwise, it is *unsaturated*. If F is connected then the factor is said to be connected. An $[a, b]$-*factor* F is said to be a *parity* $[a, b]$-*factor* if $a \equiv b (mod\, 2)$ and $deg_F(x) \equiv a (mod\, 2)$ for all $x \in V(G)$. In particular, a parity $[a, b]$-factor is an *even factor* when $a \equiv b \equiv 0 (mod\, 2)$.

For $S \subseteq V(G)$, let $|S|$ be the number of vertices in S and let $G[S]$ be the subgraph of G induced by S. We write $G - S$ for $G[V(G) \backslash S]$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Denote by $\alpha(G)$ the stability number of a graph G, by $\kappa(G)$ its vertex connectivity. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \backslash uv \in E(G)\}$. For a set $A \subseteq V(G)$, $N_G(A)$ denotes the set of neighbors in G of vertices in A. Given disjoint subsets $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of the edges in G with one extremity in A and the other one in B.

2. Known Results

Many authors have investigated $[a,b]$-factors in [1, 12, 5, 6, 8]. Tutte [13] gave the well-known necessary and sufficient condition for existence of an $[a,b]$-factor, where $a < b$.

Condition [13] *A graph G has an $[a,b]$-factor if and only if $b|S| - a|T| + \sum_{v \in T} d_{G \backslash S}(v) \geq 0$, for all pairs of disjoint subsets S and T of $V(G)$.*

This condition is also a corollary of the (g,f) factor theorem of Lovász in [9]. However, in practise, this condition remains difficult to verify. Studying connected factors was initiated by M. Kano [4]. Only few results are known which relate the stability number and factors. Neumann-Lara, Rivera-Campo [10] have proved the following theorem.

Theorem 2.1. [10] *If G is κ-connected graph, $b \geq 2$ is an integer, and $\alpha(G) \leq 1 + \kappa(b-1)$ then G has a spanning tree of maximum degree b (i.e., a connected $[1,b]$-factor).*

A similar result for connected $[2,b]$-factors was shown by Brandt [3]:

Theorem 2.2. [3] *Let $b \geq 2$ an even integer and $\kappa \geq 2$. If G is k-connected and $\alpha(G) \leq \frac{\kappa b}{2}$ then G has a 2-connected $[2,b]$-factor.*

The following results involving the stability number and the minimum degree of a graph was given by M. Kouider and Zbigniew Lonc [7]:

Theorem 2.3. [7] *Let $b \geq a+1$ and let G be a graph with the minimum degree δ. If* $\alpha(G) \leq \begin{cases} 4b(\delta - a + 1))/(a+1)^2, & \text{for } a \text{ odd;} \\ 4b(\delta - a + 1)/a(a+2), & \text{for } a \text{ even.} \end{cases}$ *then G has an $[a,b]$-factor.*

This theorem is related to the following results of Nishimura [11]:

1. Let G be a graph and let a be an odd positive integer. If
   (a) $\kappa(G) \geq (a+1)^2/2$,
   (b) $\alpha(G) \leq 4a\kappa(G)/(a+1)^2$, and
   (c) $|V(G)|$ is even,
   then G has an $[a,a]$-factor.
2. Let G be a graph and let a be an even positive integer. If
   (a) $\kappa(G) \geq (a(a+1))/2$,

(b) $\alpha(G) \leq 4\kappa(G)/a + 2$, and then G has an $[a, a]$-factor.

A similar sufficient conditions for existence of connected and of 2-connected $[a, b]$-factors was given by M. Kouider and Zbigniew Lonc [7]:

Theorem 2.4. [7] *Let G be a $\kappa(G)$-connected graph, $a \geq 2$, $b \geq a + 3$, and $(a, b) \neq (2, 5), (2, 7), (3, 6), (4, 7)$. If $\delta(G) \geq \frac{10(a+1)\kappa}{9(a-1)} + a$ and $\alpha(G) \leq \begin{cases} \frac{4\kappa b}{(a+1)^2}, & \text{for } a \text{ odd;} \\ \frac{4\kappa b}{a(a+2)}, & \text{for } a \text{ even.} \end{cases}$ then G has a connected $[a, b]$-factor.*

An analogous result of existence of 2-connected $[a, b]$-factor was given in the same paper:

Theorem 2.5. [7] *Let $b \geq a + 1$ and let one of the following two conditions be satisfied*

(i) $a \geq 4$ and $(a, b) \neq (4, 7)$ or

(ii) $a = 3$ and b is divisible by 4. If G is $\kappa(G)$-connected graph such that $\kappa \geq 2$, $\delta(G) \geq 2\kappa + a$, and $\alpha(G) \leq \dfrac{4b\kappa}{(a + 1)^2}$, then G has a 2-connected $[a, b]$-factor.

3. Main Results

We have established a sufficient condition for κ-connected graph to have an even $[2, b]$-factor which involves the stability number, the minimum degree of the graph and the integer b.

Theorem 3.1. *Let $b \geq 6$ be an even integer and let G be a κ-connected graph such that $b \leq \kappa$ and $\alpha(G) < (b - 1)(\delta - 1)/5$ then G contains an even $[2, b]$-factor.*

We will use the followings Lemmas (3.3 and 3.4) cited as exercises without proofs in the Bollobás's book [2] (p. 84) in the proof of our main theorem:

Lemma 3.2. *Each tree T contains as subgraph a path joining two leaves of T and which has at most one vertex of degree at least three in T.*

Proof. We prove this observation by induction on the order n of T. If $n \in \{2, 3\}$; then $T = P_n$; if $n = 4, T = P_4$ or $T = K_{1,3}$ and the lemma is satisfied. Suppose that $n > 4$ and each tree T' of order $n' < n$ verifies the observation. Let us consider a tree T, we set $T' = T - l$ where l is a leaf in T. As T' verifies the induction hypothesis, it contains a path P which joins two leaves l_1, l_2 of T', and has at most one vertex s of degree at least three. If l isn't a neighbor of any internal vertex of P, then P is the desired subgraph in T. Otherwise, if l is adjacent to some $l_i, i \in \{1, 2\}$, we consider then $P \cup \{l\}$. If l is adjacent to one vertex y of $P - \{l_1, l_2\}$, we consider the (l, l_1)-path (or (l, l_2)-path) which contains at most one vertex of degree at least three y in T. □

Lemma 3.3. *Each tree T with $2k$ leaves, contains k edge-disjoint paths joining pairwise distinct leaves, and each path has at most one vertex in common with the other paths.*

Proof. We prove this lemma by induction on k. If $k = 1$, $T = P_n$ and P_n joins two leaves. Suppose $k > 1$, and the result is true for any tree T' with $2k'$ leaves, $k' < k$. Let us consider a tree T with $2k$ leaves. It follows from the Lemma 3.2 that the tree T contains a path P joining two leaves and having at most one vertex y of degree at least three, the vertex y exists as $T \neq P_n$. Let $T' = (T-P)\cup y$. If $d_T(y) \geq 4$, the tree T' has exactly $2k-2$ leaves, by applying the induction hypothesis, there exists $(k-1)$ edge-disjoint paths joining distinct leaves, we add P, and we get the result. Suppose $d_T(y) = 3$, then T' has $2k-1$ leaves, we delete then the leaf y from T' and if the number of leaves is still unchanged we delete the new leaf created and we continue this process until we get a tree with $2k-2$ leaves and we apply the induction hypothesis thus we add P and we get k edge-disjoint paths. □

Lemma 3.4. *Let G be a graph and x be a vertex which is not a cutvertex and has degree at least $2k$, then there exists k edge-disjoint cycles containing x.*

Proof. Let $N'(x)$ be a set of $2k$ neighbours of x. The proof is by induction on k. If $k = 1$, the result is obvious.

Now, k is at least 2. As $G - x$ is connected, there exists a spanning tree of $G - x$, in particular this tree covers $N(x)$.

Among all the trees of $G - x$ covering $N'(x)$, we choose one tree T with the minimum number of edges. It follows that every pendant vertex of T is neighbor of x. Let P be a path connecting 2 neighbors of x, say u_1 and u_2 constructed by the precedent lemma. The path P has at most one vertex c in common with the graph $T' = \{T - P\} \cup \{c\}$.

If $|N'(x) \cap P| \geq 3$ and $c \notin N'(x)$, there exists $i \in \{1, 2\}$ and a vertex $u \in N'(x)$ such that $P_1 = P(u_i, u)$ does not contain c and does not contain any vertex of $N'(x)$ as internal vertex. As the path P, the subpath $P_1 = P(u_i, u)$ has at most the vertex c in common with the graph $T' = \{T - P_1\} \cup c$ and T' is connected.

If $|N'(x) \cap P| = 2$, let $P_1 = P$. In any case if u, u' are the extremities of P_1, we get a cycle $[u, x, u'] \cup P_1$. The remaining graph $G' = \{G - P_1\} \cup c$ has exactly $2k - 2$ vertices of $N'(x)$. We apply the induction hypothesis to G'. If $u = c \in N'(x)$, we consider the graph $G' = \{G - P_1\} \cup c$ with $N'(x) := N'(x) - \{c, u'\}$ which means, we kept c in G' without considering it as element of $N'(x)$, so G' has exactly $2k-2$ vertices of $N'(x)$ and we apply the induction hypothesis to G'. □

3.1. Proof of the main theorem

Proof. We prove this theorem by contradiction. Define F to be an even $[0, b]$-factor. Since $\delta \geq \kappa \geq 2$, F has a cycle. Moreover, assume that F has a minimum number of isolated vertices, it follows from the definition of F that these isolated vertices form a forest. There is a vertex x_0 of G such that $d_F(x_0) = 0$ and x_0 has at most one isolated vertex in F as neighbor in $G-F$; then x_0 has at least $\delta(G)-1$ neighbors in $G-F$ which are non isolated vertices in F.

Suppose that F consists on several components denoted $F_1, F_2, \ldots, F_p$. We define the operation φ_1 on F as follows (See figure 1: the dashed edges are those of $G-F$ and the white vertices are those whose degree has changed after implementation of φ_1):

Let u, v, w be a vertices of G, such that w is unsaturated by F and u, v are neighbors of w in $G-F$ and $d_F(w) = 2$.

a. If u and v are adjacent in F, we delete then from F, the edge uv and we add the edges of $G-F$, wu and wv.

b. If u and v are joined in F by a path of length three having only a vertices of degree strictly superior than four, we delete then the (u,v)-path and we add the edges wu and wv of $G-F$.

c. If u and v are unsaturated vertices in F which are adjacent in $G-F$, we add the triangle wuv in F.

d. If u and v have a neighbors u' and v' respectively in F and u' and v' are adjacent in $G-F$, we delete then from F the edges uu' and vv' and we add the edges wu, wv and $u'v'$ of $G-F$.

By the operations a, b, and, d, the degree of the vertices u' and v' in F still unchanged, even for u' and v'. In all cases, the degree of the vertex w increases by two. By φ_1, if a vertex has a degree at least four in F, it still of degree at least four.

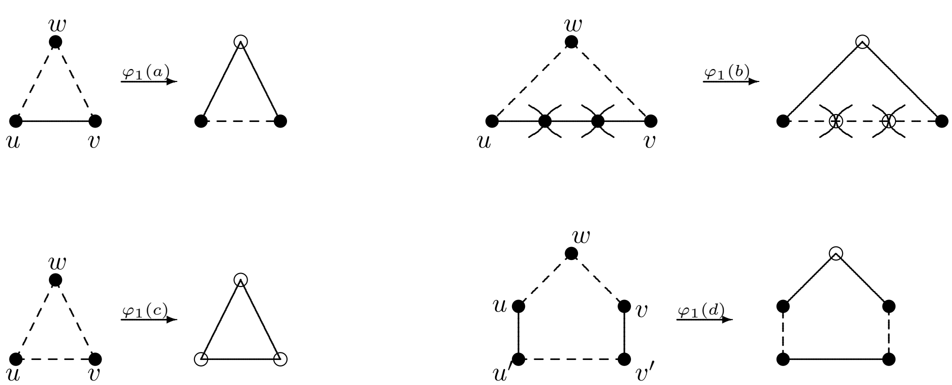

Figure 1: The insertion operation φ_1

A vertex of degree two in F is said to be *insertible*, if we can apply to this vertex, one of the operation a, b, c, d of φ_1. φ_1 is called *insertion operation.* The vertices of degree two which we cannot insert by φ_1 are called *noninsertible.* We apply the operation φ_1 and we reiterate it as possible. At the end, there is no insertible vertex.

For all saturated vertex s in F, we apply the following operation φ_2 (See Figure 2: the dashed edges are those of $G-F$ and the white vertices are those whose degree has changed after implementation of φ_2):

For each pair of vertices (u, v), neighbors of s in F:

a. If u and v have degree at least six in F and if they are adjacent in F, we delete then from F the triangle uvs.

b. If u and v are adjacent in $G-F$, we delete then from F the edges su and sv and we add the edge uv in F.

By applying φ_2, the vertex s becomes unsaturated, the vertices u and v retain their degree in F by $\varphi_2(b)$ while their degree decreases by $\varphi_2(a)$. As $b \geq 6$, in any case, we do not create a vertex of degree two by applying this operation. We apply φ_2 and we reiterate φ_1 and φ_2. For each iteration, φ_1 reduces the number of vertices of degree two and φ_2 leaves it unchanged. The iteration process then is finite. We cannot have thereafter vertices of degree strictly superior than four, as neighbors in F of a saturated vertex, which are adjacent in G.

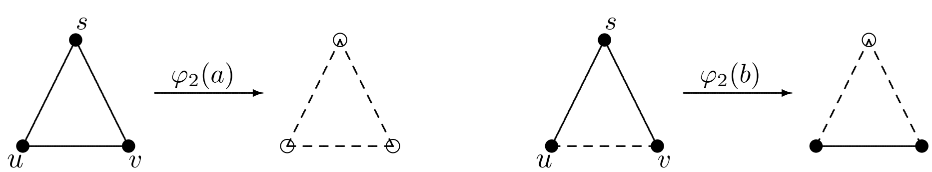

Figure 2: φ_2

We number the neighbors of x by $\{x_1, x_2, \ldots\}$. We denote each induced path of F delimited by two vertices u and v by $P[u, v]$.

Case 1. F has at least one component F_i such that $|F_i| > b$.

As the graph G is κ-connected and $b \leq \kappa$, by Menger's theorem in [14], there exists at least b vertex-disjoint paths joining the vertex x_0 to at least b disjoint vertices of the component F_i. Let $\{F_k, 1 \leq k \leq r\}$ be the set of different components of F. Let $i \leq r$ be a fixed integer. Let $(P_j(x_0, F_i))_j$ be the set of b internally disjoint paths from x_0 to F_i.

Let us first apply the following operations to the paths $(P_j(x_0, F_i))_j$: Let j, k be integers.

- (O_1) If a path P_j arrives at a point a on a component F_k and leaves the component at one point a' such that $|P_j \cap F_k| \geq 2$, and if there is no other path that uses the component F_k, we modify then P_j to a path P'_j such that $P'_j \cap F_k = P[a, a']$.

- (O_2) If one of the paths P' from x_0 to F_k contains a saturated internal vertex s such that $\{s\} = P'_i \cap F_t$, s is necessarily adjacent to two edges of $G - F$ in the path, we add so a minimum induced cycle $\mathcal{C}_k$ in F_t to the path and we choose $\mathcal{C}_k$ to be such that all its vertices have degree at least four in F if possible.

We denote the resultant paths by $(\mathcal{P}_j(x_0, F_i))_j$.

Proposition 3.5. *Let $(\mathcal{P}_j(x_0, F_i))_j$ be the set of b internally disjoint paths from x_0 to F_i. Then these paths form with F a set of at least $\frac{b-1}{3}$ elementary cycles of G, such that two by two the cycles have at most one vertex in common.*

Proof. Let us first contract each component F_k into one vertex f_k. Let t be a fixed integer. The union of a set of paths $(R_j(x_0, F_t))_j$, is called a fiber $\mathcal{F}_t$ if these paths are internally vertex disjoint. By induction, we show the following lemma.

Lemma 3.6. *There exists R_j, a family of b' subpaths of $\cup_{1 \leq j \leq b'}(\mathcal{P}_j(x_0, f_i))_j$, such that R_j are two by two internally vertex disjoint and they form a set of fibers $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_r$. Furthermore, each fiber contains at least two subpaths except possibly $\mathcal{F}_i$.*

Proof. By induction on b'. If $b' = 2$, in $\mathcal{P}_1 \cup \mathcal{P}_2$, there exists an elementary cycle containing x_0. We get a fiber with 2 paths. If $b' = 3$, if the third path $\mathcal{P}_3$ meets the fiber $\mathcal{F}_1$ in f_r, we get a fiber $\mathcal{F}_r$ of three paths; if not, $\mathcal{P}_3$ forms a fiber $\mathcal{F}_i$.

Suppose the lemma is true for $b' = b_0$. Let $b' = b_0 + 1$. So $\mathcal{P}_1, \ldots, \mathcal{P}_{b'-1}$ form a set of fibers as described in the Lemma. Consider a path $\mathcal{P}_{b'}[x_0, f_i]$. Either it does not meet the other fibers; so it forms alone a fiber $\mathcal{F}_i$. Or crossing from x_0, $\mathcal{P}_{b'}$ meets first a subpath R in the fibers, let f_l be the common vertex. Let $\mathcal{F}_j$ be the fiber of R. Either $\mathcal{F}_j$ contains at least 3 subpaths, then we substract R_{x_0} from F_j; $R[x_0, f_l] \cup \mathcal{P}_{b'}[x_0, f_l]$ form a new fiber $\mathcal{F}_l$. Or $\mathcal{F}_j$ contains at most 2 subpaths, then the construction is that of case $b' = 2$ or 3. □

Let us consider fibers $\mathcal{F}_k$ with exactly two subpaths $\mathcal{P}_1, \mathcal{P}_2$. We decontract the vertex f_k and let c_1 (resp. c_2) be the intersection of $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) and F_k. The union of $\mathcal{P}_1, \mathcal{P}_2$ and an induced path $P[c, c']$ of F_k forms a cycle in G.

We consider now fibers $\mathcal{F}_k$ with at least three subpaths. We decontract the vertices f_k and we use the algorithm given in the proof of the Lemma 3.4 to form edge-disjoint paths of F_k, having at most one common vertex. We replace then each used path by its induced path and add the subpaths of the fibers to form a new cycle in F_k. As the subpaths are two by two internally vertex disjoint following the Lemma 3.6, these cycles are pairwise edge-disjoint and each two cycles have at most one common vertex in F. The cycles formed then contain paths of F and edges of $G - F$.

So, if there are $2r_l + k_l, 0 \leq k_l \leq 1, r_l \neq 0$ paths arriving on a component F_k, we define in this case r_l edge-disjoint cycles containing the vertex x_0. The total number of paths which is at least b equals $\sum_{l=1}^{p}(2r_l + k_l)$ and the number of edge-disjoint cycles defined is then given by $\sum_{l=1}^{p} r_l$, we have $2\sum_{l=1}^{p} r_l \leq b \leq \sum_{l=1}^{p}(2r_l + k_l) = \sum_{l=1}^{p} 2r_l + \sum_{l=1}^{p} k_l \leq$

$\sum_{l=1}^{p}(2r_l)+p \leq \sum_{l=1}^{p} 2r_l+(\sum_{l=1,l\neq i}^{p} r_l)+1 \leq 3(\sum_{l=1}^{p} r_l)+1$, then $\frac{b-1}{3} \leq \sum_l r_l \leq \frac{b}{2}$, which means that we can define at least $\lceil \frac{b-1}{3} \rceil$ edge-disjoint cycles containing the vertex x_0. □

We assume that a cycle C_j contains the edges x_0x_j and $x_0x_{f(j)}$ such that $j < f(j)$. We fixe an orientation on each cycle C_j such that the vertices x_0, x_j and $x_{f(j)}$ appear in this order. For $u \in V(C_j)$, u^+(resp. u^-) denotes its successor (resp. predecessor). We assume that the vertices of each cycle are ordered by $\prec$. We denote any segment of C_j delimited by two vertices u and v by uC_jv if the vertices from u to v are ordered in the chosen direction of C. Otherwise, it will be denoted $u\overline{C}_jv$. A segment in a cycle may contain both edges of F and $G-F$. We denote by $F \cap C_j$ the subgraph of F whose vertices are those of C_j and whose edges are those of $E(F) \cap E(C_j)$.

Remark 3.7. *By the operation (O_2), there is no saturated vertex s internal to a path of $G-F$, so, any saturated vertex in a cycle C_j is adjacent to an edge of $E(F) \cap E(C_j)$.*

If each vertex in C_i has either degree at least four in F, or its degree in $F \cap C_i$ is at most one, we define then the *complementarity* operation which transforms any edge of $G-F$ to an edge of F and conversely. As consequences of this operation, we have:

Claim 3.8. *By the complementarity operation, the vertices adjacent to both an edge of F and another of $G-F$, conserve their degree, the degree of saturated vertices don't increase. The degree of other vertices, either increases by two or decreases by two.*

Lemma 3.9. *For any integer $i \geq 1$, C_i contains a vertex y such that $d_{F\cap C_i}(y) = d_F(y) = 2$.*

Proof. If each vertex of C_i has either degree at least four in F or its degree in $F \cap C_i$ is at most one, we perform then the complementarity operation in C_i, it follows then for the Claim 3.8, that there will be no violation of the parity of the vertices degrees in the cycle. We introduce then the vertex x_0 in the cycle, which is a contradiction. □

A vertex t_j in the cycle C_i is said to be *pseudo-insertible* (See Figure 3: the dashed edges are those of $G-F$) , if $d_{C_i}(t_j) = d_F(t_j) = d_{F\cap C_i}(t_j) = 2$ and if there exists a vertex t_j^0 in C_i such that $t_j^0 \prec t_j$ and such that there is a path P_j from t_j to t_j^0 in which each vertex except t_j has degree at least four in F and which verifies one of the following conditions:

a. P_j is an edge of $G-F$, and if furthermore, $(t_j^0)^-t_j$ is an edge of $G-F$ (ie, t_j^0 is an extremity of segment of F) then we suppose that t_j^0 is one extremity of a triangle T_j whose all its vertices have degree at least four in F.

b. P_j is a path of G of length two; $P_j = t_jvt_j^0$ while $v \notin C_i$, t_jv is an edge of $G-F$ and t_j^0v is an edge of F.

c. $P_j = t_j v \ldots t_j^0$ is a path of G of length three or four, and $(t_j^0)^- t_j^0$ is an edge of $G - F$ (ie, t_j^0 is an extremity of segment of F); where $t_j v$ is an edge of $G - F$ and $t_j^0 \ldots v$ is a path of F of length two or three in which each internal vertex u verify $u \prec t_j$ if $u \in C_i$.

We suppose that $d_{C_i}(t_j^0, t_j)$ is maximum under these conditions.

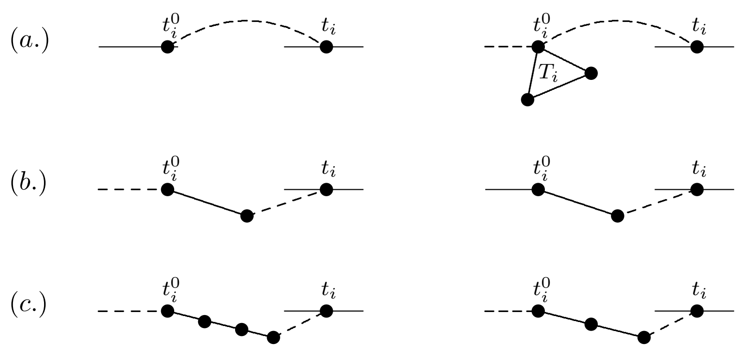

Figure 3: The definition of the pseudo-insertible vertex.

A *pseudo-insertible segment* in a cycle (resp. *pseudo-insertible cycle*) is a segment (resp. a cycle) in which any vertex is either pseudo-insertible or it has degree at least four in F or its degree in $F \cap C_i$ is at most one in C_i.

Let us consider a pseudo-insertible segment $x_i C_i t_j$, in which we define the pseudo-insertion operation φ_3 as follows:

1. If t_j is a pseudo-insertible vertex, then we replace the segment $t_j C_i t_j^0$ by

$$\begin{cases} P_j \cup T_j & \text{if } P_j \text{ and } (t_j^0)^- t_j \text{ are edges of } G - F; \\ P_j & \text{otherwise} \end{cases}$$

   a. If each vertex of the segment $x_i C_i t_{j^0}$ has either degree at least four in F or its degree in $F \cap C_i$ is at most one, we don't do any changement in the segment.

   b. Otherwise, we consider the nearest pseudo-insertible vertex from the vertex t_j^0, say t_k such that $t_k \prec t_j^0$ and we reiterate $\varphi_3(i)$.

2. If t_j is not a pseudo-insertible vertex, we do $t_j^0 := t_j$ and we apply then $\varphi_3(a)$ and $\varphi_3(b)$.

In φ_3, let us consider paths and triangles replacing two different segments $t_k^0 C_i t_k$ and $t_j^0 C_i t_j$ where $j < k$.

Claim 3.10. *The path P_k (or $P_k \cup T_k$ if a triangle is used) which replaces $t_k^0 C_i t_k$ is vertex disjoint from P_j (and $P_j \cup T_j$ if a triangle is used) which replaces $t_j^0 C_i t_j$ unless may be in t_j^0. In this case $t_k^0 C_i t_j$ is not elementary.*

Proof. **Case 1.** $t_k^0 \prec t_k \prec t_j^0 \prec t_j$.

If there is an intersection, then there exists a path of F of extremities t_i^0 and t_j^0.

- If t_k^0 is the final extremity of a segment of $F \cap C_i$, then $P_k \cup T_k$ cannot intersect $P_j \cup T_j$, as for any component F_k, $C_i \cap F_k$ has no more than one segment, the same holds for P_k and P_j when P_j has length superior than three in G. We deduce then that the path P_j has length two.

  The vertex t_j^0 cannot be in common to $P_k \cup T_k$ and P_j and t_k^0 cannot be internal to P_j, otherwise the edge of F, $t_k^0 t_j^0$ contradicts that $t_k^0 C_i t_j^0$ is an induced path of F. Thus, the intersection of $P_k \cup T_k$ (resp. P_k) and P_j gives a path of G of length two (resp. at most four), of extremities t_j and t_k^0, which contradicts the maximality of t_j^0.

- Otherwise, P_k is of length two. The intersection would give a path of extremities t_j and t_k^0, this would be of length two, which contradicts the maximality of t_j^0.

Case 2. $t_k^0 = t_j^0$.

Then C_i is not elementary and t_k^0 is a saturated vertex such that a cycle $\mathcal{C}$ was added to s and $t_j^0 = s$ equals to $C_i \cap F_{k'}$ for some k'.

Either P_j is of length at least two, the two paths meet outside C_i. By maximality of t_j^0, in the insertion we can omit $\mathcal{C}$.

Either P_j is an edge of $G - F$. As $s^- s \in E(G - F)$, necessarily P_k has length at least two. P_k and P_j meet only in s. □

It follows from the pseudo-insertion operation, the following result concerning a pseudo-insertible segment $x_i \dots v_q$.

Lemma 3.11.

(a) We get a new segment $x_i \dots v_q$ involving a sequence of edges of G such that any vertex of the new segment which is different from the pseudo-insertible vertices, has either degree at least four in F or its degree in $F \cap C_i$ is at most one.

(b) For any integer $i \geq 1$, C_i contains a vertex of degree two in F which is not pseudo-insertible.

Proof. (a) The proof is by induction on the number k of the pseudo-insertible vertices $t_1, \ldots, t_k$ used by φ_3 in the segment $x_i C_i v_q$.

It follows then, the following order: $t_k^0 \prec t_k \prec \cdots \prec t_1^0 \prec t_1$.

If $k = 1$, we replace the segment $t_1 C_i t_1^0$ either by the path P_1 or by $P_1 \cup T_1$. Let A be the replaced segment. We obtain thus the new segment $x_i C_i A$ in which each vertex has either a degree at least four in F or a degree in $F \cap C_i$ at most one.

Suppose that the assertion is true at the order $k-1$, let us verify at the order k.

Case 1. $x_i C_i v_q$ is elementary.

It follows from the proof of the Claim 3.10 (Case 1), that two different paths P_i and P_j used by φ_3 are vertex disjoint when C_i is elementary. If one path P_i has length three or four in G, by its definition, it doesn't use any vertex of degree at least four in $t_1 C_i t_i$. When we use $k-1$ pseudo-insertible vertices, we obtain a new segment $t_{k-1}^0 \ldots v_q$ in which each vertex has either degree at least four in F or its degree in $F \cap C_i$ is at most one. We reapply φ_3, we thus use the vertex t_k and we obtain the segment $A C_i t_{k-1}^0 \ldots v_q$ where A is either the path P_k or $P_k \cup T_k$. As the vertex t_k is pseudo-insertible, t_k will be an extremity of an edge of F. The vertex t_k^0, and the vertices of a triangle T_k if T_k is used, have degree at least four in F. So, we get the result.

Case 2. $x_i C_i v_q$ is not elementary.

There exists a saturated vertex s to which we have added a minimum induced cycle $\mathcal{C}_k \subseteq F_k$ in the segment $x_i C_i v_q$. If all vertices of $\mathcal{C}_k$ have degree at least four in F, we don't do any changement to $\mathcal{C}_k$. If $s = t_j^0$ for some j, where $t_j \notin F_k$ and P_j has length at least two in G, then we omit $\mathcal{C}_k$ when we perform φ_3 and then the degree of the vertex s in $F \cap C_i$ is one.

$\mathcal{C}_k$ is a pseudo-insertible cycle, and $s = t_j^0 = t_l^0$ where $t_l \in \mathcal{C}_k$ for some l (See the proof of the Claim3.10, Case 2) and P_j is an edge of $G - F$, or if $s = t_l^0$, then we can verify that s isn't internal to a path of $G - F$ in the new segment.

(b) Let us consider a cycle C_i. Following the Lemma 3.9, this cycle contains at least a noninsertible vertex.

Assume that any vertex in C_i of degree two in F is pseudo-insertible, we consider then the nearest pseudo-insertible vertex t_k from $x_{f(i)}$ in the sens $x_{f(i)} \overline{C}_i t_k$. Following the definition of the vertex t_k, each vertex of the segment $t_k C_i x_{f(i)}$ has either degree at least four in F or its degree in $G - F$ is at most one. As the segment $x_i C_i t_k$ is pseudo-insertible, we apply the operation φ_3, we get a new segment following a., by adding the edge $x_0 x_i$ and the segment $t_k C_i x_{f(i)} x_0$, we define a new cycle $\mathcal{C}$ in which every vertex has either degree at least four in F, or degree in $F \cap C_i$ at most one. We apply the complementarity operation to this cycle. When we apply the complementarity to one pseudo-insertible segment, there can exist a vertex s disturbed twice following the Claim 3.10 (See Case 2 in the proof), but its degree is still unchanged following the Case 2, as the vertex s has degree at least one in $F \cap C_i$. We don't create any isolated vertex. It follows from the complementarity, the insertion of the vertex x_0 in the factor F. This is a contradiction. □

When we apply φ_3 simultaneously to two cycles C_i and $C_j, i \neq j$, there can exist either a vertex or a path of G which can be used in φ_3 in both cycles. If it isn't the case, we say that φ_3 is done independently.

Claim 3.12. *Let $x_1C_1u_1$ and $x_2C_2u_2$ be two pseudo-insertible segments without common vertex (except possibly $u_1 = u_2$). If φ_3 does not occur independently in the two segments; then, there exists a path $R(t_i^0, t_j^0)$ of F between these two segments, in which each vertex has degree at least four in F. If R is chosen such that $d(x_i, t_i^0) + d(x_i, t_j^0)$ is minimum, then there is no path of G, $Q(a, b)$ with $a < t_i^0$, $b < t_j^0$ between the two segments.*

Proof. Let $x_0C_1u_1$ and $x_0C_2u_2$ be two pseudo-insertible segments. Suppose that there exists an intersection on the paths used by φ_3 in both segments. It means there exists a path P_i (resp. P_j) in $x_1C_1u_1$ (resp. $x_2C_2u_2$) such that $P_i \cap P_j \neq \emptyset$.

So, if P_i intersects C_i and C_j or if $P_i \cap P_j \neq \emptyset$ then there exists a path of F between the two cycles, all its internal vertices are of degree at least four in F. □

Lemma 3.13. *Let $x_1C_1u_1$ and $x_2C_2u_2$ be two pseudo-insertible segments without common vertex (except possibly $u_1 = u_2$). There doesn't exist a path of F connecting these segments, in which each internal vertex has degree at least four in F.*

Proof. Let $P(u, v)$ a path of F between $x_1C_1u_1$ and $x_2C_2u_2$, chosen among all paths of G between these segments such that $d_{C_1}(x_1, u) + d_{C_2}(x_2, v)$ is minimum.

As $P(u, v)$ is a path of F, then C_1 et C_2 have respectively the segments $P[c, c']$ and $P[d, d']$ on the same component F_k of F. It follows that $u \in P[c, c']$ and $v \in P[d, d']$.

Case 1. $u_1 \neq u_2$.

Following the choice of the path P and Claim 3.12, the pseudo-insertion operation φ_3 will be done independently in the two segments x_1C_1u and x_2C_2v. We add thereafter the path P. The resultant cycle $\mathcal{C}$ has each vertex, either of degree at least four in F or its degree in $F \cap C_i$ is at most one. We apply the complementarity to $\mathcal{C}$.

So, if u or v is an extremity of a segment of $F \cap \mathcal{C}$, it will retain its degree. Otherwise, u or v is internal to a path of F which means that its degree is at least four in F and thereafter, it decreases by two. The vertex x_0 will be inserted into F, which is a contradiction.

Case 2. $u_1 = u_2$.

Following the choice of paths in the proof of the Lemma 3.4 applied for the cycles which have segments on the same component of F, we know that the induced paths $P[c, c']$ and $P[d, d']$ have at most one vertex in common.

Let $[c, c'] \cap [d, d'] = \{w = u_1 = u_2\}$, the vertex w may be a neighbor of x_0 or there can exists a path of G from x_0 to w such that this path or the edge x_0w belongs to another cycle defined on F_k.

If $w \prec \{u,v\}$ and if furthermore $w = v$ or u, we perform then φ_3 in the segments x_1C_1w et x_2C_2w which are pseudo-insertibles, φ_3 is done independently. We apply the complementarity to the obtained cycle. As the vertex w has degree at least four in F, it degree decreases by two and the vertex x_0 will be introduced into F which is a contradiction. So, either $u \prec w$ or $v \prec w$ and in this case $P[c,u] \cap P[d,v] = \emptyset$ and we return back to the first case. □

For any integer $i \geq 1$, let w_i be the first non pseudo-insertible vertex along C_i. It follows from the definition of the vertex w_i that for any vertex u in C_i, such that $u \prec w_i$, the segment x_iC_iu is pseudo-insertible.

We deduce from the previous lemma, the following result:

Claim 3.14. *Let i,j be two integers such that $i \neq j$. The segments $x_iC_iw_i$ and $x_jC_jw_j$ are vertex disjoint.*

Proof. It follows from proof of the Lemma 3.13, Case 2 that either $w_i \prec w$ or $w_j \prec w$. □

Lemma 3.15. *Let i,j be two integers such that $i \neq j$. If there exists an edge u_1u_2 of $G-F$ between the segments $x_iC_iw_i$ and $x_jC_jw_j$ then $\exists i \in \{1,2\}$ such that u_i is saturated, $u_iu_i^+ \in E(F)$, and $u_iu_i^- \in E(G-F)$.*

Proof. Let (u_1,u_2) an edge of $G-F$ between $x_1C_1u_1$ and $x_2C_2u_2$, chosen among all paths of G between these segments such that $d_{C_i}(x_1,u_1) + d_{C_j}(x_2,u_2)$ is minimum, and for which for every $i \in \{1,2\}$, u_i is unsaturated, or, if u_i is saturated for some i, then either $u_iu_i^+ \notin E(F)$, or $u_iu_i^- \notin E(G-F)$.

It follows from the Claim 3.14 that the segments $x_iC_iw_i$ and $x_jC_jw_j$ are vertex disjoint.

Let us consider the pseudo-insertible segments $x_1C_1u_1$ and $x_2C_2u_2$. As there is no paths of F between these segments following the Lemma 3.13, we then apply φ_3 which is done independently following Lemma 3.12, we get then a new segments to which we add the edges x_0x_i, x_0x_j and u_1u_2 of $G-F$, we obtain then a cycle denoted $\mathcal{C}$. As the vertices $u_i, i \in \{1,2\}$ are either both unsaturated, or if one of them is saturated with $u_iu_i^+ \notin E(F)$, or $u_iu_i^- \notin E(G-F)$, we cannot get in the cycle $\mathcal{C}$, a saturated vertex adjacent to two edges of $G-F$, thus by applying the complementarity, we won't increase the degree of a saturated vertex in $\mathcal{C}$:

If u_i is unsaturated for some i, then its degree increases by two; if it has degree at least four, it retain its degree if $u_i^-u_i \in E(F)$, otherwise, its degree increases by two.

If u_i is saturated for some i, such that either $u_iu_i^+ \notin E(F)$, or $u_iu_i^- \notin E(G-F)$, then u_i retains its degree.

The vertex x_0 is then added to the factor F, which is a contradiction. □

We can prove with the same manner as in the proof of the previous lemma the following result:

Lemma 3.16. *There is no path of $G - F$ of length at most two with an unsaturated internal vertex between w_i and w_j for $i \neq j$.*

Lemma 3.17. *Let i, j be two integers such that $i \neq j$, w_j cannot be a neighbor in $G - F$ to a saturated vertex $s \in x_iC_iw_i$ such that the vertex s is one extremity of a triangle T in F, whose all its vertices have degree at least four in F.*

Proof. Let i, j be two integers such that $i \neq j$ and s be a saturated vertex in $x_iC_iw_i$ such that $s \in T$, where T is a triangle of F whose all its vertices have degree at least four in F and such that $d_{C_i}(x_i, s)$ is minimum and sw_j is chosen among all paths of G between x_iC_is and $x_jC_jw_j$ such that $d_{C_i}(x_i, s) + d_{C_j}(x_j, w_j)$ is minimum.

From the Lemma 3.15, we deduce that if the edge sw_j exists then s^-s is an edge of $G - F$ in C_j. We apply φ_3 to the pseudo-insertible segments $x_jC_jw_j$ and x_iC_is. φ_3 is done independently following the Lemma 3.12 and the Lemma 3.13. We add thereafter the edges sw_j, x_0x_j and x_0x_i, we obtain then a cycle denoted $\mathcal{C}$. If $T \subset C_j$, we apply the complementarity to $\mathcal{C}$, otherwise, the complementarity will be applied for $\mathcal{C} \cup \mathcal{T}$. In both cases, the vertices s and w_j conserve their degree, and x_0 is inserted to the factor F. This is a contradiction. □

Lemma 3.18. *Let i, j be two integers such that $i \neq j$. There does not exist a path $P = w_ia \ldots v$ of G where $v \in x_jC_jw_j$ with $d_F(v) \geq 4$, w_ia is an edge of $G - F$ and $P(a, v)$ is a path of F in which each vertex different from v doesn't belong to $x_jC_jw_j$ and has degree at least four in F.*

Proof. Let i, j be two integers such that $i \neq j$ and v be a vertex in $x_jC_jw_j$ such that $P = w_ia \ldots v$ is a path as described in the lemma, chosen among all paths of G between $x_iC_iw_i$ and $x_jC_jw_j$ such that $d_{C_j}(x_j, v) + d_{C_i}(x_i, w_i)$ is minimum.

We deduce from Lemma 3.14 that the segments x_jC_jv and $x_iC_iw_i$ are vertex disjoint. Following the choice of the path P and Claim 3.12 and Lemma 3.13, the pseudo-insertion operation φ_3 will be done independently in these segments. So, we apply φ_3 in x_jC_jv and $x_iC_iw_i$.

We remark that each path P_i used by φ_3 in $x_iC_iw_i$ is vertex disjoint from P, otherwise there will be a path of F between the two segments, contradicting Lemma 3.13. However, if there is a path P_j used by φ_3 in x_jC_jv which intersect P, there will contradict the definition of P. So, φ_3 don't perturb any vertex of P

We add thereafter the path P and the edges x_0x_i, x_0x_j. The resultant cycle $\mathcal{C}$ has each vertex, either of degree at least four in F or its degree in $F \cap C_i$ is at most one. We apply the complementarity to $\mathcal{C}$.

The degree of each vertex of $p(a, v)$ except v, decreases by two, v, w_j conserve their degree. The vertex x_0 will be inserted into F which is a contradiction. □

Lemma 3.19. $\cup_{i\geq 1} w_i$ *is an independent set in* G.

Proof. For any integer i, j such that $i \neq j$, then $w_i \neq w_j$, since every vertex w_i is defined on a cycle C_i, and the cycles $C_i, i \geq 1$ are edge-disjoint and the vertices w_i have degree two in F. If w_i and w_j are in the same component of F, they are of distance at least two. From the Lemma 3.15 and the Lemma 3.16, we deduce that $w_i w_j$ is not an edge of $G - F$. □

For any saturated vertex s neighbor of w_i in $G - F$ and for any pair of vertices u, v of degree at least four in F such that s, u, v form a triangle T of F. From Lemma 3.17, we can by using again $\varphi_2(a)$ to T, reduce to the case when $min(d_F(u), d_F(v)) = 2$, we deduce then the following result.

Claim 3.20. *By applying again* $\varphi_2(a)$, *the segments* $x_i C_i w_i, i \geq 1$ *are still pseudo-insertible.*

Proof. Let s be a saturated vertex adjacent to w_i in $G - F$ and u, v be vertices of degree at least four in F which form with s a triangle T in F. We apply $\varphi_2(a)$ to T. By the way, the vertices u, v may become of degree two, but they don't belong to $\cup_{i\geq 1} x_i C_i w_i$. In fact, it follows from the Lemma 3.17, that the vertex s doesn't belong to $x_j C_j w_j$, for any $j \neq i$. So, if $s \in x_i C_i w_i$, then w_i will be pseudo-insertible, which is a contradiction. However, if u or v are in the cycle C_i, we have necessarily $w_i \prec u, v$ otherwise w_i will be pseudo-insertible too. If u or v are in the cycle $C_j, j \neq i$, it follows from the Lemma 3.18 that neither u, nor v are in $x_j C_j w_j$, thus the claim holds. □

We denote by S_i (resp. $\overline{S}_i$) the set of the neighbors of w_i in $G - F$, which are saturated by F (resp. unsaturated by F).

Lemma 3.21. $\overline{S}_i$ *is an independent set in* G *of cardinality at least* $(\delta - 2) - \frac{2\alpha}{b-1}$.

Proof. Let u and v be vertices of $\overline{S}_i$. If u and v are adjacent in F, we apply then the operation $\varphi_1(a)$ and if they are adjacent in $G - F$, we apply $\varphi_1(c)$. In both cases, we insert the vertex w_i, which contradicts its definition.

For any integer i, S_i is a stable in F, since if there exists two vertices of S_i adjacent in F, we apply $\varphi_1(a)$, which implies the insertion of the vertex w_i, which is a contradiction.

We set $L_i = N_F(S_i)$.

Claim 3.22. $F[L_i] = G[L_i]$.

Proof. Suppose the existence of an edge of $G - F$ joining two vertices u, v of L_i. The vertices u, v cannot be in the neighborhood of the same vertex because of the operation $\varphi_2(b)$ carried out, as a result, they will be in the neighborhood of two disjoint vertices, we therefore apply $\varphi_1(d)$, which involves the insertion of the vertex w_i, which is a contradiction. □

We have $L_i = N_F(S_i)$, we set L_i^4 the set of vertices of L_i which have degree at least four in F.

We define also the set $L_2 = L_i - L_i^4$ and $L_2'' = \{u \in L_2$ such that $d_{F[L_i]}(u) = 1\}$ and $L_2' = L_2 - L_2''$.

Claim 3.23. *L_2' is an independent set in G.*

Proof. It follows from the Claim 3.22 and the definition. □

Let L_i^6 be the set of vertices of L_i^4 which have degree at least six in F, we prove the following result.

Claim 3.24. *L_i^6 is an independent set in G.*

Proof. Let u and v be two vertices in L_i^6. If u and v are neighbors of the same vertex, we know after the implementation of the operation φ_2 that u and v are neither adjacent in F, nor in $G-F$. If, however, u and v are in the neighborhood of two disjoint vertices u' and v' respectively, and if u and v are adjacent in $G-F$, we therefore apply $\varphi_1(d)$, which implies the insertion of the vertex w_i, which is a contradiction. If u and v are adjacent in F, we apply $\varphi_1(b)$, which implies also the insertion of the vertex w_i, which is a contradiction. □

Let us consider the set $S_i \cup L_i^4$. From the Claim 3.24, we deduce that each cycle in $S_i \cup L_i^4$, has at least two vertices of degree exactly four in F.

From the Lemma 3.15, it follows that it can exist a saturated vertex $s \in S_i$, neighbor of w_i in $G-F$ such that $s \in \cup_{j\neq i} x_j C_j w_j$. In this case, $s^-s \in E(G-F), s^+s \in E(F)$ and s isn't an extremity of a triangle whose all its vertices have degree at least four in F following the Lemma 3.17. If $j = i$, as the vertex w_i is not pseudo-insertible, it can exist a vertex $s \in x_i C_i w_i \cap S_i$ such that $s^-s \in E(G-F)$ and s isn't an extremity of triangle in which each vertex has degree at least four in F. We set S_0 the set of such vertices s in $\cup_{j\neq 1} x_j C_j w_j$ and S_0^+ the set of the successors of each vertex of S_0. It is obvious that $S_0^+ \subset L_i$.

Remark 3.25. *There cannot exist a cycle $\mathcal{C} \subset S_i \cup L_i^4$ which contains two vertices s_1, s_2 of $x_i C_i w_i \cap S_0$, because in this case $s_1^- s_1$ and $s_2^- s_2$ are in $G-F$.*

However a cycle $\mathcal{C} \subset S_i \cup L_i^4$ may contains one vertex s_1 in $x_i C_i w_i \cap S_0$ and one vertex s_2 of $S_i \cap C_i$ such that $w_i \prec s_2$, but in this case there exists at most one such couple (s_1, s_2) following the remark given above. Let us consider the set $S_i' = S_i - \{s_1\}$ instead of S_i.

We eliminate one by one each cycle in $S_i' \cup L_i^4$ and we get.

Claim 3.26. *By eliminating one by one each cycle in $S_i' \cup L_i^4$, the segments $x_j C_j w_j, j \geq 1$ are still pseudo-insertible.*

Proof. Let us eliminate one by one each cycle in $S'_i \cup L^4_i$.

Suppose that there exists a cycle $\mathcal{C} \subset S'_i \cup L^4_i$ which contains either a vertex $s \in S_0$ or s^+ (Which means that we can disturb one vertex of $x_jC_jw_j$ when we delete this cycle). Suppose first that $s \in \mathcal{C} \cap x_iC_iw_i$. Without loss of generality we suppose that $s^+ \in \mathcal{C}$ too. As the vertex s isn't an extremity of a triangle in F with vertices of degree at least four in F, thus it exists a vertex $l_0 \in (S'_i \cup L^4_i) \cap \mathcal{C}$, such that l_0 is adjacent to s^+ and not adjacent to s. So, $l_0 \notin S_0$ following the remark 3.25 and $l_0 \notin x_jC_jw_j, j \neq i$ following the Lemma 3.13. As l_0 is of distance at most two from w_i, then $l_0 \notin x_iC_iw_i$, otherwise w_i is pseudo-insertible. So, the path of G containing l_0 and which joins w_i and s has length three or four which implies that w_i is pseudo-insertible which is a contradiction.

If the vertex s is in $x_jC_jw_j, j \neq i$. Without loss of generality, we suppose that both s, s^+ are in $\mathcal{C}$. As s isn't an extremity of a triangle of F, there will exist a vertex $l_1 \in \mathcal{C} \cup (L^4_i \cup S'_i)$ such that l_1 is adjacent to s^+ and not adjacent to s. The vertex $l_1 \notin x_jC_jw_j$, by the Lemma 3.15and 3.18 and $l_1 \notin x_{j'}C_{j'}w_{j'}, j' \neq j$ by the Lemma 3.13. Each internal vertex of the path of G of length at most two, joining w_i to l_1 is external to $\cup_{j \neq 1} x_jC_jw_j$ (The same argument to that of l_1). We get then a path of G between w_i and s^+ of length three which contradicts the Lemma 3.18. □

From the Claim 3.26, we deduce the following result.

Claim 3.27. *$S'_i \cup L^4_i$ is a forest in F.*

We have $e_G(S'_i, L^4_i) \leq |S'_i| + |L^4_i|$ as $S'_i \cup L^4_i$ is a forest and we have also $e_G(S'_i, L_2) \leq 2|L'_2| + |L''_2|$. So, $e_G(S'_i, L_i) = e_G(S'_i, L^4_i) + e_G(S'_i, L_2) \leq |S'_i| + |L^4_i| + 2|L'_2| + |L''_2|$. As $e_G(S'_i, L_i) = b|S'_i| \leq |S'_i| + |L^4_i| + 2|L'_2| + |L''_2|$, we deduce that $(b-1)|S'_i| \leq |L^4_i| + 2|L'_2| + |L''_2|$.

On the other hand, we have $\alpha(G) \geq \alpha(F[L_i]) \geq \alpha(F[L^4_i \cup L''_2]) + |L'_2| \geq \frac{|L^4_i| + |L''_2|}{2} + |L'_2|$ which implies that $2\alpha(G) \geq |L^4_i| + 2|L'_2| + |L''_2| \geq (b-1)|S'_i|$. So, $|S'_i| \leq \frac{2}{b-1}\alpha(G)$.

Hence, $|S_i| \leq \frac{2}{b-1}\alpha(G) + 1$ and then $|\overline{S}_i| \geq \delta - 1 - \frac{2\alpha(G)}{b-1}$. □

Lemma 3.28. *$\bigcup_{i \geq 1} \overline{S}_i$ is an independent set in G.*

Proof. It follows from Lemmas 3.19 and 3.16 that for any integer $i \neq j$, w_iw_j cannot be an edge of $G - F$ and that for any integers, i, j such that $i \neq j$, $\overline{S}_i \cap \overline{S}_j = \emptyset$. □

According to Lemma 3.21 and 3.5, $\bigcup_{i \geq 1} \overline{S}_i$ is a stable in G of cardinality at least $\frac{b-1}{3}(\delta - 1 - \frac{2\alpha(G)}{b-1})$. We have $\frac{b-1}{3}(\delta - 1 - \frac{2\alpha(G)}{b-1}) > \alpha(G) \Leftrightarrow \delta - 1 > \frac{5\alpha(G)}{(b-1)} \Leftrightarrow \alpha(G) < \frac{(b-1)(\delta-1)}{5}$.

Case 2. All the components F_i of F have at most b vertices.

This means that the factor F has no saturated vertex. Consider then one set of κ vertices neighbors of x_0, $X_0 = \{x_1, \ldots, x_\kappa\}$, as consequence of the Menger's theorem and the expansion lemma of whitney in [14], there exist $\lfloor \kappa/2 \rfloor$ vertex-disjoint paths between a subset of $\lfloor \kappa/2 \rfloor$ vertices of X_0 and its complement in X_0. We can form then $\lfloor \kappa/2 \rfloor$ vertex-disjoint cycles which have only the vertex x_0 in common. We remark that such cycle cannot contains only the edges of $G - F$, otherwise the vertex x_0 will be joined to the factor F by adding the edges of the cycle in F since F hasn't saturated vertices, which is a contradiction. We denote these cycles by $C_1, \ldots, C_{\lfloor \kappa/2 \rfloor}$, we fixe an orientation on each cycle C_i such that the vertices x_0, x_i appear in this order and we return back to the first case. We prove with the same manner as in Lemma 3.9 and in Lemma 3.11 that each cycle C_i contains a noninsertible vertex w_i. We consider then the set of neighbors of w_i, not saturated in F which we denote by $\overline{S}_i$, and we prove with same manner as in the Lemma 3.28 that $\bigcup_{i \geq 1} \overline{S}_i$ is a stable in G with cardinality at least $\lfloor \kappa/2 \rfloor(\delta - 2)$. As $\alpha(G) < \frac{(b-1)(\delta-1)}{5} \leq \frac{(\kappa-1)(\delta-1)}{5} < \lfloor \kappa/2 \rfloor(\delta - 2)$ there is a contradiction. □

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